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CHARACTERIZING REGULAR AND INTRA-REGULAR SEMIGROUPS IN TERMS OF PICTURE FUZZY BI-IDEALS

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Abstract. *The concept of picture fuzzy sets, was introduced by Cuong and Kreinovich in 2013, which is direct extensions of the fuzzy sets and the intuitionistic fuzzy sets. In 2020, Yiarayong applied this concept to the algebraic structure of semigroups and characterized different classes regular and intra-regular semigroups by picture fuzzy left (right) ideals. In this paper, the concepts of picture fuzzy quasi-ideals and picture fuzzy (generalized) bi-ideals are introduced. Then, picture fuzzy subsemigroups, picture fuzzy left (right) ideals and picture fuzzy (generalized) bi-ideals are considered by picture fuzzy products. Finally, some further characterizations of regular and intra-regular semigroups in terms of picture fuzzy left (right) ideals, picture fuzzy quasi-ideals and picture fuzzy (generalized) bi-ideals are obtained.*
Keywords: Picture fuzzy set, Picture fuzzy bi-ideal, Picture fuzzy generalized bi-ideal, Regular semigroup, Intra-regular semigroup

1. **Introduction.** An algebraic structure consisting of a nonempty set S together with an associative binary operation, namely, a semigroup [9], which is a generalization of groups. The semigroup was assigned to research various aspects of mathematics, for example, language theory, automata theory, combinatorics and other branches of applied mathematics (see, e.g., [2, 24, 26, 30, 34]). Regularities are interesting and important properties to study on semigroups. In 1968, Lajos [20] characterized the regular semigroups by means of left ideals and right ideals of semigroups. Next, he has shown that a semigroup S is regular if and only if the relation $BSB = B$ holds for each bi-ideal B of S, see [22]. In [21, 23], Lajos and Szasz characterized some classes of semigroups, that is, intra-regular semigroups in terms of left ideals and right ideals of semigroups. Also, Kehayopulu et al. [13] characterized the intra-regular ordered semigroup using left ideals and right ideals of ordered semigroups. Moreover, the intra-regular semigroup (with out order) was characterized in terms of left ideals, right ideals, quasi-ideals and bi-ideals of semigroups by Lee [25].

The concept of fuzzy sets was introduced by Zadeh [39] in 1965, as a function from a nonempty set X to the unit interval $[0, 1]$. The theory of fuzzy sets has been shown to be a useful method for describing situations where the data is imprecise or ambiguous, for instance, Syafaruddin et al. [35] studied the adaptive neuro-fuzzy inference system (ANFIS) method based optimal power point of PV modules. The concept of fuzzy groups, introduced by Rosenfeld [31], was the first inspired application to many algebraic structures. Kuroki [15, 18] also proposed the notion of fuzzy subsemigroups. In addition,

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he characterized some classes of semigroups by means of fuzzy left ideals, fuzzy right ideals and introduced the concept of fuzzy generalized bi-ideals in semigroups [16, 17]. In 2010, regular ordered semigroups and intra-regular ordered semigroups are characterized in terms of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals of ordered semigroups by Xie and Tang [36]. Atanassov [3, 4] developed the concept of intuitionistic fuzzy sets as a generalization of the concept of fuzzy sets. The degree of membership of an element in a given set is determined by fuzzy sets, while intuitionistic fuzzy sets provide both membership and non-membership degrees. Many mathematicians have looked into this theory (see, e.g., [1, 11, 27, 28, 40]). In 2002, Kim and Jun [14] considered the intuitionistic fuzzification of the concept of several ideals in semigroups and investigated some properties of such ideals. After that Hong and Fang [10] discussed some theorems which characterize intra-regular semigroups in terms of intuitionistic fuzzy left ideals, intuitionistic fuzzy right ideals and intuitionistic fuzzy bi-ideals. Furthermore, intra-regular ordered semigroups were characterized in terms of intuitionistic fuzzy interior ideals of ordered semigroups by Shabir and Khan [33]. Hur et al. [12] characterized regular semigroups using intuitionistic fuzzy left, right, two-sided ideals and bi-ideals of semigroups.

The concept of picture fuzzy sets was first introduced by Cuong and Kreinovich [6] in 2013, as direct extensions of the fuzzy sets and the intuitionistic fuzzy sets. Following that there were a lot of studies into the concept of picture fuzzy sets (see, e.g., [5, 7, 8, 38]). Recently, Yiarayong [37] applied the concept of picture fuzzy sets to semigroup theory and characterized different classes regular semigroups and intra-regular semigroups by means of picture fuzzy left ideals and picture fuzzy right ideals of semigroups. To more characterize the classifications of regular and intra-regular semigroups by using the concept of picture fuzzy sets. The purpose of this paper is to present some further characterizations of regular and intra-regular semigroups in terms of picture fuzzy left (right) ideals, picture fuzzy quasi-ideals and picture fuzzy (generalized) bi-ideals of semigroups.

2. **Preliminaries.**

First of all, we will recall some types of ideals of semigroups, which are necessary for the next section.

A *semigroup* is an algebraic system $(S, \cdot)$ consisting of a nonempty set S together with an associative binary operation "$\cdot$" on S. A nonempty subset A of S is called a *subsemigroup* of S if $AA \subseteq A$. A nonempty subset A of S is called a *left* (*right*) *ideal* of S if $SA \subseteq A$ ($AS \subseteq A$). A nonempty subset A of S is called an *ideal* of S if it is both a left ideal and a right ideal of S. A nonempty subset Q of S is called a *quasi-ideal* of S if $QS \cap SQ \subseteq Q$. A subsemigroup B of S is called a *bi-ideal* of S if $BSB \subseteq B$. A nonempty subset G of S is called a *generalized bi-ideal* of S if $GSG \subseteq G$.

Now, we review the notion of fuzzy sets, defined by Zadeh [39]. Let X be a nonempty set. A *fuzzy set* of X is a mapping $\mu : X \to [0, 1]$. The *intersection* and the *union* of any two fuzzy sets μ and λ of a nonempty set X, denoted by $\mu \cap \lambda$ and $\mu \cup \lambda$, respectively, are defined by letting $x \in X$,

$$(\mu \cap \lambda)(x) = \min\{\mu(x), \lambda(x)\} \text{ and } (\mu \cup \lambda)(x) = \max\{\mu(x), \lambda(x)\}.$$

Atanassov [3, 4] introduced the concept of intuitionistic fuzzy sets, which is an extension of fuzzy sets. An *intuitionistic fuzzy set* $\mathcal{A}$ on a universe X is an object of the form

$$\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x)\rangle \mid x \in X\}$$

where $\mu_{\mathcal{A}}(x) : X \to [0, 1]$ and $\eta_{\mathcal{A}}(x) : X \to [0, 1]$ denote the degree of membership and the degree of non-membership, respectively, of each $x \in X$ to the set $\mathcal{A}$ and also $0 \leq \mu_{\mathcal{A}}(x) + \eta_{\mathcal{A}}(x) \leq 1$ for all $x \in X$.

