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ENHANCED STABILITY CONDITIONS OF TIME-DELAYED GENERALIZED NEURAL NETWORKS VIA A NOVEL LYAPUNOV FUNCTIONAL

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Abstract. *This paper is devoted to discussion of the stability analysis of time-delayed generalized neural networks. A modified Lyapunov-Krasovskii functional (LKF) is constructed by combining the delay-product-type functional with delay-dependent matrices. The augmented LKF contains more coupling information of the nonlinearity, the time delay intervals and other state variables, which can further reduce the conservativeness of stability criteria. Moreover, to further show the validity of the modified LKF, two corollaries are also given under other related simplified LKFs. Finally, some common numerical examples are presented to show the effectiveness of the proposed approach.*

Keywords: Delay-product-type functional, Delay-dependent matrices, Lyapunov stability theory, Linear matrix inequality, Neural networks

1. Introduction.

At present, the research and application of neural networks are very popular. It is involved in many aspects, such as pattern recognition, associative memory, optimization, image and signal processing, and other science and engineering problems [1]. The most important problem on artificial neural networks is that it usually has to be stable. However, time delays are often unavoidable, such as in the communication between neurons, in the switching speed of amplifiers, which are often considered as the main cause of low performance and even instability [2-4]. The bigger maximum admissible delay upper bound is, the higher endurability on transmission delays of the neural network is. Therefore, the stability of time-delayed neural networks (DNNs) is always a hot topic, and the main purpose is maximizing the delay upper bound as soon as possible for stable DNNs [5-19], where many important delay-independent and dependent stability criteria were derived via different LKFs and various inequality techniques. The maximum admissible delay upper bounds for DNNs stability are getting larger and larger as the conservativeness of stability criteria decrease. In a word, the main efforts to obtain stability criteria of DNNs and other time-delayed systems based on Lyapunov method can be summarized as follows: one is finding an appropriate LKF, that is, a positive definite functional accompanying by a negative definite time derivative along the trajectory of the system, for example, LKF with delayed partitioning method [7], LKF with augmented terms [8, 9, 12-17], and LKF with triple-integral and quadruple-integral terms [5-7, 10, 11, 20]; the

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other is reducing the upper bounds of the time derivative of LKF as much as possible by developing various inequality techniques, such as linear matrix inequality (LMI) conditions based on Jensen inequality [21], LMI conditions based on Wirtinger-based inequality [22], LMI conditions based on Bessel-Legendre inequality [18, 19, 23], and additional functions based on inequalities [24]. Besides, further to increase the freedom of solving the LMI, there are some other methods, for instance, the generalized zero equality [25], the one or second-order reciprocally convex combinations [26, 27], and the free-weighting-matrix approach [28].

Recently, a new LKF and a generalized free-weighting-matrix were used to obtain some improvement delay-dependent stability criteria for DNNs in [29], where Zhang et al. considered the effect of the LKFs while discussing the relationship between the selectivity of inequality techniques and the conservativeness of stability criteria. The comparative results illustrate the integral inequality technique that brings the upper bound of the derivative of integral terms closer to the true value does not always deduce a less conservative stability condition if the LKF is not properly constructed. Particularly, it is important to reduce the conservativeness for the novel LKF with the two terms $h(t)P_a$ and $(h - h(t))P_b$, which are named as delay-product-type terms. Compared with the general LKF, P_a and P_b were just symmetrical, not always positive definite, which can extend the freedom for checking the feasibility of stable conditions based on LMI. Zhang et al. [18,19] extended the above two delay-product-type terms, which is named LKF with delay-dependent matrices, and derived some new stability conditions for DNNs. However, these delay-dependent matrices are only limited to non-integral terms of LKFs. Lately, to fully utilize the information of delay derivative, another effective LKF was constructed by Kwon et al. [30] with delay-dependent Lyapunov matrices in single-integral terms. Kwon et al. pointed out that the stability conditions based on this type of LKF are less conservative than those based on LKFs with delay-independent Lyapunov matrices in single-integral terms.

As mentioned above, the two types of LKFs only improve one class of Lyapunov matrices, respectively, that is, only the non-integral item or the single integral item. It is natural to wonder about whether both classes of Lyapunov matrices can be improved, simultaneously. Inspired by the above analysis, the contributions of this paper can be summarized as follows.

- Improve the LKF, where the delay-product-type function in the non-integral term and two delay-dependent matrices in the single-integral term are considered simultaneously. In this way, more information about the delay and delay derivatives is introduced into the LKF, further reducing the conservativeness of LKF construction.
- Inspired from [29, 31, 32], the Lyapunov matrices in the non-integral term are just symmetrical, not always positive definite. When proving the positive definiteness of the LKF, the non-integral and the single-integral term are handled as a whole, which reduces the constraints on the LMI decision variables in the stability criterion. This can increase the degree of freedom of solving LMIs and reduce the conservativeness of the stability criterion.
- Based on the quadratic generalized free-weighting matrix inequality (QGFMI), the integrated vector of the double-integral term is augmented with $x(s)$ and $g(W_2x(s))$, which further makes full use of the coupling information between the nonlinearity and other state variables.

In conclusion, it is still challenging to solve the above problem, which provides a motivation to derive less conservative stability criteria for DNNs.

The stability of DNNs is studied in this paper, where an enhanced stability condition is obtained based on LMI by constructing a novel LKF. Our criterion is an improvement on some ones recently proposed. Through the comparison and analysis of numerical examples, it is illustrated that this paper improves some existing methods and achieves better results in terms of stability than some existing ones.

Notation: The matrix P is a positive definite matrix if $P > 0$, vice versa. I and 0 represent corresponding dimension unit matrix and zero matrix. The diagonal matrix is represented by $\operatorname{diag}\{\cdots\}$, and e_i $(i = 1, \ldots, m)$ are block entry matrices with $e_3^T = \begin{bmatrix} 0 & 0 & I & \underbrace{0 \cdots 0}_{m-3} \end{bmatrix}$, where m is the length of the vector $\xi(t)$ in theorems and corollaries. $*$ denotes the symmetric terms in a block matrix. $F[h(t), d(t)]$, $G[x(t)]$ denote F, G are the function of $h(t)$, $d(t)$ and $x(t)$, respectively. $Sym\{B\} = B + B^T$.

2. **Problem Formulation and Preliminary.**

The DNN system considered in this paper is based on the following description:

$$\dot{z}(t) = -Az(t) + W_0 f(W_2 z(t)) + W_1 f(W_2 z(t - h(t))) + J. \tag{1}$$

Here, $z(t) \in \mathbb{R}^n$ is the neuron state vector of the above DNN. $f(\alpha) \in \mathbb{R}^n$ denotes the neuron activation function (NAF); The positive definite diagonal matrix $A \in \mathbb{R}^{n\times n}$ denotes the feedback gain; W_0, W_1, $W_2 \in \mathbb{R}^{n\times n}$ represent the constant matrices of interconnection weight and $J \in \mathbb{R}^n$ represents a constant input vector. $h(t)$ represents the corresponding time-varying delay and satisfies some constraints hypothesis described as follows:

$$0 \le h(t) \le h, \quad \left|\dot{h}(t)\right| \le \mu, \quad \forall t \ge 0 \tag{2}$$

with two nonnegative constants h and μ. Suppose the nonlinear function, that is NAF, $f(\alpha)$ satisfies the following hypothesis.

$$k_i^- \le \frac{f_i(\alpha_1) - f_i(\alpha_2)}{\alpha_1 - \alpha_2} \le k_i^+, \quad \forall \alpha_1 \ne \alpha_2,\ i = 1, 2, \ldots, n. \tag{3}$$

The two real scalars k_i^- and k_i^+ could be arbitrary. As noted in [33], when $k_i^- = 0$ and $k_i^+ > 0$, assumption (3) describes a globally Lipschitz continuous and monotone nondecreasing case. And when $k_i^+ > k_i^- > 0$, the assumption represents the case of globally Lipschitz continuous and monotone increasing. Define the error signal as $x(\cdot) = z(\cdot) - z^*$ and $g(x) = f(x + z^*) - f(z^*)$. Thus, the error system of the DNN (1) can be redescribed as the following system.

$$\dot{x}(t) = -Ax(t) + W_0 g(W_2 x(t)) + W_1 g(W_2 x(t - h(t))) \tag{4}$$

with the transformed NAF $g(\cdot)$ with $g_i(0) = 0$ satisfying

$$k_i^- \le \frac{g_i(\alpha_1) - g_i(\alpha_2)}{\alpha_1 - \alpha_2} \le k_i^+, \quad \forall \alpha_1 \ne \alpha_2,\ i = 1, 2, \ldots, n, \tag{5}$$

which derive the following conditions for $\alpha_2 = 0$

$$k_i^- \le \frac{g_i(\alpha)}{\alpha} \le k_i^+, \quad i = 1, 2, \ldots, n. \tag{6}$$

Remark 2.1. *The application of DNN systems is very popular, such as pattern recognition, associative memory, optimization, image and signal processing. For example, a practical model, called the quadruple-tank process system (QTPS), as first inspected by Johansson [34], can clearly use the DNN model, which has attracted a great deal of research attention in recent years [35, 36]. Therefore, it is very necessary to study the DNN system.*

The most important work to be dealt with in this paper is to obtain a less conservative stability condition than some recent results for the DNN (1) based on constraints (2)-(6) via Lyapunov stability theory. The following main lemmas are necessary for this purpose.

Lemma 2.1. [37]. *For given vectors* α_1, α_2 *and positive real scalars* λ *satisfying* $0 < \lambda < 1$, *matrices* $(0 < R_1, R_2 \in \mathbb{R}^{n\times n})$, *and arbitrary matrices* $U_{01}, U_{02} \in \mathbb{R}^{n\times n}$, *the following inequality holds*

$$\frac{\alpha_1^T R_1 \alpha_1}{\lambda} + \frac{\alpha_2^T R_2 \alpha_2}{1-\lambda} \geq \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}^T \begin{bmatrix} R_1 + (1-\lambda)T_1 & (1-\lambda)U_{01} + \lambda U_{02} \\ * & R_2 + \lambda T_2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix},$$

where $T_1 = R_1 - U_{02}R_2^{-1}U_{02}^T$, $T_2 = R_2 - U_{01}^T R_1^{-1} U_{01}$.

Lemma 2.2. (QGFMI [30]) *For a positive definite matrix* R, *two matrices* X, Y *with appropriate dimension, and a function* $\{\omega(s) \mid s \in [a,b]\}$, *we have the following inequality*

$$\begin{aligned} &-\int_a^b \omega^T(s) R \omega(s) ds \\ &\leq \begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix}^T \begin{bmatrix} (b-a)XR^{-1}X^T & X[I \quad 0] \\ * & \dfrac{b-a}{3} YR^{-1}Y^T + Sym\{Y[-I \quad 2I]\} \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_1 \end{bmatrix}, \end{aligned}$$

where η_0 *is an any vector, and* $\eta_1^T = \left[\int_a^b \omega^T(s)ds \quad \frac{1}{b-a}\int_a^b \int_\theta^b \omega^T(s) ds d\theta \right]$.

3. Main Results.

In this section, a novel LKF will be constructed at first. A new stability criterion is derivated according to the proposed LKF. In addition, to prove the validity of the LKF, two stability criteria based on relatively simple LKFs with the same inequality technique are also given as corollaries. When analyzing the stability of DNNs, the main aim is reducing the conservativeness of stability criteria via LKFs with broader form and much tighter integral inequality techniques application. It is foreseeable that broader LKFs or much tighter integral inequality techniques may reduce the conservativeness. Firstly, a novel LKF is composed of the following form:

$$V(t) = \sum_{i=1}^{4} V_i(t) \tag{7}$$

with

$$\begin{aligned} V_1(t) &= \zeta^T(t) P \zeta(t) + h(t)\zeta_1^T(t) P_a \zeta_1(t) + (h - h(t))\zeta_2^T(t) P_b \zeta_2(t), \\ V_2(t) &= \int_{t-h(t)}^{t} \gamma(s)^T Q_1(t) \gamma(s) ds + \int_{t-h}^{t-h(t)} \gamma(s)^T Q_2(t) \gamma(s) ds, \\ V_3(t) &= 2\sum_{i=1}^{n} \int_0^{W_{2i}x(t)} \left[h_{1i}\left(g_i(s) - k_i^- s\right) + h_{2i}\left(k_i^+ s - g_i(s)\right)\right] ds \\ &\quad + 2\sum_{i=1}^{n} \int_0^{W_{2i}x(t-h(t))} \left[h_{3i}\left(g_i(s) - k_i^- s\right) + h_{4i}\left(k_i^+ s - g_i(s)\right)\right] ds \\ &\quad + 2\sum_{i=1}^{n} \int_0^{W_{2i}x(t-h)} \left[h_{5i}\left(g_i(s) - k_i^- s\right) + h_{6i}\left(k_i^+ s - g_i(s)\right)\right] ds, \\ V_4(t) &= \int_{-h}^{0} \int_{t+\theta}^{t} \gamma^T(s) R \gamma(s) ds d\theta, \end{aligned}$$

where $P \in \mathbb{R}^{7n\times 7n}$, $(P_a, P_b \in \mathbb{R}^{5n\times 5n})$ are real symmetric matrices and $(R, Q_1(t) = Q_{10} - h(t)Q_{11}, Q_2(t) = Q_{20} + (h - h(t))Q_{21} \in \mathbb{R}^{3n\times 3n})$ are real positive definite matrices, $h_{pi} > 0$, $(p = 1, \dots, 6)$. Moreover, notations of several symbols and matrices can be found in Appendix A.

Remark 3.1. *The novelties of the LKF* (7) *lie in the following four aspects and these modified measures are a major contribution to reducing the conservativeness of stability criteria for DNNs.*

- *It is different from some existing ones due to* $V_1(t)$ *and* $V_2(t)$*, which are inspired from the LKFs of [18, 29, 30], where* $h(t)P_a$*,* $(h - h(t))P_b$ *of* $V_1(t)$ *and* $Q_1(t)$*,* $Q_2(t)$ *of* $V_2(t)$ *are delay-dependent matrices. Especially,* $Q_1(t)$*,* $Q_2(t)$ *are used to describe the coupling between some vectors as compared with [5, 11, 14, 18, 19, 29], that is* $x(s)$*,* $\dot{x}(s)$ *and* $g(W_2x(s))$ *are coupled by* $Q_1(t)$ *for* $s \in [t - h(t), t]$ *and by* $Q_2(t)$ *for* $s \in [t - h, t - h(t)]$*. Moreover, the delay-dependent and delay-derivative dependent couplings are contained in* $V_2(t)$ *and its derivative, respectively, which further makes full use of the coupling information between the time-varying delay* $h(t)$*, its derivative* $\dot{h}(t)$ *and some state vectors.*
- *Some delay-product-type terms are added into* $V_1(t)$*. Compared with some general LKFs proposed in [5, 11, 14, 16, 17, 30],* P_a *and* P_b *were just symmetrical, not always positive definite, which can extend the freedom for checking the feasibility of stable conditions based on LMI.*
- *Unlike some existing LKFs proposed in [5, 12], the integrated vector in double-integral term* $V_4(t)$ *includes some extra state vectors, such as* $x(s)$ *and* $g(W_2x(s))$*, which increases some useful coupling information between the nonlinearity and some other state variables, for example,* $g(W_2x(t))$*,* $x(t)$*,* $x(t - h(t))$*,* $\dot{x}(t)$*,* $v_3(t)$*, and* $v_4(t)$*.*
- *Bounding the time derivative of* $V_2(t)$ *and* $V_4(t)$ *by Lemma 2.2 can show some special terms related to the vectors* $v_3(t)$ *and* $v_4(t)$ *which are merged into* $V_1(t)$*, that is* $\zeta(t)$*,* $\zeta_1(t)$ *and* $\zeta_2(t)$*.*

Theorem 3.1. *The DNN* (1) *with the conditions* (2)-(3) *is globally asymptotically stable for given non-negative scalars of* h*,* μ*, if there exist symmetric matrices* $P \in \mathbb{R}^{7n\times 7n}$*,* $(P_a, P_b \in \mathbb{R}^{5n\times 5n})$*,* $(Q_a, Q_b, R_a, R_b \in \mathbb{R}^{n\times n})$*, positive definite matrices* $(R, Q_1(t), Q_2(t) \in \mathbb{R}^{3n\times 3n})$*, positive definite diagonal matrices* $H_p = \mathrm{diag}\{h_{p1}, h_{p2}, \dots, h_{pn}\}$*,* $(\Lambda_{ik}, \Delta_{ir} \in \mathbb{R}^{n\times n})$ *and any matrices* $(U_{0i} \in \mathbb{R}^{3n\times 3n}$*,* $U_j \in \mathbb{R}^{n\times n})$*,* $N \in \mathbb{R}^{4n\times 2n}$*,* $X_j \in \mathbb{R}^{13n\times 3n}$*,* $Y_j \in \mathbb{R}^{6n\times 3n}$ $(p = 1, \dots, 6$*;* $i, r = 1, 2$*;* $k = 1, 2, 3$*;* $j = 1, 2, 3, 4)$ *such that LMIs* (8)-(14) *hold:*

$$\bar{Q}_a > 0, \quad \bar{Q}_b > 0, \quad \bar{R}_a > 0, \quad \bar{R}_b > 0, \tag{8}$$

$$\Omega_{1[0]} > 0,\ \Omega_{1[h]} > 0,\ \Omega_{2[0]} > 0,\ \Omega_{2[h]} > 0, \tag{9}$$

$$\begin{bmatrix} \Omega_{4[0]} + \frac{1}{h}J^T\Omega_{3[0]}J & E_1U_{02} \\ * & h\Omega_{2[0]} \end{bmatrix} > 0, \quad \begin{bmatrix} \Omega_{4[h]} + \frac{1}{h}J^T\Omega_{3[h]}J & E_2U_{01}^T \\ * & h\Omega_{1[h]} \end{bmatrix} > 0, \tag{10}$$

$$\begin{bmatrix} \Pi_{[0,-\mu]} & \Omega_{a[1,4]} & \Omega_{b[3,4]} \\ * & -h\bar{R}_b & 0 \\ * & * & -3h\bar{R}_b \end{bmatrix} < 0, \tag{11}$$

$$\begin{bmatrix} \Pi_{[h,-\mu]} & \Omega_{a[1,3]} & \Omega_{b[2,3]} \\ * & -h\bar{R}_a & 0 \\ * & * & -3h\bar{R}_a \end{bmatrix} < 0, \tag{12}$$

$$\begin{bmatrix} \Pi_{[0,\mu]} & \Omega_{a[1,4]} & \Omega_{b[3,4]} & 2\mu\Omega_{a[1,2]} & 2\mu\Omega_{b[3,2]} \\ * & -h\bar{R}_b & 0 & 0 & 0 \\ * & * & -3h\bar{R}_b & 0 & 0 \\ * & * & * & -2\mu h\bar{Q}_b & 0 \\ * & * & * & * & -6\mu h\bar{Q}_b \end{bmatrix} < 0, \tag{13}$$

$$\begin{bmatrix} \Pi_{[h,\mu]} & \Omega_{a[1,3]} & \Omega_{b[2,3]} & 2\mu\Omega_{a[1,1]} & 2\mu\Omega_{b[2,1]} \\ * & -h\bar{R}_a & 0 & 0 & 0 \\ * & * & -3h\bar{R}_a & 0 & 0 \\ * & * & * & -2\mu h\bar{Q}_a & 0 \\ * & * & * & * & -6\mu h\bar{Q}_a \end{bmatrix} < 0, \tag{14}$$

where the notations of several symbols and matrices can be found in Appendix B.