The concept of picture fuzzy sets was first introduced by Cuong and Kreinovich [6], in 2013, as direct generalizations of the fuzzy sets and the intuitionistic fuzzy sets. A *picture fuzzy set* $\mathcal{A}$ on a universe X is defined as the form

$$\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\rangle \mid x \in X\}$$

where $\mu_{\mathcal{A}}(x) : X \to [0,1]$, $\eta_{\mathcal{A}}(x) : X \to [0,1]$ and $\nu_{\mathcal{A}}(x) : X \to [0,1]$ denote the degree of positive membership, the degree of neutral membership and the degree of negative membership, respectively, for each $x \in X$ to the set $\mathcal{A}$ such that $\mu_{\mathcal{A}}$, $\eta_{\mathcal{A}}$ and $\nu_{\mathcal{A}}$ satisfy the following condition: $0 \leq \mu_{\mathcal{A}}(x)+\eta_{\mathcal{A}}(x)+\nu_{\mathcal{A}}(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we shall use the symbol $\mathcal{A}$ instead of the picture fuzzy set $\mathcal{A} = \{\langle x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x)\rangle \mid x \in X\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be any two picture fuzzy sets on a universe X [8]. We define

(*i*) $\mathcal{A} \subseteq \mathcal{B}$ iff $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x)$, $\eta_{\mathcal{A}}(x) \geq \eta_{\mathcal{B}}(x)$ and $\nu_{\mathcal{A}}(x) \geq \nu_{\mathcal{B}}(x)$ for all $x \in X$,
(*ii*) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$,
(*iii*) $\mathcal{A} \cap \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \cap \mu_{\mathcal{B}})(x), (\eta_{\mathcal{A}} \cup \eta_{\mathcal{B}})(x), (\nu_{\mathcal{A}} \cup \nu_{\mathcal{B}})(x)\rangle \mid x \in X\}$,
(*iv*) $\mathcal{A} \cup \mathcal{B} = \{\langle x, (\mu_{\mathcal{A}} \cup \mu_{\mathcal{B}})(x), (\eta_{\mathcal{A}} \cap \eta_{\mathcal{B}})(x), (\nu_{\mathcal{A}} \cap \nu_{\mathcal{B}})(x)\rangle \mid x \in X\}$.

Now, we denote by $\mathcal{PFS}(S)$ the collection of picture fuzzy sets on a semigroup S with $\mathcal{S} = \{\langle x, 1, 0, 0\rangle \mid x \in S\}$ and $\emptyset = \{\langle x, 0, 0, 1\rangle \mid x \in S\}$ [37]. Let X be any subset of a semigroup S. The *picture characteristic function* [37] of X is denoted by $\mathcal{C}^X = \{\langle x, \mu_{\mathcal{C}^X}(x), \eta_{\mathcal{C}^X}(x), \nu_{\mathcal{C}^X}(x)\rangle \mid x \in S\}$, where

$$\mu_{\mathcal{C}^X}(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{otherwise,} \end{cases} \quad \eta_{\mathcal{C}^X}(x) = \begin{cases} 0 & \text{if } x \in X \\ 1 & \text{otherwise,} \end{cases} \quad \nu_{\mathcal{C}^X}(x) = \begin{cases} 0 & \text{if } x \in X \\ 1 & \text{otherwise.} \end{cases}$$

We note that if $X = S$ ($X = \emptyset$), then $\mathcal{C}^X = \mathcal{S}$ ($\mathcal{C}^X = \emptyset$).

A picture fuzzy set $\mathcal{A}$ on a semigroup S is called a *picture fuzzy subsemigroup* [37] of S if it satisfies the following conditions:

(*i*) $\mu_{\mathcal{A}}(xy) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\}$;
(*ii*) $\eta_{\mathcal{A}}(xy) \leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y)\}$;
(*iii*) $\nu_{\mathcal{A}}(xy) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(y)\}$,

for all $x, y \in S$.

A picture fuzzy set $\mathcal{A}$ on a semigroup S is called a *picture fuzzy left* (*right*) *ideal* [37] of S if satisfying the following conditions:

(*i*) $\mu_{\mathcal{A}}(xy) \geq \mu_{\mathcal{A}}(y)$ ($\mu_{\mathcal{A}}(xy) \geq \mu_{\mathcal{A}}(x)$);
(*ii*) $\eta_{\mathcal{A}}(xy) \leq \eta_{\mathcal{A}}(y)$ ($\eta_{\mathcal{A}}(xy) \leq \eta_{\mathcal{A}}(x)$);
(*iii*) $\nu_{\mathcal{A}}(xy) \leq \nu_{\mathcal{A}}(y)$ ($\nu_{\mathcal{A}}(xy) \leq \nu_{\mathcal{A}}(x)$),

for all $x, y \in S$.

A picture fuzzy set $\mathcal{A}$ on a semigroup S is said to be a *picture fuzzy ideal* of S if it is both a picture fuzzy left ideal and a picture fuzzy right ideal of S.

Next, we will introduce the notions of picture fuzzy bi-ideals and picture fuzzy generalized bi-ideals of semigroups as follows.

Definition 2.1. *A picture fuzzy subsemigroup $\mathcal{A}$ on a semigroup S is called a picture fuzzy bi-ideal of S if it satisfies the following conditions:*

(*i*) $\mu_{\mathcal{A}}(xyz) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(z)\}$;
(*ii*) $\eta_{\mathcal{A}}(xyz) \leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(z)\}$;
(*iii*) $\nu_{\mathcal{A}}(xyz) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(z)\}$,

for all $x, y, z \in S$.

Definition 2.2. *A picture fuzzy set $\mathcal{A}$ on a semigroup S is called a picture fuzzy generalized bi-ideal of S if it satisfies the following conditions:*

(i) $\mu_{\mathcal{A}}(xyz) \geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(z)\}$;
(ii) $\eta_{\mathcal{A}}(xyz) \leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(z)\}$;
(iii) $\nu_{\mathcal{A}}(xyz) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(z)\}$,

for all $x, y, z \in S$.