Proof: Construct an LKF (7). First step, because the positive definiteness of the Lyapunov matrices P, P_a and P_b is not required, the positive definiteness of the LKF (7) should be proved. The P_a- and P_b- related terms can be reorganized as

$$\begin{aligned} & h(t)\zeta_1^T(t)P_a\zeta_1(t) + (h-h(t))\zeta_2^T(t)P_b\zeta_2(t) \\ = & \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \\ v_3^T(t) \\ 0 \end{bmatrix}^T [h(t)P_a] \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \\ v_3^T(t) \\ 0 \end{bmatrix} \\ & + \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \\ v_4^T(t) \\ 0 \end{bmatrix}^T [(h-h(t))P_b] \begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \\ v_4^T(t) \\ 0 \end{bmatrix} \\ & + 2\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \\ v_3^T(t) \\ 0 \end{bmatrix}^T P_a \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \omega_1(t) \end{bmatrix} + 2\begin{bmatrix} x(t) \\ x(t-h(t)) \\ x(t-h) \\ v_4^T(t) \\ 0 \end{bmatrix}^T P_b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \omega_2(t) \end{bmatrix} \\ & + \frac{\omega_1^T(t)EP_aE^T\omega_1(t)}{h(t)} + \frac{\omega_2^T(t)EP_bE^T\omega_2(t)}{(h-h(t))}, \end{aligned} \tag{15}$$

where $E = [0 \quad 0 \quad 0 \quad 0 \quad I]$.

Based on $Q_1(t) > 0$, $Q_2(t) > 0$ and Jensens inequality, the $V_2(t)$ term can be estimated as

$$V_2(t) \geq \frac{\eta_1^T(t)Q_1(t)\eta_1(t)}{h(t)} + \frac{\eta_2^T(t)Q_2(t)\eta_2(t)}{(h-h(t))}. \tag{16}$$

According to $\Omega_{i[h(t)]} > 0$ $(i = 1, 2, 3)$ and Lemma 2.1, we can obtain Inequality (17) by simple derivation based on (15) and (16)

$$\begin{aligned}
&V_2(t) + \frac{\omega_1^T(t)EP_aE^T\omega_1(t)}{h(t)} + \frac{\omega_2^T(t)EP_bE^T\omega_2(t)}{h-h(t)} \\
\geq\ & \frac{\eta_1^T(t)\left[\text{diag}\left\{EP_aE^T,\, 0,\, 0\right\} + Q_1(t)\right]\eta_1(t)}{h(t)} + \frac{\eta_2^T(t)\left[\text{diag}\left\{EP_aE^T,\, 0,\, 0\right\} + Q_2(t)\right]\eta_2(t)}{h-h(t)} \\
\geq\ & \Delta^T(t)\frac{\Omega_{3[h(t)]}}{h}\Delta(t) - \frac{(1-\alpha)}{h}\eta_1^T(t)U_{02}\Omega_{2[h(t)]}^{-1}U_{02}^T\eta_1(t) - \frac{\alpha}{h}\eta_2^T(t)U_{01}^T\Omega_{1[h(t)]}^{-1}U_{01}\eta_2(t). \qquad (17)
\end{aligned}$$

It follows from (5), (7), (9), (10), (15)-(17) and $h_{pi} > 0$, $R > 0$ that $V(t) > 0$. Thus, the LKF (7) is positive definite.

Second step, taking the time derivative of $V(t)$ along the trajectory of the DNN (1), the following formula can be obtained

$$\begin{aligned}
\dot{V}_1(t) &= \dot{h}(t)\zeta_1^T(t)P_a\zeta_1(t) - \dot{h}(t)\zeta_2^T(t)P_b\zeta_2(t) + 2\zeta^T(t)P\dot{\zeta}(t) + 2h(t)\zeta_1^T(t)P_a\dot{\zeta}_1(t) \\
&\quad + 2(h-h(t))\zeta_2^T(t)P_b\dot{\zeta}_2(t) \\
&= \dot{h}(t)\zeta_1^T(t)P_a\zeta_1(t) - \dot{h}(t)\zeta_2^T(t)P_b\zeta_2(t) \\
&\quad + 2\zeta^T(t)P\begin{bmatrix} e_4^T \\ h_de_5^T \\ e_6^T \\ e_1^T - h_de_2^T \\ h_de_2^T - e_3^T \\ e_{11}^T - h_de_{12}^T \\ h_de_{12}^T - e_{13}^T \end{bmatrix}\xi(t) + 2h(t)\zeta_1^T(t)P_a\begin{bmatrix} e_4^T \\ h_de_5^T \\ e_6^T \\ e_{11}^T - h_de_{12}^T \\ \dfrac{e_1^T - h_de_2^T - \dot{h}(t)e_7^T}{h(t)} \end{bmatrix}\xi(t) \\
&\quad + 2(h-h(t))\zeta_2^T(t)P_b\begin{bmatrix} e_4^T \\ h_de_5^T \\ e_6^T \\ h_de_{12}^T - e_{13}^T \\ \dfrac{h_de_2^T - e_3^T + \dot{h}(t)e_8^T}{h-h(t)} \end{bmatrix}\xi(t), \qquad (18) \\
\dot{V}_2(t) &= \gamma^T(t)Q_1(t)\gamma(t) + h_d\gamma^T(t-h(t))\left[Q_2(t) - Q_1(t)\right]\gamma(t-h(t)) \\
&\quad - \gamma^T(t-h)Q_2(t)\gamma(t-h) - \dot{h}(t)\int_{t-h(t)}^{t}\gamma^T(s)Q_{11}\gamma(s)ds \\
&\quad - \dot{h}(t)\int_{t-h}^{t-h(t)}\gamma^T(s)Q_{21}\gamma(s)ds, \qquad (19) \\
\dot{V}_3(t) &= 2g^T(W_2x(t))(H_1 - H_2)W_2\dot{x}(t) + 2x^T(t)W_2^T(K_1H_2 - K_2H_1)W_2\dot{x}(t) \\
&\quad + 2h_dg^T(W_2x(t-h(t)))(H_3 - H_4)W_2\dot{x}(t-h(t)) \\
&\quad + 2h_dx^T(t-h(t))W_2^T(K_1H_4 - K_2H_3)W_2\dot{x}(t-h(t)) \\
&\quad + 2g^T(W_2x(t-h))(H_5 - H_6)W_2\dot{x}(t-h) \\
&\quad + 2x^T(t-h)W_2^T(K_1H_6 - K_2H_5)W_2\dot{x}(t-h), \qquad (20)
\end{aligned}$$

$$\dot{V}_4(t) = \gamma^T(t)(hR)\gamma(t) - \int_{t-h(t)}^{t} \gamma^T(s)R\gamma(s)ds - \int_{t-h}^{t-h(t)} \gamma^T(s)R\gamma(s)ds. \tag{21}$$

The following zero equations are satisfied for additional symmetric matrices Q_a, Q_b, R_a and R_b:

$$\begin{aligned} 0 &= \dot{h}(t)\xi^T(t)[e_1\ e_2\ e_3]\mathrm{diag}\{Q_a,\ Q_b - Q_a,\ -Q_b\}[e_1\ e_2\ e_3]^T\xi(t) \\ &\quad - 2\dot{h}(t)\int_{t-h(t)}^{t} x^T(s)Q_a\dot{x}(s)ds - 2\dot{h}(t)\int_{t-h}^{t-h(t)} x^T(s)Q_b\dot{x}(s)ds, \quad (22) \\ 0 &= \xi^T(t)[e_1\ e_2\ e_3]\mathrm{diag}\{R_a,\ R_b - R_a,\ -R_b\}[e_1\ e_2\ e_3]^T\xi(t) \\ &\quad - 2\int_{t-h(t)}^{t} x^T(s)R_a\dot{x}(s)ds - 2\int_{t-h}^{t-h(t)} x^T(s)R_b\dot{x}(s)ds, \quad (23) \\ 0 &= \mu\int_{t-h(t)}^{t} \gamma^T(s)\bar{Q}_a\gamma(s)ds - \mu\int_{t-h(t)}^{t} \gamma^T(s)\bar{Q}_a\gamma(s)ds \\ &\quad + \mu\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{Q}_b\gamma(s)ds - \mu\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{Q}_b\gamma(s)ds. \quad (24) \end{aligned}$$

Taking the zero inequalities in $\dot{V}_2$ and $\dot{V}_4$, we have the following integral terms for $\bar{\mu} = \mu + \dot{h}(t)$.

$$\begin{aligned} \varphi = &-\bar{\mu}\int_{t-h(t)}^{t} \gamma^T(s)\bar{Q}_a\gamma(s)ds - \bar{\mu}\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{Q}_b\gamma(s)ds \\ &- \int_{t-h(t)}^{t} \gamma^T(s)\bar{R}_a\gamma(s)ds - \int_{t-h}^{t-h(t)} \gamma^T(s)\bar{R}_b\gamma(s)ds. \end{aligned} \tag{25}$$

It follows from Lemma 2.2 with an augmented vector $\gamma(s)$ that

$$\begin{aligned} &-\bar{\mu}\int_{t-h(t)}^{t} \gamma^T(s)\bar{Q}_a\gamma(s)ds \\ &\le \bar{\mu}\begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix}^T \begin{bmatrix} h(t)X_1\bar{Q}_a^{-1}X_1^T & X_1F_1 \\ * & \frac{d(t)}{3}Y_1\bar{Q}_a^{-1}Y_1^T + Sym\{Y_1F_2\} \end{bmatrix} \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix}, \\ &-\bar{\mu}\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{Q}_b\gamma(s)ds \\ &\le \bar{\mu}\begin{bmatrix} G_1^T \\ G_3^T \end{bmatrix}^T \begin{bmatrix} (h-h(t))X_2\bar{Q}_b^{-1}X_2^T & X_2F_1 \\ * & \frac{h-h(t)}{3}Y_2\bar{Q}_b^{-1}Y_2^T + Sym\{Y_2F_2\} \end{bmatrix} \begin{bmatrix} G_1^T \\ G_3^T \end{bmatrix}, \\ &-\int_{t-h(t)}^{t} \gamma^T(s)\bar{R}_a\gamma(s)ds \\ &\le \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix}^T \begin{bmatrix} h(t)X_3\bar{R}_a^{-1}X_3^T & X_3F_1 \\ * & \frac{h(t)}{3}Y_3\bar{R}_a^{-1}Y_3^T + Sym\{Y_3F_2\} \end{bmatrix} \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix}, \\ &-\int_{t-h}^{t-h(t)} \gamma^T(s)\bar{R}_b\gamma(s)ds \end{aligned}$$

$$\leq \begin{bmatrix} G_1^T \\ G_3^T \end{bmatrix}^T \begin{bmatrix} (h-h(t))X_4\bar{R}_b^{-1}X_4^T & X_4F_1 \\ * & \dfrac{h-h(t)}{3}Y_4\bar{R}_b^{-1}Y_4^T + Sym\{Y_4F_2\} \end{bmatrix} \begin{bmatrix} G_1^T \\ G_3^T \end{bmatrix}.$$