Obviously, every picture fuzzy left (right) ideal of a semigroup S is a picture fuzzy bi-ideal, and every picture fuzzy bi-ideal of S is a picture fuzzy generalized bi-ideal. In general, the converse of statements is not true. We can show it by the following examples.

Example 2.1. *Let $S = \{a, b, c, d\}$. Define the binary operation $\cdot$ on S as follows:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	b
d	a	a	b	c

Then, $(S, \cdot)$ is a semigroup [22]. *Next, we define a picture fuzzy set $\mathcal{A}$ on S as follows:*

$$\begin{array}{llll} \mu_{\mathcal{A}}(a) = 0.6, & \mu_{\mathcal{A}}(b) = 0.3, & \mu_{\mathcal{A}}(c) = 0.4, & \mu_{\mathcal{A}}(d) = 0.1; \\ \eta_{\mathcal{A}}(a) = 0.1, & \eta_{\mathcal{A}}(b) = 0.3, & \eta_{\mathcal{A}}(c) = 0.2, & \eta_{\mathcal{A}}(d) = 0.4; \\ \nu_{\mathcal{A}}(a) = 0.1, & \nu_{\mathcal{A}}(b) = 0.4, & \nu_{\mathcal{A}}(c) = 0.3, & \nu_{\mathcal{A}}(d) = 0.5. \end{array}$$

By routine computations, we obtain that $\mathcal{A}$ is a picture fuzzy bi-ideal of S, but it is not a picture fuzzy left ideal, because

$$\mu_{\mathcal{A}}(dc) = \mu_{\mathcal{A}}(b) = 0.3 < 0.4 = \mu_{\mathcal{A}}(c),$$
$$\eta_{\mathcal{A}}(dc) = \eta_{\mathcal{A}}(b) = 0.3 > 0.2 = \eta_{\mathcal{A}}(c),$$
$$\nu_{\mathcal{A}}(dc) = \nu_{\mathcal{A}}(b) = 0.4 > 0.3 = \nu_{\mathcal{A}}(c).$$

Also, $\mathcal{A}$ is not a picture fuzzy right ideal of S, because

$$\mu_{\mathcal{A}}(cd) = \mu_{\mathcal{A}}(b) = 0.3 < 0.4 = \mu_{\mathcal{A}}(c),$$
$$\eta_{\mathcal{A}}(cd) = \eta_{\mathcal{A}}(b) = 0.3 > 0.2 = \eta_{\mathcal{A}}(c),$$
$$\nu_{\mathcal{A}}(cd) = \nu_{\mathcal{A}}(b) = 0.4 > 0.3 = \nu_{\mathcal{A}}(c).$$

Example 2.2. *Let $S = \{a, b, c, d\}$ and define the binary operation $\cdot$ on S by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Then, $(S, \cdot)$ is a semigroup [32]. *Define a picture fuzzy set $\mathcal{A}$ as follows:*

$$\begin{array}{llll} \mu_{\mathcal{A}}(a) = 0.7, & \mu_{\mathcal{A}}(b) = 0.1, & \mu_{\mathcal{A}}(c) = 0.5, & \mu_{\mathcal{A}}(d) = 0.3; \\ \eta_{\mathcal{A}}(a) = 0.2, & \eta_{\mathcal{A}}(b) = 0.5, & \eta_{\mathcal{A}}(c) = 0.3, & \eta_{\mathcal{A}}(d) = 0.4; \\ \nu_{\mathcal{A}}(a) = 0.1, & \nu_{\mathcal{A}}(b) = 0.4, & \nu_{\mathcal{A}}(c) = 0.2, & \nu_{\mathcal{A}}(d) = 0.3. \end{array}$$

By routine calculations, we have that $\mathcal{A}$ is a picture fuzzy generalized bi-ideal of S, but it is not a picture fuzzy bi-ideal, because

$$\mu_{\mathcal{A}}(cc) = \mu_{\mathcal{A}}(b) = 0.1 < 0.5 = \mu_{\mathcal{A}}(c) = \min\{\mu_{\mathcal{A}}(c), \mu_{\mathcal{A}}(c)\},$$
$$\eta_{\mathcal{A}}(cc) = \eta_{\mathcal{A}}(b) = 0.5 > 0.3 = \eta_{\mathcal{A}}(c) = \max\{\eta_{\mathcal{A}}(c), \eta_{\mathcal{A}}(c)\},$$

$$\nu_{\mathcal{A}}(cc) = \nu_{\mathcal{A}}(b) = 0.4 > 0.2 = \nu_{\mathcal{A}}(c) = \max\{\nu_{\mathcal{A}}(c), \nu_{\mathcal{A}}(c)\}.$$

That is, $\mathcal{A}$ is not a picture fuzzy subsemigroup of S.

Let $\mathcal{A}$ and $\mathcal{B}$ be any two picture fuzzy sets on a semigroup S. The *picture fuzzy product* [37] of $\mathcal{A}$ and $\mathcal{B}$ is defined by

$$\mathcal{A} \circ \mathcal{B} := \{\langle x, (\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}})(x), (\eta_{\mathcal{A}} \circ \eta_{\mathcal{A}})(x), (\nu_{\mathcal{A}} \circ \nu_{\mathcal{A}})(x)\rangle \mid x \in S\}$$

where

$$(\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}})(x) = \begin{cases} \sup\limits_{x=yz} [\min\{\mu_{\mathcal{A}}(y), \mu_{\mathcal{B}}(z)\}] & \text{if } x \text{ is expressible } x = yz, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\eta_{\mathcal{A}} \circ \eta_{\mathcal{A}})(x) = \begin{cases} \inf\limits_{x=yz} [\max\{\eta_{\mathcal{A}}(y), \eta_{\mathcal{B}}(z)\}] & \text{if } x \text{ is expressible } x = yz, \\ 1 & \text{otherwise,} \end{cases}$$

$$(\nu_{\mathcal{A}} \circ \nu_{\mathcal{A}})(x) = \begin{cases} \inf\limits_{x=yz} [\max\{\nu_{\mathcal{A}}(y), \nu_{\mathcal{B}}(z)\}] & \text{if } x \text{ is expressible } x = yz, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 2.1. *Let S be a semigroup and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be picture fuzzy sets on S. If $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{C} \subseteq \mathcal{D}$, then $\mathcal{A} \circ \mathcal{C} \subseteq \mathcal{B} \circ \mathcal{D}$.*

Proof: The proof is straightforward. □

Lemma 2.2. *Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S. Then, the following statements are equivalent:*