For any appropriately dimensioned matrices $\bar{U} = \begin{bmatrix} U_1^T & U_2^T & U_3^T & U_4^T \end{bmatrix}^T \in \mathbb{R}^{4n\times 2n}$, $N \in \mathbb{R}^{4n\times 2n}$ it is true that

$$0 = 2\xi^T(t)\left[e_1\ e_4\ e_{11}\ e_{12}\right]\bar{U}\left[-Ae_1^T + W_0e_{11}^T + W_1e_{12}^T - e_4^T\right]\xi(t), \tag{26}$$

$$0 = 2\xi^T(t)[e_7\ e_8\ e_{14}\ e_{15}]N[e_{14} - h(t)e_7\ e_{15} - (h-h(t))e_8]^T\xi(t). \tag{27}$$

If the nonlinear constraint conditions (5) and (6) are partitioned according to time-varying delay, we can obtain

$$\begin{aligned} \lambda_k(s) \triangleq{}& 2\left[g(W_2x(s)) - K_2W_2x(s)\right]^T\left(\frac{h(t)}{h}\Lambda_{1k} + \frac{h-h(t)}{h}\Lambda_{2k}\right)[K_1W_2x(s) \\ & - g(W_2x(s))] \geq 0, \end{aligned} \tag{28}$$

$$\begin{aligned} \delta_i(s_1,s_2) \triangleq{}& 2\left[g(W_2x(s_1)) - g(W_2x(s_2)) - K_2W_2(x(s_1)-x(s_2))\right]^T\left(\frac{h(t)}{h}\Delta_{1i}\right. \\ & \left. + \frac{h-h(t)}{h}\Delta_{2i}\right) \times [K_1W_2(x(s_1)-x(s_2)) - g(W_2x(s_1)) \\ & + g(W_2x(s_2))] \geq 0, \end{aligned} \tag{29}$$

where Λ_{rk} and Δ_{ri} $(i,r=1,2;\ k=1,2,3)$ are positive definite diagonal matrices.

Thus, the following inequalities hold

$$\lambda_1(t) + \lambda_2(t-h(t)) + \lambda_3(t-h) \geq 0, \tag{30}$$

$$\delta_1(t,t-h(t)) + \delta_2(t-h(t),t-h) \geq 0. \tag{31}$$

Finally, from the above derivation, we have

$$\begin{aligned} \dot{V}(t) \leq \xi^T(t)\Bigg\{&\Pi_{[h(t),\dot{h}(t)]} + h(t)\left[G_1X_3\bar{R}_a^{-1}X_3^TG_1^T + \frac{1}{3}G_2Y_3\bar{R}_a^{-1}Y_3^TG_2^T\right] \\ &+ (h-h(t))\left[G_1X_4\bar{R}_b^{-1}X_4^TG_1^T + \frac{1}{3}G_3Y_4\bar{R}_b^{-1}Y_4^TG_3^T\right] \\ &+ \bar{\mu}h(t)\left[G_1X_1\bar{Q}_a^{-1}X_1^TG_1^T + \frac{1}{3}G_2Y_1\bar{Q}_a^{-1}Y_1^TG_2^T\right] \\ &+ \bar{\mu}(h-h(t))\left[G_1X_2\bar{Q}_b^{-1}X_2^TG_1^T + \frac{1}{3}G_3Y_2\bar{Q}_b^{-1}Y_2^TG_3^T\right]\Bigg\}\xi(t). \end{aligned}$$

Therefore, LMIs (11)-(14) hold, which implies that $\dot{V}(t) < 0$ by the transformation of **Schur complement equivalence**. This shows that DNN (1) is stable from Lyapunov stability theory, which completes the proof.

Remark 3.2. *Most of literature requires the positivity of all matrices involved when Lyapunov functional is a positive definite function. Recently, some relaxed stability conditions for time-delayed linear systems are proposed in [31, 32], where symmetric matrices for the non-integral item in constructed Lyapunov functional do not require to be positive definite matrices, and the non-integral item and the single integral items are considered as a whole for explaining* $V(t) > 0$. *Inspired from [31, 32], the matrices* P, P_a *and* P_b *of the above LKF* (7) *are just symmetrical, not always positive definite. When proving* $V(t) > 0$, *we calculate* $V_1(t)$ *and* $V_2(t)$ *as a whole applying Lemma 2.1, which may expand the feasible regions of LMIs.*

Remark 3.3. *To illustrate the effectiveness of proving $V(t) > 0$ via handling $V_1(t)$ and $V_2(t)$ as a whole based on Lemma 2.1, the following corollary can be obtained by requiring all of the matrices in $V_1(t)$ to be positive definite, that is, letting $P, P_a, P_b > 0$.*

Corollary 3.1. *The DNN* (1) *satisfying the conditions* (2) *and* (3) *is globally asymptotically stable for given values of $h \geq 0$, $\mu \geq 0$, if there exist positive definite matrices $P \in \mathbb{R}^{7n\times 7n}$, $(P_a, P_b \in \mathbb{R}^{5n\times 5n})$, $(R, Q_1(t), Q_2(t) \in \mathbb{R}^{3n\times 3n})$, symmetric matrices $(Q_a, Q_b, R_a, R_b \in \mathbb{R}^{n\times n})$, positive definite diagonal matrices $H_p = \operatorname{diag}\{h_{p1}, h_{p2}, \ldots, h_{pn}\}$, $(\Lambda_{ik}, \Delta_{ir} \in \mathbb{R}^{n\times n})$ and any matrices $U_j \in \mathbb{R}^{n\times n}$, $N \in \mathbb{R}^{4n\times 2n}$, $X_j \in \mathbb{R}^{13n\times 3n}$, $Y_j \in \mathbb{R}^{6n\times 3n}$ $(p = 1, \ldots, 6$; $i, r = 1, 2$; $k = 1, 2, 3$; $j = 1, 2, 3, 4)$ such that the LMIs* (8), (11)-(14) *hold.*

Remark 3.4. *To illustrate the effectiveness of the delay-dependent matrices of $V_2(t)$ in the LKF proposed in this paper, the following corollary can be obtained by choosing the LKF with delay-independent matrices in $V_2(t)$, that is, letting $Q_{11} = Q_{21} = 0$.*

Corollary 3.2. *The DNN* (1) *satisfying the conditions* (2) *and* (3) *is globally asymptotically stable for given values of $h \geq 0$, $\mu \geq 0$, if there exist symmetric matrices $P \in \mathbb{R}^{7n\times 7n}$, $(P_a, P_b \in \mathbb{R}^{5n\times 5n})$, $(R_a, R_b \in \mathbb{R}^{n\times n})$, positive definite matrices $(R, Q_{10}, Q_{20} \in \mathbb{R}^{3n\times 3n})$, positive definite diagonal matrices $H_p = \operatorname{diag}\{h_{p1}, h_{p2}, \ldots, h_{pn}\}$, $(\Lambda_{ik}, \Delta_{ir} \in \mathbb{R}^{n\times n})$ and any matrices $(U_{0i} \in \mathbb{R}^{3n\times 3n}$, $U_j \in \mathbb{R}^{n\times n})$, $N \in \mathbb{R}^{4n\times 2n}$, $X_i \in \mathbb{R}^{13n\times 3n}$, $Y_i \in \mathbb{R}^{6n\times 3n}$ $(p = 1, \ldots, 6$; $i, r = 1, 2$; $k = 1, 2, 3$; $j = 1, 2, 3, 4)$ such that LMIs* (9) *and* (10) *and LMIs* (32)-(34) *hold for $\dot{h}(t) \in \{-\mu, \mu\}$:*

$$\bar{R}_a > 0, \ \bar{R}_b > 0, \tag{32}$$

$$\begin{bmatrix} \bar{\Pi}_{[0,\dot{h}(t)]} & \Omega_{a[1,4]} & \Omega_{b[3,4]} \\ * & -h\bar{R}_b & 0 \\ * & * & -3h\bar{R}_b \end{bmatrix} < 0, \tag{33}$$