(i) *$\mathcal{A}$ is a picture fuzzy subsemigroup of S;*
(ii) *$\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$.*

Proof: (i) $\Rightarrow$ (ii) Let $a \in S$. Obviously, $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$ for all $x, y \in S$ such that $a \neq xy$. On the other hand, if there exist $b, c \in S$ such that $a = bc$. Thus,

$$\begin{aligned} (\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}})(a) &= \sup_{a=bc} [\min\{\mu_{\mathcal{A}}(b), \mu_{\mathcal{A}}(c)\}] \\ &\leq \sup_{a=bc} [\mu_{\mathcal{A}}(bc)] = \sup_{a=bc} [\mu_{\mathcal{A}}(a)] \\ &= \mu_{\mathcal{A}}(a) \end{aligned}$$

and

$$\begin{aligned} (\eta_{\mathcal{A}} \circ \eta_{\mathcal{A}})(a) &= \inf_{a=bc} [\max\{\eta_{\mathcal{A}}(b), \eta_{\mathcal{A}}(c)\}] \\ &\geq \inf_{a=bc} [\eta_{\mathcal{A}}(bc)] = \inf_{a=bc} [\eta_{\mathcal{A}}(a)] \\ &= \eta_{\mathcal{A}}(a). \end{aligned}$$

Also, we obtain that $(\nu_{\mathcal{A}} \circ \nu_{\mathcal{A}})(a) \geq \nu_{\mathcal{A}}(a)$. Therefore, $\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$.

(ii) $\Rightarrow$ (i) Let $x, y \in S$. Take $a = xy$. By assumption, we have

$$\begin{aligned} \mu_{\mathcal{A}}(xy) = \mu_{\mathcal{A}}(a) &\geq (\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}})(a) \\ &= \sup_{a=bc} [\min\{\mu_{\mathcal{A}}(b), \mu_{\mathcal{A}}(c)\}] \\ &\geq \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(y)\} \end{aligned}$$

and

$$\begin{aligned} \eta_{\mathcal{A}}(xy) = \eta_{\mathcal{A}}(a) &\leq (\eta_{\mathcal{A}} \circ \eta_{\mathcal{A}})(a) \\ &= \inf_{a=bc} [\max\{\eta_{\mathcal{A}}(b), \eta_{\mathcal{A}}(c)\}] \end{aligned}$$

$$\leq \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(y)\}.$$

Similarly, we can show that $\nu_{\mathcal{A}}(xy) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(y)\}$. Consequently, $\mathcal{A}$ is a picture fuzzy subsemigroup of S. □

Lemma 2.3. *Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S. Then, the following conditions are equivalent:*

(i) $\mathcal{A}$ is a picture fuzzy left ideal of S;

(ii) $\mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$.

Proof: $(i) \Rightarrow (ii)$ Let $a \in S$. It is clear that $\mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$ for all $x, y \in S$ such that $a \neq xy$. Otherwise, there exist $b, c \in S$ such that $a = bc$. Then,

$$\begin{aligned}(\mu_{\mathcal{S}} \circ \mu_{\mathcal{A}})(a) &= \sup_{a=bc}[\min\{\mu_{\mathcal{S}}(b), \mu_{\mathcal{A}}(c)\}] \\ &\leq \sup_{a=bc}[\min\{1, \mu_{\mathcal{A}}(bc)\}] \\ &= \sup_{a=bc}[\min\{1, \mu_{\mathcal{A}}(a)\}] \\ &= \mu_{\mathcal{A}}(a)\end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{S}} \circ \eta_{\mathcal{A}})(a) &= \inf_{a=bc}[\max\{\eta_{\mathcal{S}}(b), \eta_{\mathcal{A}}(c)\}] \\ &\geq \inf_{a=bc}[\max\{0, \eta_{\mathcal{A}}(bc)\}] \\ &= \inf_{a=bc}[\max\{0, \eta_{\mathcal{A}}(a)\}] \\ &= \eta_{\mathcal{A}}(a).\end{aligned}$$

Similarly, we have that $(\nu_{\mathcal{S}} \circ \nu_{\mathcal{A}})(a) \geq \nu_{\mathcal{A}}(a)$. Hence, $\mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$.

$(ii) \Rightarrow (i)$ Let $x, y \in S$. Put $a = xy$. By hypothesis, we have

$$\begin{aligned}\mu_{\mathcal{A}}(xy) = \mu_{\mathcal{A}}(a) &\geq (\mu_{\mathcal{A}} \circ \mu_{\mathcal{A}})(a) \\ &= \sup_{a=bc}[\min\{\mu_{\mathcal{S}}(b), \mu_{\mathcal{A}}(c)\}] \\ &\geq \min\{\mu_{\mathcal{S}}(x), \mu_{\mathcal{A}}(y)\} \\ &= \min\{1, \mu_{\mathcal{A}}(y)\} = \mu_{\mathcal{A}}(y)\end{aligned}$$

and

$$\begin{aligned}\eta_{\mathcal{A}}(xy) = \eta_{\mathcal{A}}(a) &\leq (\eta_{\mathcal{A}} \circ \eta_{\mathcal{A}})(a) \\ &= \inf_{a=bc}[\max\{\eta_{\mathcal{S}}(b), \eta_{\mathcal{A}}(c)\}] \\ &\leq \max\{\eta_{\mathcal{S}}(x), \eta_{\mathcal{A}}(y)\} \\ &= \max\{0, \eta_{\mathcal{A}}(y)\} = \eta_{\mathcal{A}}(y).\end{aligned}$$

Similarly, we can show that $\nu_{\mathcal{A}}(xy) \leq \nu_{\mathcal{A}}(y)$. Therefore, $\mathcal{A}$ is a picture fuzzy left ideal of S. □

The proof of the following lemma is similar to that of Lemma 2.3.

Lemma 2.4. *Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S. Then, the following conditions are equivalent:*

(i) $\mathcal{A}$ is a picture fuzzy right ideal of S;

(ii) $\mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$.

Lemma 2.5. *Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S. Then, the following statements are equivalent:*

(i) *$\mathcal{A}$ is a picture fuzzy generalized bi-ideal of S;*
(ii) $\mathcal{A} \circ \mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$.

Proof: $(i) \Rightarrow (ii)$ Assume that $\mathcal{A}$ is a picture fuzzy generalized bi-ideal of S. Let $a \in S$. If there is no $a = bcd$ for all $b, c, d \in S$, then it is well done. Suppose that there exist $x, y, z \in S$ such that $a = xyz$. Let $k = xy$. Thus,

$$\begin{aligned}(\mu_{\mathcal{A}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{A}})(a) &= \sup_{a=kz}[\min\{(\mu_{\mathcal{A}} \circ \mu_{\mathcal{S}})(k), \mu_{\mathcal{A}}(z)\}] \\ &= \sup_{a=kz}\left[\min\left\{\sup_{k=xy}[\min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{S}}(y)\}], \mu_{\mathcal{A}}(z)\right\}\right] \\ &= \sup_{a=kz}\left[\sup_{k=xy}[\min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{S}}(y), \mu_{\mathcal{A}}(z)\}]\right] \\ &= \sup_{a=kz}\left[\sup_{k=xy}[\min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(z)\}]\right] \\ &\leq \sup_{a=kz}\left[\sup_{k=xy}[\mu_{\mathcal{A}}(xyz)]\right] \\ &= \sup_{a=kz}[\mu_{\mathcal{A}}(kz)] = \mu_{\mathcal{A}}(a)\end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{A}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{A}})(a) &= \inf_{a=kz}[\max\{(\eta_{\mathcal{A}} \circ \eta_{\mathcal{S}})(k), \eta_{\mathcal{A}}(z)\}] \\ &= \inf_{a=kz}\left[\max\left\{\inf_{k=xy}[\max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{S}}(y)\}], \eta_{\mathcal{A}}(z)\right\}\right] \\ &= \inf_{a=kz}\left[\inf_{k=xy}[\max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{S}}(y), \eta_{\mathcal{A}}(z)\}]\right] \\ &= \inf_{a=kz}\left[\inf_{k=xy}[\max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(z)\}]\right] \\ &\geq \inf_{a=kz}\left[\inf_{k=xy}[\eta_{\mathcal{A}}(xyz)]\right] \\ &= \inf_{a=kz}[\eta_{\mathcal{A}}(kz)] = \eta_{\mathcal{A}}(a).\end{aligned}$$

Similarly, we have that $(\nu_{\mathcal{A}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{A}})(a) \geq \nu_{\mathcal{A}}(a)$. Therefore, $\mathcal{A} \circ \mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$.

$(ii) \Rightarrow (i)$ Let $x, y, z \in S$. Put $a = xyz$. By assumption, we have

$$\begin{aligned}\mu_{\mathcal{A}}(xyz) &= \mu_{\mathcal{A}}(a) \\ &\geq (\mu_{\mathcal{A}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{A}})(a) \\ &= \sup_{a=pq}[\min\{(\mu_{\mathcal{A}} \circ \mu_{\mathcal{S}})(p), \mu_{\mathcal{A}}(q)\}] \\ &\geq \min\{(\mu_{\mathcal{A}} \circ \mu_{\mathcal{S}})(xy), \mu_{\mathcal{A}}(z)\} \\ &= \min\left\{\sup_{xy=uv}[\min\{\mu_{\mathcal{A}}(u), \mu_{\mathcal{S}}(v)\}], \mu_{\mathcal{A}}(z)\right\} \\ &\geq \min\{\min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{S}}(y)\}, \mu_{\mathcal{A}}(z)\} \\ &= \min\{\mu_{\mathcal{A}}(x), \mu_{\mathcal{A}}(z)\}\end{aligned}$$

and

$$\begin{aligned}\eta_{\mathcal{A}}(xyz) &= \eta_{\mathcal{A}}(a)\\ &\leq (\eta_{\mathcal{A}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{A}})(a)\\ &= \inf_{a=pq}[\max\{(\eta_{\mathcal{A}} \circ \eta_{\mathcal{S}})(p), \eta_{\mathcal{A}}(q)\}]\\ &\leq \max\{(\eta_{\mathcal{A}} \circ \eta_{\mathcal{S}})(xy), \eta_{\mathcal{A}}(z)\}\\ &= \max\left\{\inf_{xy=uv}[\max\{\eta_{\mathcal{A}}(u), \eta_{\mathcal{S}}(v)\}], \eta_{\mathcal{A}}(z)\right\}\\ &\leq \max\{\max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{S}}(y)\}, \eta_{\mathcal{A}}(z)\}\\ &= \max\{\eta_{\mathcal{A}}(x), \eta_{\mathcal{A}}(z)\}.\end{aligned}$$

Also, we have that $\nu_{\mathcal{A}}(xyz) \leq \max\{\nu_{\mathcal{A}}(x), \nu_{\mathcal{A}}(z)\}$. Consequently, $\mathcal{A}$ is a picture fuzzy generalized bi-ideal of S. □

By Lemma 2.2 and Lemma 2.5, we have the following corollary.

Corollary 2.1. *Let $\mathcal{A}$ be a picture fuzzy set on a semigroup S. Then, the following statements are equivalent:*

(*i*) *$\mathcal{A}$ is a picture fuzzy bi-ideal of S;*
(*ii*) *$\mathcal{A} \circ \mathcal{A} \subseteq \mathcal{A}$ and $\mathcal{A} \circ \mathcal{S} \circ \mathcal{A} \subseteq \mathcal{A}$.*

Lemma 2.6. [37] *Let A be a nonempty subset of a semigroup S. Then, the following properties hold:*

(*i*) *A is a subsemigroup of S if and only if $\mathcal{C}^A$ is a picture fuzzy subsemigroup of S;*
(*ii*) *A is a left ideal of S if and only if $\mathcal{C}^A$ is a picture fuzzy left ideal of S;*
(*iii*) *A is a right ideal of S if and only if $\mathcal{C}^A$ is a picture fuzzy right ideal of S;*
(*iv*) *A is an ideal of S if and only if $\mathcal{C}^A$ is a picture fuzzy ideal of S.*

Lemma 2.7. *Let A be a nonempty subset of a semigroup S. Then, A is a generalized bi-ideal of S if and only if $\mathcal{C}^A$ is a picture fuzzy generalized bi-ideal of S.*