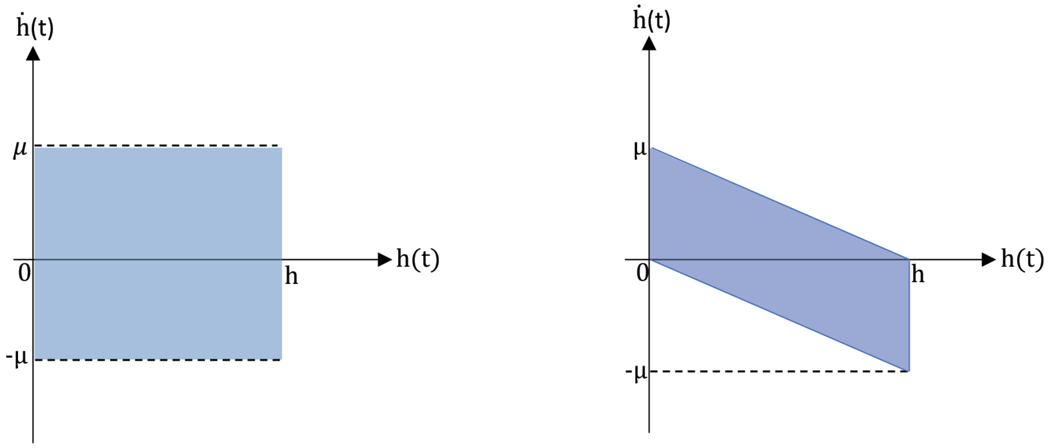
$$\begin{bmatrix} \bar{\Pi}_{[h,\dot{h}(t)]} & \Omega_{a[1,3]} & \Omega_{b[2,3]} \\ * & -h\bar{R}_a & 0 \\ * & * & -3h\bar{R}_a \end{bmatrix} < 0, \tag{34}$$

where $\bar{\Pi}_{[h(t),\dot{h}(t)]} = Sym\left\{\Pi_{1[h(t),\dot{h}(t)]} + \bar{\Pi}_{3[h(t),\dot{h}(t)]}\right\} + \bar{\Pi}_{2[h(t),\dot{h}(t)]}$,

$$\begin{aligned} \bar{\Pi}_{2[h(t),\dot{h}(t)]} &= [e_1\ e_4\ e_{11}][Q_{10} + hR][e_1\ e_4\ e_{11}]^T + h_d[e_2\ e_5\ e_{12}][Q_{20} - Q_{10}][e_2\ e_5\ e_{12}]^T \\ &\quad - [e_3\ e_6\ e_{13}]Q_{20}[e_3\ e_6\ e_{13}]^T + e_1R_ae_1^T + e_2[R_b - R_a]e_2^T - e_3R_be_3^T \\ &\quad + \dot{h}(t)\Pi_6P_a\Pi_6^T - \dot{h}(t)\Pi_7P_b\Pi_7^T, \\ \bar{\Pi}_{3[h(t),\dot{h}(t)]} &= G_1X_1F_1G_2^T + G_2Y_1F_2G_2^T + G_1X_2F_1G_3^T + G_3Y_2F_2G_3^T, \end{aligned}$$

$\Pi_{1[h(t),\dot{h}(t)]}$ *and other related notations are defined in Theorem 3.1.*

Remark 3.5. *It is worth noting that the author. Seuret and Gouaisbaut of [38] pointed out that the delay set is a polyhedral set, and two main characterizations of the allowable delay set were given, that is a usual assumptive delay set $\mathcal{H}_1$ satisfying $\left[h(t), \dot{h}(t)\right] \in \mathcal{H}_1 = [0, h] \times [-\mu, \mu]$ and another new allowable delay set $\mathcal{H}_2$ satisfying $\left[h(t), \dot{h}(t)\right] \in \mathcal{H}_2 = \{(0, 0), (0, \mu), (h, 0), (h, -\mu)\}$. Figure 1 depicts the graphical interpretation of the above two delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$, where we can find that once the values of h, μ are given, $\mathcal{H}_2$ is included in $\mathcal{H}_1$. In the next section of this paper, the two allowable delay sets, that is the usual delay set $\mathcal{H}_1$ and the refined allowable delay set $\mathcal{H}_2$, will be used to show the effectiveness of Theorem 3.1.*



A. Graph representing $\mathcal{H}_1$

B. Graph representing $\mathcal{H}_2$

FIGURE 1. Graphical interpretation of $\mathcal{H}_1$ and $\mathcal{H}_2$

Remark 3.6. *Besides, the original forms of Inequalities* (9)-(14) *and* (33)-(34) *are not LMIs due to their dependence on the two time-varying delay parameters* $h(t)$ *and* $\dot{h}(t)$*. Indeed, the matrix inequalities in the conditions can be rewritten as the following form:*

$$\Xi_1 + \dot{h}(t)[\Xi_2 + h(t)\Xi_3] < 0, \tag{35}$$

where Ξ_i*,* $i = 1, 2, 3$ *are time-independent matrix functions. In the light of the convex combination technique proposed in [39], the original forms of Inequalities* (9)-(14) *and* (33)-(34) *hold if the following LMIs hold for the above two allowable delay sets* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively,*

$$\mathcal{H}_1:\ \Xi_1 + \dot{h}(t)[\Xi_2 + h(t)\Xi_3]_{\{[h(t),\dot{h}(t)]=[0,h]\times[-\mu,\mu]\}} < 0, \tag{36}$$

$$\mathcal{H}_2:\ \Xi_1 + \dot{h}(t)[\Xi_2 + h(t)\Xi_3]_{\{(h(t),\dot{h}(t))=\{(0,0),(0,\mu),(h,0),(h,-\mu)\}\}} < 0, \tag{37}$$

which implies that the solution of Inequalities (9)-(14) *and* (33)-(34) *becomes the feasibility-checking of the LMIs. The corollaries are the same.*

4. **Numerical Examples.**

In this section, we give three examples to show the effectiveness of the criteria proposed in this paper for the two allowable delay sets $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, by comparing maximal admissible delay upper bounds (MADUBs), the conservativeness of the criteria is checked. And the index of the number of decision variables (NoVs) is applied to showing the complexity of criteria. '$-$' in tables denotes the data are not given in the corresponding papers.

4.1. **Conservativeness comparison.**

The system parameters of the examples are as follows.

Example 4.1.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ W_0 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix},\ W_1 = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ K_1 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.8 \end{bmatrix},\ K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ J = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}.$$

Example 4.2.

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & 0.7 \end{bmatrix}, \ W_0 = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \ W_1 = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ K_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.8 \end{bmatrix}, \ K_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ J = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}.$$

Example 4.3.

$$A = \begin{bmatrix} 7.3458 & 0 & 0 \\ 0 & 0 & 6.9987 \\ 0 & 0 & 5.5949 \end{bmatrix}, \ W_0 = 0_3, \ W_1 = I_3,$$

$$W_2 = \begin{bmatrix} 13.6014 & -2.9616 & -0.6936 \\ 7.4736 & 21.6810 & 3.2100 \\ 0.7290 & -2.6344 & -20.1300 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.3680 & 0 & 0 \\ 0 & 0.1795 & 0 \\ 0 & 0 & 0.2876 \end{bmatrix}, \ K_2 = 0_3, \ J = [0.4 \ 0.2 \ 0.3]^T.$$

Table 1. MADUBs h for different μ and delay sets (Example 4.1)

Delay sets	Methods	Constraint of $\dot{h}(t)$	μ		NoVs
			0.8	0.9	
$\mathcal{H}_1$	[8, Th. 5]	$\left\vert\dot{h}(t)\right\vert \le \mu$	4.0534	2.6447	$34.5n^2 + 20.5n$
	[9, Th. 1]	$\dot{h}(t) \le \mu$	4.8167	3.4245	$64.5n^2 + 17.5n$
	[11, Th. 1]	$\dot{h}(t) \le \mu$	4.9045	2.6237	$38n^2 + 20n$
	[14, Th. 9]	$\dot{h}(t) \le \mu$	5.0945	3.4978	$142.5n^2 + 16.5n$
	[15, Th. 2]	$\dot{h}(t) \le \mu$	5.4428	3.6482	$128n^2 + 20n$
	[17, Th. 3]	$\left\vert\dot{h}(t)\right\vert \le \mu$	6.7001	4.0707	$217.5n^2 + 13.5n$
	[16, Pro. 1]	$\left\vert\dot{h}(t)\right\vert \le \mu$	7.0936	4.4353	$115n^2 + 22n$
	[5, Th. 1]	$\left\vert\dot{h}(t)\right\vert \le \mu$	7.5173	5.3993	$42n^2 + 27n$
	[19, Pro. 4]	$\left\vert\dot{h}(t)\right\vert \le \mu$	10.4938	6.0434	$284.5n^2 + 38.5n$
	Co. 3.2	$\left\vert\dot{h}(t)\right\vert \le \mu$	10.0576	5.3998	$208n^2 + 14n$
	Co. 3.1	$\left\vert\dot{h}(t)\right\vert \le \mu$	10.7136	5.8569	$314n^2 + 14n$
	Th. 3.1	$\left\vert\dot{h}(t)\right\vert \le \mu$	11.7993	6.9503	$332n^2 + 18n$
$\mathcal{H}_2$	Co. 3.2	$\left\vert\dot{h}(t)\right\vert \le \mu$	30.1576	28.3679	$208n^2 + 14n$
	Co. 3.1	$\left\vert\dot{h}(t)\right\vert \le \mu$	35.7791	33.6218	$314n^2 + 14n$
	Th. 3.1	$\left\vert\dot{h}(t)\right\vert \le \mu$	38.8567	37.2187	$332n^2 + 18n$

In Tables 1-3, the MADUBs obtained by Theorem 3.1 and Corollary 3.1 are listed and compared with some recent results for different constraints of $\dot{h}(t)$. The following is a summary of the results.

1) Definitely, we can find that the augmented LKF (7) proposed in this paper and the application of Lemma 2.2 to bounding the time derivative of V_2 and V_4 can obtain less conservative results than some recent results. The MADUBs obtained by Theorem 3.1 are larger for all of the different constraints of $\dot{h}(t)$ than some existing literature.
2) Theorem 3.1 and Corollary 3.1 are derived by choosing a related LKF with requiring or not requiring the positive definiteness of matrices P, P_a and P_b, respectively. However, the MADUBs calculated based on Theorem 3.1 are larger than those calculated based on Corollary 3.1. That is, when proving $V(t) > 0$, $V_1(t)$ and $V_2(t)$ are handled as a whole without requiring the positive definiteness of matrices P, P_a and P_b, which can expand the feasible regions of LMIs. This matches the explanation in Remark 3.2.

Table 2. MADUBs h for different μ and delay sets (Example 4.2)

Delay sets	Methods	Constraint	μ				NoVs
		of $\dot{h}(t)$	0.4	0.45	0.5	0.55	
$\mathcal{H}_1$	[6, Th. 1]	$\dot{h}(t) \leq \mu$	7.6697	6.7287	6.4126	6.2569	$15n^2 + 16n$
	[29, Th. 1]	$\dot{h}(t) \leq \mu$	8.3498	7.3817	7.0219	6.8156	$73n^2 + 13n$
	[10, Th. 1]	$\dot{h}(t) \leq \mu$	8.5669	7.6260	7.2809	7.0683	$90n^2 + 14n$
	[14, Th. 9]	$\dot{h}(t) \leq \mu$	9.6800	8.5192	8.0535	7.7707	$142.5n^2 + 16.5n$
	[5, Th. 3]	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	9.7094	7.7523	6.8570	6.2977	$42n^2 + 27n$
	[12, Th. 3]	$\dot{h}(t) \leq \mu$	10.1095	8.6732	8.1733	7.8993	$46n^2 + 42n$
	[13, Th. 1]	$\dot{h}(t) \leq \mu$	10.2367	9.0586	8.5986	8.3181	$79.5n^2 + 15.5n$
	[15, Th. 1]	$\dot{h}(t) \leq \mu$	10.4371	9.1910	8.6957	8.3806	$128n^2 + 20n$
	[17, Th. 1]	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	15.1061	10.7374	9.3840	8.6354	$194.5n^2 + 10.5n$
	[29, Th. 3]	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	13.8671	11.1174	10.0050	9.4157	$79.5n^2 + 15.5n$
	[18, $N = 2$]	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	23.8409	17.6941	14.8593	–	$131n^2 + 24n$
	Co. 3.2	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	20.8821	14.5312	10.4411	9.4497	$208n^2 + 14n$
	Co. 3.1	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	23.7758	17.6859	14.9666	12.1985	$314n^2 + 14n$
	Th. 3.1	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	25.2563	19.4758	18.7761	16.2294	$332n^2 + 18n$
$\mathcal{H}_2$	[18, $N = 2$]	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	77.4833	46.6448	46.6448	–	$131n^2 + 24n$
	Co. 3.2	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	45.5598	40.1191	40.1190	40.1190	$208n^2 + 14n$
	Co. 3.1	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	50.9972	45.5577	45.5573	44.8991	$314n^2 + 14n$
	Th. 3.1	$\left\lvert\dot{h}(t)\right\rvert \leq \mu$	53.5382	47.5583	47.5581	47.5499	$332n^2 + 18n$

Table 3. MADUBs h for different μ and delay sets (Example 4.3)

Delay sets	Methods	Constraint of $\dot{h}(t)$	μ				NoVs
			0.0	0.1	0.5	0.9	
$\mathcal{H}_1$	[12, Th. 3]	$\dot{h}(t) \le \mu$	1.8899	1.1115	0.4807	–	$46n^2 + 42n$
	[13, Th. 1]	$\dot{h}(t) \le \mu$	1.9261	1.1205	0.4614	0.3963	$185.5n^2 + 21.5n$
	[29, Th. 3]	$\lvert\dot{h}(t)\rvert \le \mu$	1.8899	1.1135	0.4922	0.4701	$79.5n^2 + 15.5n$
	[17, Th. 1]	$\lvert\dot{h}(t)\rvert \le \mu$	1.8899	1.1193	0.4590	0.3945	$194.5n^2 + 10.5n$
	[16, Pro. 1]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.1454	0.5806	–	$115n^2 + 22n$
	[18, $N = 2$]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.1511	0.5836	–	$131n^2 + 24n$
	Co. 3.2	$\lvert\dot{h}(t)\rvert \le \mu$	1.8998	1.1500	0.5671	0.4413	$208n^2 + 14n$
	Co. 3.1	$\lvert\dot{h}(t)\rvert \le \mu$	1.8999	1.1501	0.5677	0.4417	$314n^2 + 14n$
	Th. 3.1	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.1996	0.6235	0.5028	$332n^2 + 18n$
$\mathcal{H}_2$	[18, $N = 2$]	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.9202	1.3348	–	$131n^2 + 24n$
	Co. 3.2	$\lvert\dot{h}(t)\rvert \le \mu$	1.9200	1.9111	1.3499	1.3001	$208n^2 + 14n$
	Co. 3.1	$\lvert\dot{h}(t)\rvert \le \mu$	1.9200	1.9124	1.3501	1.3003	$314n^2 + 14n$
	Th. 3.1	$\lvert\dot{h}(t)\rvert \le \mu$	1.9349	1.9310	1.4997	1.4677	$332n^2 + 18n$

3) Theorem 3.1 and Corollary 3.2 are derived by choosing a related LKF with and without delay-dependent matrices in $V_2(t)$ based on the same inequality technique. However, because the MADUBs calculated based on Theorem 3.1 are better than the ones calculated based on Corollary 3.2, LKF with delay-dependent matrices approach is more effective than the LKF with delay-independent matrices approach in single-integral term, which matches the explanation in Remark 3.1.
4) With the increasing of μ for Theorem 3.1, the increasing trend of the MADUBs is becoming more and more obvious, which shows that more information about delay and its derivative in $V_2(t)$ is beneficial to further reduce the conservativeness of the stability criterion for the fast-varying delays.
5) The appropriate selection of the delay set, such as $\mathcal{H}_2$, makes a big difference on increasing the MADUBs, which matches the description in [38] and Remark 3.5.
6) It can be known from the comparative analysis of NoVs in the tables that the conservativeness of our criteria is reduced at the cost of increasing decision variables compared with those in the literature.

4.2. **Simulation verification.** To confirm the obtained result from Tables 1-3, the simulation result is shown in Figures 2-4. As you can see from Figures 2-4, the state responses of the DNN (4) converge to zero, which verifies the DNN (1) is stable at the equilibrium points.