Proof: Assume that A is a generalized bi-ideal of S. Suppose that $\mu_{\mathcal{C}^A}(abc) < \min\{\mu_{\mathcal{C}^A}(a), \mu_{\mathcal{C}^A}(c)\}$ for some $a, b, c \in S$. Then, $\mu_{\mathcal{C}^A}(abc) = 0$ and $\min\{\mu_{\mathcal{C}^A}(a), \mu_{\mathcal{C}^A}(c)\} = 1$. Also, $\mu_{\mathcal{C}^A}(a) = 1$ and $\mu_{\mathcal{C}^A}(c) = 1$. We obtain that $abc \notin A$ and $a, c \in A$. Since $ASA \subseteq A$, we have that $abc \in A$, which is a contradiction. Thus,

$$\mu_{\mathcal{C}^A}(xyz) \geq \min\{\mu_{\mathcal{C}^A}(x), \mu_{\mathcal{C}^A}(z)\}$$

for all $x, y, z \in S$. If $\eta_{\mathcal{C}^A}(abc) > \max\{\eta_{\mathcal{C}^A}(a), \eta_{\mathcal{C}^A}(c)\}$ for some $a, b, c \in S$, then $\eta_{\mathcal{C}^A}(abc) = 1$ and $\max\{\eta_{\mathcal{C}^A}(a), \eta_{\mathcal{C}^A}(c)\} = 0$. This implies that $\eta_{\mathcal{C}^A}(a) = 0$ and $\eta_{\mathcal{C}^A}(c) = 0$. That is, $abc \notin A$ and $a, c \in A$. By assumption, we get that $abc \in A$. This is a contradiction. Hence,

$$\eta_{\mathcal{C}^A}(xyz) \leq \max\{\eta_{\mathcal{C}^A}(x), \eta_{\mathcal{C}^A}(z)\}$$

for all $x, y, z \in S$. Similarly, we can show that $\nu_{\mathcal{C}^A}(xyz) \leq \max\{\nu_{\mathcal{C}^A}(x), \nu_{\mathcal{C}^A}(z)\}$ for all $x, y, z \in S$. Therefore, $\mathcal{C}^A$ is a picture fuzzy generalized bi-ideal of S.

Conversely, assume that $\mathcal{C}^A$ is a picture fuzzy generalized bi-ideal of S. Let $x, y, z \in S$ such that $x, z \in A$. Then, $\mu_{\mathcal{C}^A}(xyz) \geq \min\{\mu_{\mathcal{C}^A}(x), \mu_{\mathcal{C}^A}(z)\} = \min\{1, 1\} = 1$. So, $xyz \in A$. Thus, $ASA \subseteq A$. Consequently, A is a generalized bi-ideal of S. □

By Lemma 2.6 (*i*) and Lemma 2.7, we have the following corollary.

Corollary 2.2. *Let A be a nonempty subset of a semigroup S. Then, A is a bi-ideal of S if and only if $\mathcal{C}^A$ is a picture fuzzy bi-ideal of S.*

3. **Characterizations of Regular Semigroups.** In this section, we study some characterizations of regular semigroups by the properties of picture fuzzy left ideals, picture fuzzy right ideal and picture fuzzy (generalized) bi-ideals of semigroups.

A semigroup S is called *regular* [20] if for every element a of S, there exists an element x in S such that $a = axa$.

Theorem 3.1. *Every picture fuzzy generalized bi-ideal of a regular semigroup S is a picture fuzzy bi-ideal of S.*

Proof: Let $\mathcal{A}$ be a picture fuzzy generalized bi-ideal of S and $a, b \in S$. Since S is regular, there exists $x \in S$ such that $b = bxb$. Then,

$$\mu_{\mathcal{A}}(ab) = \mu_{\mathcal{A}}(a(bxb)) = \mu_{\mathcal{A}}(a(bx)b) \geq \min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{A}}(b)\}$$

and

$$\eta_{\mathcal{A}}(ab) = \eta_{\mathcal{A}}(a(bxb)) = \eta_{\mathcal{A}}(a(bx)b) \leq \max\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{A}}(b)\}.$$

In a same way, we have that $\nu_{\mathcal{A}}(ab) \leq \max\{\nu_{\mathcal{A}}(a), \nu_{\mathcal{A}}(b)\}$. Hence, $\mathcal{A}$ is a picture fuzzy subsemigroup of S. This shows that $\mathcal{A}$ is a picture fuzzy bi-ideal of S. □

Now, we recall the characterization of regular semigroups by their picture fuzzy left ideals and picture fuzzy right ideals which occurred in [37] as follows.

Lemma 3.1. [37] *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$\mathcal{R} \cap \mathcal{L} = \mathcal{R} \circ \mathcal{L}$, for every picture fuzzy left ideal $\mathcal{L}$ and every picture fuzzy right ideal $\mathcal{R}$ of S.*

Lemma 3.2. [22] *Let S be a semigroup. Then, S is regular if and only if $B = BSB$, for every bi-ideal B of S.*

Next, we give some more characterizations of regular semigroups by picture fuzzy bi-ideals and picture fuzzy generalized bi-ideals of semigroups.

Theorem 3.2. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular;*

(ii) *$\mathcal{B} = \mathcal{B} \circ \mathcal{S} \circ \mathcal{B}$, for every picture fuzzy bi-ideal $\mathcal{B}$ of S.*

Proof: (i) ⇒ (ii) Let $\mathcal{B}$ be a picture fuzzy bi-ideal of S. By Corollary 2.1, $\mathcal{B} \circ \mathcal{S} \circ \mathcal{B} \subseteq \mathcal{B}$. Otherwise, let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa$. Then,

$$\begin{aligned}
(\mu_{\mathcal{B}} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{B}})(a) &= \sup_{a=yz}[\min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{S}})(y), \mu_{\mathcal{B}}(z)\}] \\
&\geq \min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{S}})(ax), \mu_{\mathcal{B}}(a)\} \\
&= \min\left\{\sup_{ax=pq}[\min\{\mu_{\mathcal{B}}(p), \mu_{\mathcal{S}}(q)\}], \mu_{\mathcal{B}}(a)\right\} \\
&\geq \min\{\min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{S}}(x)\}, \mu_{\mathcal{B}}(a)\} \\
&= \mu_{\mathcal{B}}(a)
\end{aligned}$$

and

$$\begin{aligned}
(\eta_{\mathcal{B}} \circ \eta_{\mathcal{S}} \circ \eta_{\mathcal{B}})(a) &= \inf_{a=yz}[\max\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{S}})(y), \eta_{\mathcal{B}}(z)\}] \\
&\leq \max\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{S}})(ax), \eta_{\mathcal{B}}(a)\} \\
&= \max\left\{\inf_{ax=pq}[\max\{\eta_{\mathcal{B}}(p), \eta_{\mathcal{S}}(q)\}], \eta_{\mathcal{B}}(a)\right\} \\
&\leq \max\{\max\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{S}}(x)\}, \eta_{\mathcal{B}}(a)\}
\end{aligned}$$

$$= \eta_{\mathcal{B}}(a).$$

Similarly, we can show that $(\nu_{\mathcal{B}} \circ \nu_{\mathcal{S}} \circ \nu_{\mathcal{B}})(a) \le \nu_{\mathcal{B}}(a)$. Hence, $\mathcal{B} \subseteq \mathcal{B} \circ \mathcal{S} \circ \mathcal{B}$. Therefore, $\mathcal{B} = \mathcal{B} \circ \mathcal{S} \circ \mathcal{B}$.