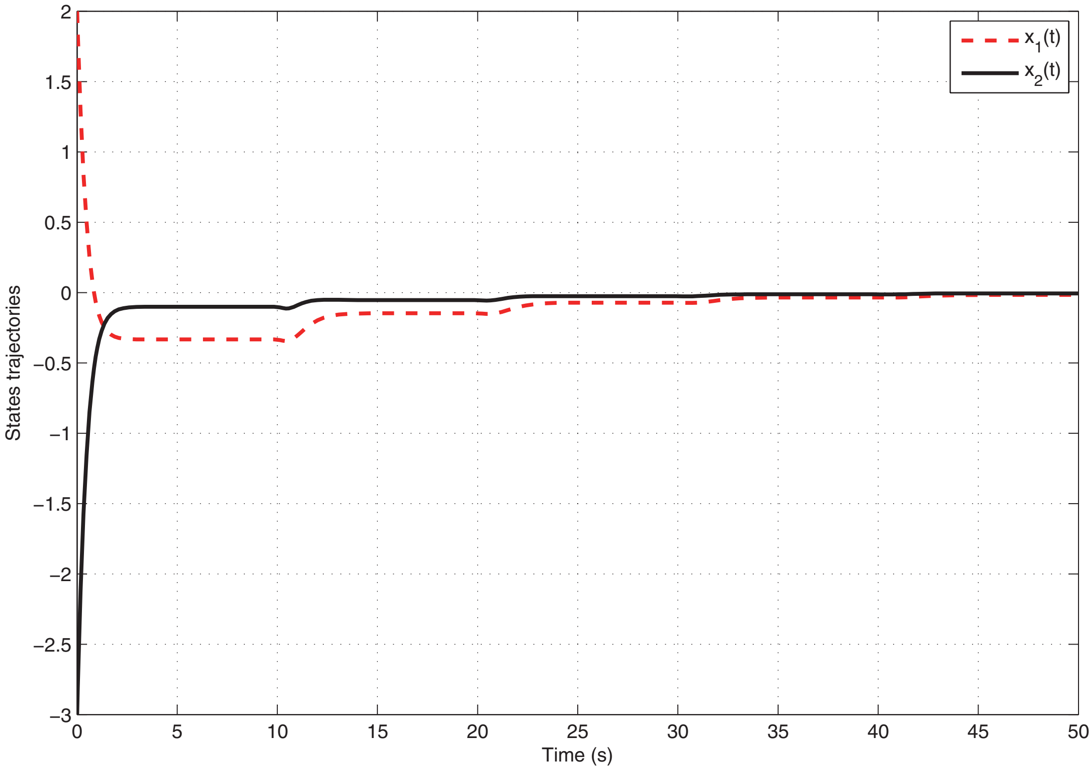


Figure 2. The state responses for Example 4.1

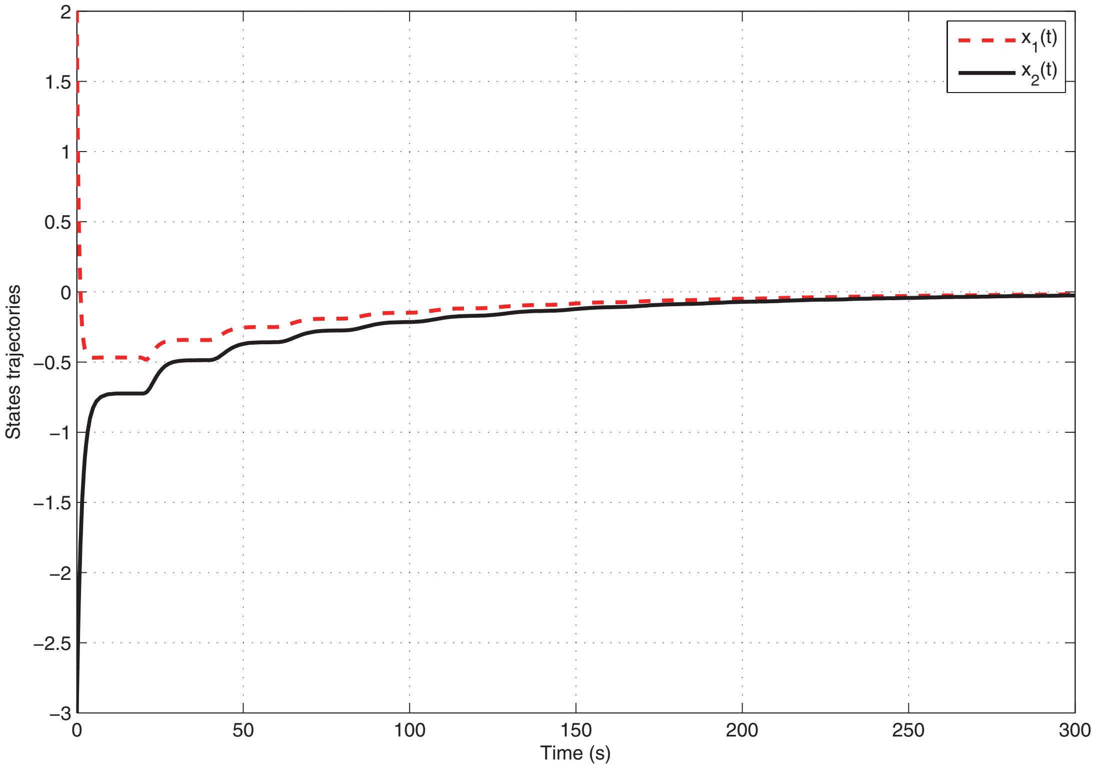


Figure 3. The error state responses for Example 4.2

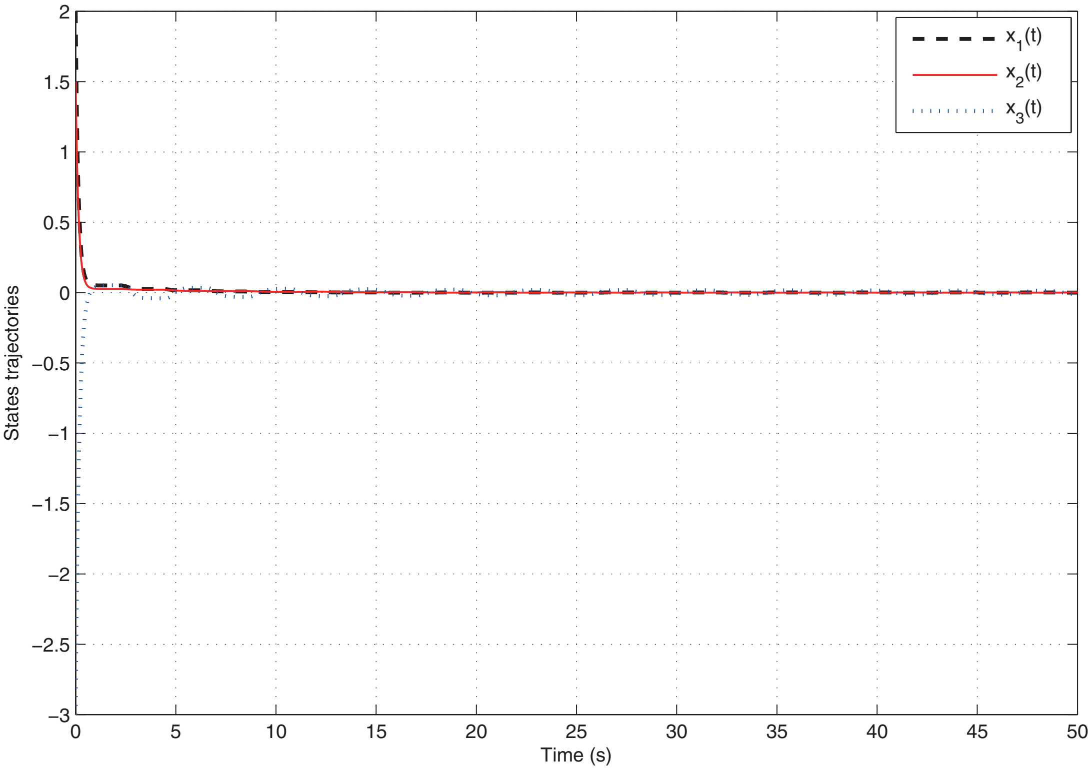


FIGURE 4. The state responses for Example 4.3

DNNs should be stable for the following conditions according to Tables 1-3.

$$\text{Example 4.1:}\quad g(x)=\begin{bmatrix}0.4\tanh(x_1)\\ 0.8\tanh(x_2)\end{bmatrix},\ x(t)=[0.6\ \ -0.8]^T,\ t\in[-38.8567,\ 0],$$
$$h(t)=\frac{38.8567}{2}+\frac{38.8567}{2}\sin\left(\frac{1.6t}{38.8567}\right);$$
$$\text{Example 4.2:}\quad g(x)=\begin{bmatrix}0.3\tanh(x_1)\\ 0.8\tanh(x_2)\end{bmatrix},\ x(t)=[2\ \ -3]^T,\ t\in[-47.5583,\ 0],$$
$$h(t)=\frac{47.5583}{2}+\frac{47.5583}{2}\sin\left(\frac{0.9t}{47.5583}\right);$$
$$\text{Example 4.3:}\quad g(x)=\begin{bmatrix}0.3680\tanh(x_1)\\ 0.1795\tanh(x_2)\\ 0.2876\tanh(x_3)\end{bmatrix},\ x(t)=[0.2\ \ 0.5\ \ -0.3]^T,\ t\in[-1.9310,\ 0],$$
$$h(t)=\frac{1.9310}{2}+\frac{1.9310}{2}\sin\left(\frac{0.2t}{1.9310}\right).$$

Remark 4.1. *The stability problem of the DDN is studied in this paper. The most important work to be dealt with in this paper is to derive a less conservative stability criterion than some recent results via Lyapunov stability theory application. In Section 4, the examples of conservative comparisons are just some numerical examples, not experiments. The corresponding simulation results just show that the DDN is still stable under the maximum time-delay bounds calculated by Theorem 3.1 of this paper. That is, the simulation graphs just show that the calculated values in tables are within the actual value range. The proposed method can be applied to other systems, for example, time-delayed Lur'e* *systems, time-delayed linear systems, and time-delayed neutral systems. This will be one of our further topics.*

5. **Conclusion.** This paper presents a new stability criterion for DNN systems with time-varying delays and sector bounded nonlinearities via a novel LKF combining the delay-product-type function with the delay-dependent matrices. Two effective matrix inequality techniques are applied to further reducing the conservativeness of the proposed criterion from some existing results. In addition, the improvement stability criteria are compared with some other cases, such as the LKF with requiring the positive definiteness of all Lyapunov matrices, the LKF with delay-independent matrices in the single-integral term. Finally, according to the comparison and discussion of numerical examples, the effectiveness of the proposed approaches is verified.

It can be known from the comparative analysis in the tables that the conservativeness of our criteria is reduced at the cost of increasing decision variables compared with those in the literature. The main reasons include augmented vectors in the LKF and the application of the generalized free-weighting-matrix integral inequalities. After consulting a lot of literature, it can be seen that the current solutions are mainly focused on developing tighter integral inequalities technique without adding additional free-weight matrices, or improving the LKF by making full use of the time-varying delay information without increasing the dimension of the quadratic term vector. These methods will be always one of our teams future research topics.

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Appendix A. Notations of several symbols and matrices for the LKF (7):

$$
\begin{aligned}
&h_d = 1 - \dot{h}(t),\ \ \gamma^T(s) = \left[x^T(s)\ \ \dot{x}^T(s)\ \ g^T(W_2x(s))\right],\ \ v_1(t) = \int_{t-h(t)}^{t} \frac{x^T(s)}{h(t)}ds,\\
&v_2(t) = \int_{t-h}^{t-h(t)} \frac{x^T(s)}{h-h(t)}ds,\ \ v_3(t) = \int_{t-h(t)}^{t} g^T(W_2x(s))ds,\ \ v_4(t) = \int_{t-h}^{t-h(t)} g^T(W_2x(s))ds,\\
&u_1(t) = \int_{t-h(t)}^{t}\int_{\theta}^{t} \frac{x^T(s)}{h(t)}dsd\theta,\ \ u_2(t) = \int_{t-h}^{t-h(t)}\int_{\theta}^{t-h(t)} \frac{x^T(s)}{h-h(t)}dsd\theta,\\
&u_3(t) = \int_{t-h(t)}^{t}\int_{\theta}^{t} \frac{g^T(W_2x(s))}{h(t)}dsd\theta,\ \ u_4(t) = \int_{t-h}^{t-h(t)}\int_{\theta}^{t-h(t)} \frac{g^T(W_2x(s))}{h-h(t)}dsd\theta,\\
&\omega_1(t) = h(t)v_1(t),\ \ \omega_2(t) = (h-h(t))v_2(t),\\
&\zeta^T(t) = \left[x^T(t)\ \ x^T(t-h(t))\ \ x^T(t-h)\ \ \omega_1(t)\ \ \omega_2(t)\ \ v_3(t)\ \ v_4(t)\right],\\
&\zeta_1^T(t) = \left[x^T(t)\ \ x^T(t-h(t))\ \ x^T(t-h)\ \ v_3(t)\ \ v_1(t)\right],\\
&\zeta_2^T(t) = \left[x^T(t)\ \ x^T(t-h(t))\ \ x^T(t-h)\ \ v_4(t)\ \ v_2(t)\right],\\
&\Delta^T(t) = \left[\omega_1^T(t)\ \ x^T(t)-x^T(t-h(t))\ \ v_3(t)\ \ \omega_2(t)\ \ x^T(t-h(t))-x^T(t-h)\ \ v_4(t)\right],\\
&\xi^T(t) = \big[x^T(t)\ \ x^T(t-h(t))\ \ x^T(t-h)\ \ \dot{x}^T(t)\ \ \dot{x}^T(t-h(t))\ \ \dot{x}^T(t-h)\ \ v_1(t)\ \ v_2(t)\ \ u_1(t)\\
&\qquad\quad u_2(t)\ \ g^T(W_2x(t))\ \ g^T(W_2x(t-h(t)))\ \ g^T(W_2x(t-h))\ \ \omega_1(t)\ \ \omega_2(t)\ \ v_3(t)\ \ v_4(t)\\
&\qquad\quad u_3(t)\ \ u_4(t)\big],\\
&\eta_1(t) = \begin{bmatrix} \omega_1(t) \\ x(t)-x(t-h(t)) \\ v_3(t) \end{bmatrix},\ \ \eta_2(t) = \begin{bmatrix} \omega_2(t) \\ x(t-h(t))-x(t-h) \\ v_4(t) \end{bmatrix},\\
&K_1 = \mathrm{diag}\left\{k_1^+, k_2^+, \ldots, k_n^+\right\},\ \ K_2 = \mathrm{diag}\left\{k_1^-, k_2^-, \ldots, k_n^-\right\}.
\end{aligned}
$$

Appendix B. Notations of several symbols and matrices for Theorem 3.1:

$$
\begin{aligned}
&\Omega_{1[h(t)]} = \mathrm{diag}\left\{EP_aE^T,\ 0,\ 0\right\} + Q_1(t),\ \ \Omega_{2[h(t)]} = \mathrm{diag}\left\{EP_bE^T,\ 0,\ 0\right\} + Q_2(t),\\
&\Omega_{3[h(t)]} = \begin{bmatrix} (2-\alpha)\Omega_{1[h(t)]} & (1-\alpha)U_{01} + \alpha U_{02} \\ * & (1+\alpha)\Omega_{2[h(t)]} \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\Omega_{4[h(t)]} = {} & P + \frac{J^T \Omega_{3[h(t)]} J}{h} + \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_2^T \\ \tilde{e}_3^T \\ \tilde{e}_6^T \\ 0 \end{bmatrix}^T [h(t) P_a] \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_2^T \\ \tilde{e}_3^T \\ \tilde{e}_6^T \\ 0 \end{bmatrix} + \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_2^T \\ \tilde{e}_3^T \\ \tilde{e}_7^T \\ 0 \end{bmatrix}^T [(h - h(t)) P_b] \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_2^T \\ \tilde{e}_3^T \\ \tilde{e}_7^T \\ 0 \end{bmatrix} \\
& + Sym \left\{ \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_2^T \\ \tilde{e}_3^T \\ \tilde{e}_6^T \\ 0 \end{bmatrix}^T P_a \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{e}_4^T \end{bmatrix} + \begin{bmatrix} \tilde{e}_1^T \\ \tilde{e}_2^T \\ \tilde{e}_3^T \\ \tilde{e}_7^T \\ 0 \end{bmatrix}^T P_b \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \tilde{e}_5^T \end{bmatrix} \right\},
\end{aligned}
$$

$$
\alpha = \frac{h(t)}{h}, \quad J^T = [E_1 \;\; E_2], \quad E_1 = [\tilde{e}_4 \;\; \tilde{e}_1 - \tilde{e}_2 \;\; \tilde{e}_6], \; E_2 = [\tilde{e}_5 \;\; \tilde{e}_2 - \tilde{e}_3 \;\; \tilde{e}_7],
$$

$$
\Pi_{[h(t),\dot{h}(t)]} = Sym \left\{ \Pi_{1[h(t),\dot{h}(t)]} + \Pi_{3[h(t),\dot{h}(t)]} \right\} + \Pi_{2[h(t),\dot{h}(t)]},
$$

$$
\begin{aligned}
\Pi_{1[h(t),\dot{h}(t)]} = {} & \Gamma_{[h(t)]} P \Psi^T_{[\dot{h}(t)]} \\
& + h(t) \left[e_1 \;\; e_2 \;\; e_3 \;\; e_{16} \;\; e_7 \right] P_a \left[e_4 \;\; h_d e_5 \;\; e_6 \;\; e_{11} - h_d e_{12} \;\; \frac{e_1 - h_d e_2 - \dot{h}(t) e_7}{h(t)} \right]^T \\
& + (h - h(t)) \left[e_1 \; e_2 \; e_3 \; e_{17} \; e_8 \right] P_b \left[e_4 \;\; h_d e_5 \;\; e_6 \;\; h_d e_{12} - e_{13} \;\; \frac{h_d e_2 - e_3 + \dot{h}(t) e_8}{h - h(t)} \right]^T \\
& + \Theta_{1[h(t)]} + \Theta_{2[\dot{h}(t)]} + \Pi_4 \Pi_5 \\
& + [e_7 \;\; e_8 \;\; e_{14} \;\; e_{15}] N [e_{14} - h(t) e_7 \;\; e_{15} - (h - h(t)) e_8]^T,
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{2[h(t),\dot{h}(t)]} = {} & [e_1 \; e_4 \; e_{11}][Q_1(t) + hR][e_1 \;\; e_4 \;\; e_{11}]^T + h_d [e_2 \;\; e_5 \;\; e_{12}][Q_2(t) - Q_1(t)][e_2 \;\; e_5 \;\; e_{12}]^T \\
& - [e_3 \;\; e_6 \;\; e_{13}] Q_2(t) [e_3 \;\; e_6 \;\; e_{13}]^T + e_1 \left[R_a + \dot{h}(t) Q_a \right] e_1^T \\
& + e_2 \left[R_b - R_a + \dot{h}(t)(Q_b - Q_a) \right] e_2^T - e_3 \left[R_b + \dot{h}(t) Q_b \right] e_3^T \\
& + \mu h(t) [e_1 \, e_4 \, e_{11}] \bar{Q}_a [e_1 \, e_4 \, e_{11}]^T + \mu (h - h(t)) [e_2 \, e_5 \, e_{12}] \bar{Q}_b [e_2 \, e_5 \, e_{12}]^T \\
& + \dot{h}(t) \Pi_6 P_a \Pi_6^T - \dot{h}(t) \Pi_7 P_b \Pi_7^T,
\end{aligned}
$$

$$
\begin{aligned}
\Pi_{3[h(t),\dot{h}(t)]} = {} & G_1 X_3 F_1 G_2^T + G_2 Y_3 F_2 G_2^T + G_1 X_4 F_1 G_3^T + G_3 Y_4 F_2 G_3^T \\
& + \left(\mu + \dot{h}(t) \right) \left(G_1 X_1 F_1 G_2^T + G_2 Y_1 F_2 G_2^T + G_1 X_2 F_1 G_3^T + G_3 Y_2 F_2 G_3^T \right),
\end{aligned}
$$

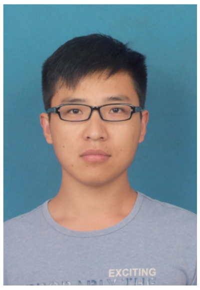
$$
\begin{aligned}
\Theta_{1[h(t)]} = {} & \sum_{i=1}^{3} \left[e_{10+i} - e_i W_2^T K_2^T \right] \left(\frac{h(t)}{h} \Lambda_{1i} + \frac{h - h(t)}{h} \Lambda_{2i} \right) \left[K_1 W_2 e_i^T - e_{10+i} \right] \\
& + \sum_{i=1}^{2} \left[(e_{10+i} - e_{11+i}) - (e_i - e_{1+i}) W_2^T K_2^T \right] \left(\frac{h(t)}{h} \Delta_{1i} \right. \\
& \left. + \frac{h - h(t)}{h} \Delta_{2i} \right) \left[K_1 W_2 (e_i - e_{1+i})^T - (e_{10+i} - e_{11+i})^T \right],
\end{aligned}
$$

$$
\Theta_{2[\dot{h}(t)]} = \left[e_{11} - e_1 W_2^T K_2^T \right] H_1 W_2 e_4^T + \left[e_1 W_2^T K_1^T - e_{11} \right] H_2 W_2 e_4^T
$$

$$+h_d\left[e_{12}-e_2W_2^TK_2^T\right]H_3W_2e_5^T+h_d\left[e_2W_2^TK_1^T-e_{12}\right]H_4W_2e_5^T$$
$$+\left[e_{13}-e_3W_2^TK_2^T\right]H_5W_2e_6^T+\left[e_3W_2^TK_1^T-e_{13}\right]H_6W_2e_6^T,$$

$$\Omega_{a[k,j]}=hG_kX_j,\ \ \Omega_{b[k,j]}=hG_kY_j,\ \ k\in[1,2,3],\ \ j\in[1,2,3,4],$$
$$\Gamma_{[h(t)]}=\left[e_1\ \ e_2\ \ e_3\ \ h(t)e_7\ \ (h-h(t))e_8\ \ e_{16}\ \ e_{17}\right],$$
$$\Psi_{[\dot h(t)]}=\left[e_4\ \ h_de_5\ \ e_6\ \ e_1-h_de_2\ \ h_de_2-e_3\ \ e_{11}-h_de_{12}\ \ h_de_{12}-e_{13}\right],$$
$$G_1=\left[e_1\ \ e_2\ \ e_3\ \ e_7\ \ e_8\ \ e_9\ \ e_{10}\ \ e_{14}\ \ e_{15}\ \ e_{16}\ \ e_{17}\ \ e_{18}\ \ e_{19}\right],$$
$$G_2=\left[e_{14}\ \ e_1-e_2\ \ e_{16}\ \ e_9\ \ e_1-e_7\ \ e_{18}\right],\ \ G_3=\left[e_{15}\ \ e_2-e_3\ \ e_{17}\ \ e_{10}\ \ e_2-e_8\ \ e_{19}\right],$$
$$\bar Q_a=Q_{11}+\begin{bmatrix}0&Q_a&0\\Q_a&0&0\\0&0&0\end{bmatrix},\ \ \bar Q_b=Q_{21}+\begin{bmatrix}0&Q_b&0\\Q_b&0&0\\0&0&0\end{bmatrix},\ \ \bar R_a=R+\begin{bmatrix}0&R_a&0\\R_a&0&0\\0&0&0\end{bmatrix},$$
$$\bar R_b=R+\begin{bmatrix}0&R_b&0\\R_b&0&0\\0&0&0\end{bmatrix},\ \ F_1=\begin{bmatrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\end{bmatrix},$$
$$F_2=\begin{bmatrix}-I&0&0&2I&0&0\\0&-I&0&0&2I&0\\0&0&-I&0&0&2I\end{bmatrix},$$
$$\Pi_4=\left[e_1U_1+e_4U_2+e_{11}U_3+e_{12}U_4\right],\ \ \Pi_5=\left[-Ae_1^T+W_0e_{11}^T+W_1e_{12}^T-e_4^T\right],$$
$$\Pi_6=\left[e_1\ \ e_2\ \ e_3\ \ e_{16}\ \ e_7\right],\ \ \Pi_7=\left[e_1\ \ e_2\ \ e_3\ \ e_{17}\ \ e_8\right].$$

Author Biography



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