$(ii) \Rightarrow (i)$ Let B be a bi-ideal of S. By Corollary 2.2, $\mathcal{C}^B$ is a picture fuzzy bi-ideal of S. Let $a \in B$. By assumption, we have

$$\sup_{a=yz}[\min\{(\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{S}})(y), \mu_{\mathcal{C}^B}(z)\}] = (\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{S}} \circ \mu_{\mathcal{C}^B})(a) = \mu_{\mathcal{C}^B}(a) = 1.$$

This means that there exist $b, c \in S$ such that $a = bc$, $(\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{S}})(b) = 1$ and $\mu_{\mathcal{C}^B}(c) = 1$. Then, $c \in B$ and $\sup_{b=pq}[\min\{\mu_{\mathcal{C}^B}(p), \mu_{\mathcal{S}}(q)\}] = (\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{S}})(b) = 1$. That is, there exist $u, v \in S$ such that $b = uv$, $\mu_{\mathcal{C}^B}(u) = 1$ and $\mu_{\mathcal{S}}(v) = 1$. Thus, $u \in B$. It follows that $a = bc = (uv)c \in BSB$. So, $B \subseteq BSB$. Since B is a bi-ideal of S, $BSB \subseteq B$. Hence, $B = BSB$. Consequently, S is regular by Lemma 3.2. □

From Theorem 3.1 and Theorem 3.2, the following theorem can be proved.

Theorem 3.3. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular;*
(ii) *$\mathcal{G} = \mathcal{G} \circ \mathcal{S} \circ \mathcal{G}$, for every picture fuzzy generalized bi-ideal $\mathcal{G}$ of S.*

Theorem 3.4. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular;*
(ii) *$\mathcal{B} \cap \mathcal{A} = \mathcal{B} \circ \mathcal{A} \circ \mathcal{B}$, for every picture fuzzy ideal $\mathcal{A}$ and every picture fuzzy bi-ideal $\mathcal{B}$ of S.*

Proof: $(i) \Rightarrow (ii)$ Let $\mathcal{A}$ and $\mathcal{B}$ be a picture fuzzy ideal and a picture fuzzy bi-ideal of S, respectively. By Lemma 2.1 and Corollary 2.1, we have that $\mathcal{B} \circ \mathcal{A} \circ \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{S} \circ \mathcal{B} \subseteq \mathcal{B}$. Since $\mathcal{A}$ is a picture fuzzy ideal of S, $\mathcal{A}$ is both a picture fuzzy left ideal and a picture fuzzy right ideal of S. Then, using Lemma 2.1, Lemma 2.3 and Lemma 2.4, we obtain that $\mathcal{B} \circ \mathcal{A} \circ \mathcal{B} \subseteq (\mathcal{S} \circ \mathcal{A}) \circ \mathcal{S} \subseteq \mathcal{A} \circ \mathcal{S} \subseteq \mathcal{A}$. It turns out that $\mathcal{B} \circ \mathcal{A} \circ \mathcal{B} \subseteq \mathcal{B} \cap \mathcal{A}$. Next, let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa = axaxa$. Thus,

$$\begin{aligned}
(\mu_{\mathcal{B}} \circ \mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(a) &= \sup_{a=yz}[\min\{\mu_{\mathcal{B}}(y), (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(z)\}] \\
&\ge \min\{\mu_{\mathcal{B}}(a), (\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(xaxa)\} \\
&= \min\left\{\mu_{\mathcal{B}}(a), \sup_{xaxa=pq}[\min\{\mu_{\mathcal{A}}(p), \mu_{\mathcal{B}}(q)\}]\right\} \\
&\ge \min\{\mu_{\mathcal{B}}(a), \min\{\mu_{\mathcal{A}}(xax), \mu_{\mathcal{B}}(a)\}\} \\
&\ge \min\{\mu_{\mathcal{B}}(a), \min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{B}}(a)\}\} \\
&= \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{A}}(a)\} \\
&= (\mu_{\mathcal{B}} \cap \mu_{\mathcal{B}})(a)
\end{aligned}$$

and

$$\begin{aligned}
(\eta_{\mathcal{B}} \circ \eta_{\mathcal{A}} \circ \eta_{\mathcal{B}})(a) &= \inf_{a=yz}[\max\{\eta_{\mathcal{B}}(y), (\eta_{\mathcal{A}} \circ \eta_{\mathcal{B}})(z)\}] \\
&\le \max\{\eta_{\mathcal{B}}(a), (\eta_{\mathcal{A}} \circ \eta_{\mathcal{B}})(xaxa)\} \\
&= \max\left\{\eta_{\mathcal{B}}(a), \inf_{xaxa=pq}[\max\{\eta_{\mathcal{A}}(p), \eta_{\mathcal{B}}(q)\}]\right\} \\
&\le \max\{\eta_{\mathcal{B}}(a), \max\{\eta_{\mathcal{A}}(xax), \eta_{\mathcal{B}}(a)\}\} \\
&\le \max\{\eta_{\mathcal{B}}(a), \max\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{B}}(a)\}\}
\end{aligned}$$

$$= \max\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{A}}(a)\}$$
$$= (\eta_{\mathcal{B}} \cup \eta_{\mathcal{B}})(a).$$

Similarly, we obtain that $(\nu_{\mathcal{B}} \circ \nu_{\mathcal{A}} \circ \nu_{\mathcal{B}})(a) \leq (\nu_{\mathcal{B}} \cup \nu_{\mathcal{B}})(a)$. This shows that $\mathcal{B} \cap \mathcal{A} \subseteq \mathcal{B} \circ \mathcal{A} \circ \mathcal{B}$. Therefore, $\mathcal{B} \cap \mathcal{A} = \mathcal{B} \circ \mathcal{A} \circ \mathcal{B}$.

$(ii) \Rightarrow (i)$ Let $\mathcal{B}$ be a picture fuzzy bi-ideal of S. Since $\mathcal{S}$ itself is a picture fuzzy ideal of S and by hypothesis, we get that $\mathcal{B} = \mathcal{B} \cap \mathcal{S} = \mathcal{B} \circ \mathcal{S} \circ \mathcal{B}$. By Theorem 3.2, S is regular. □

The following result can be achieved by Theorem 3.1 and Theorem 3.4.

Theorem 3.5. *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) *S is regular;*

(ii) *$\mathcal{G} \cap \mathcal{A} = \mathcal{G} \circ \mathcal{A} \circ \mathcal{G}$, for every picture fuzzy ideal $\mathcal{A}$ and every picture fuzzy generalized bi-ideal $\mathcal{G}$ of S.*

In the following theorem, we give some characterization of regular semigroups in terms of picture fuzzy left ideals, picture fuzzy right ideal and picture fuzzy (generalized) bi-ideals of semigroups.

Theorem 3.6. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular;*

(ii) *$\mathcal{G} \cap \mathcal{L} \subseteq \mathcal{G} \circ \mathcal{L}$, for every picture fuzzy generalized bi-ideal $\mathcal{G}$ and every picture fuzzy left ideal $\mathcal{L}$ of S;*

(iii) *$\mathcal{B} \cap \mathcal{L} \subseteq \mathcal{B} \circ \mathcal{L}$, for every picture fuzzy bi-ideal $\mathcal{B}$ and every picture fuzzy left ideal $\mathcal{L}$ of S;*

(iv) *$\mathcal{R} \cap \mathcal{G} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{G} \circ \mathcal{L}$, for every picture fuzzy generalized bi-ideal $\mathcal{G}$, every picture fuzzy left ideal $\mathcal{L}$ and every picture fuzzy right ideal $\mathcal{R}$ of S;*

(v) *$\mathcal{R} \cap \mathcal{B} \cap \mathcal{L} \subseteq \mathcal{R} \circ \mathcal{B} \circ \mathcal{L}$, for every picture fuzzy bi-ideal $\mathcal{B}$, every picture fuzzy left ideal $\mathcal{L}$ and every picture fuzzy right ideal $\mathcal{R}$ of S.*

Proof: $(i) \Rightarrow (ii)$ Let $\mathcal{G}$ and $\mathcal{L}$ be a picture fuzzy generalized bi-ideal and a picture fuzzy left ideal of S, respectively, and let $a \in S$. Since S is regular, there exists $x \in S$ such that $a = axa$. Then,

$$(\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(a) = \sup_{a=yz}[\min\{\mu_{\mathcal{G}}(y), \mu_{\mathcal{L}}(z)\}]$$
$$\geq \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(xa)\}$$
$$\geq \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(a)\}$$
$$= (\mu_{\mathcal{G}} \cap \mu_{\mathcal{L}})(a)$$

and

$$(\eta_{\mathcal{G}} \circ \eta_{\mathcal{L}})(a) = \inf_{a=yz}[\max\{\eta_{\mathcal{G}}(y), \eta_{\mathcal{L}}(z)\}]$$
$$\leq \max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{L}}(xa)\}$$
$$\leq \max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{L}}(a)\}$$
$$= (\eta_{\mathcal{G}} \cup \eta_{\mathcal{L}})(a).$$

From a similar proof of the case above, we have that $(\nu_{\mathcal{G}} \circ \nu_{\mathcal{L}})(a) \leq (\nu_{\mathcal{G}} \cup \nu_{\mathcal{L}})(a)$. This shows that $\mathcal{G} \cap \mathcal{L} \subseteq \mathcal{G} \circ \mathcal{L}$.

$(ii) \Rightarrow (iii)$ Since every picture fuzzy bi-ideal is a picture fuzzy generalized bi-ideal of S, we obtain that (iii) holds.

$(iii) \Rightarrow (i)$ Let $\mathcal{L}$ and $\mathcal{R}$ be a picture fuzzy left ideal and a picture fuzzy right ideal of S, respectively. Since every picture fuzzy right ideal of S is a picture fuzzy bi-ideal, it follows that $\mathcal{R}$ is also a picture fuzzy bi-ideal of S. By hypothesis, we have $\mathcal{R}\cap\mathcal{L} \subseteq \mathcal{R}\circ\mathcal{L}$. On the other hand, $\mathcal{R}\circ\mathcal{L} \subseteq \mathcal{R}\cap\mathcal{L}$. Hence, $\mathcal{R}\cap\mathcal{L} = \mathcal{R}\circ\mathcal{L}$. By Lemma 3.1, S is regular.

$(i) \Rightarrow (iv)$ Let $\mathcal{G}$, $\mathcal{L}$ and $\mathcal{R}$ be a picture fuzzy generalized bi-ideal, a picture fuzzy left ideal and a picture fuzzy right ideal of S, respectively. Let $a \in S$. Then, there exists $x \in S$ such that $a = axa$. Thus,

$$\begin{aligned}(\mu_{\mathcal{R}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(a) &= \sup_{a=yz}[\min\{\mu_{\mathcal{R}}(y), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(z)\}] \\ &\geq \min\{\mu_{\mathcal{R}}(ax), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{L}})(a)\} \\ &\geq \min\left\{\mu_{\mathcal{R}}(a), \sup_{a=pq}[\min\{\mu_{\mathcal{G}}(p), \mu_{\mathcal{L}}(q)\}]\right\} \\ &\geq \min\{\mu_{\mathcal{R}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(xa)\}\} \\ &\geq \min\{\mu_{\mathcal{R}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(a)\}\} \\ &= \min\{\mu_{\mathcal{R}}(a), \mu_{\mathcal{G}}(a), \mu_{\mathcal{L}}(a)\} \\ &= (\mu_{\mathcal{R}} \cap \mu_{\mathcal{G}} \cap \mu_{\mathcal{L}})(a)\end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{R}} \circ \eta_{\mathcal{G}} \circ \eta_{\mathcal{L}})(a) &= \inf_{a=yz}[\max\{\eta_{\mathcal{R}}(y), (\eta_{\mathcal{G}} \circ \eta_{\mathcal{L}})(z)\}] \\ &\leq \max\{\eta_{\mathcal{R}}(ax), (\eta_{\mathcal{G}} \circ \eta_{\mathcal{L}})(a)\} \\ &\leq \max\left\{\eta_{\mathcal{R}}(a), \inf_{a=pq}[\max\{\eta_{\mathcal{G}}(p), \eta_{\mathcal{L}}(q)\}]\right\} \\ &\leq \max\{\eta_{\mathcal{R}}(a), \max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{L}}(xa)\}\} \\ &\leq \max\{\eta_{\mathcal{R}}(a), \max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{L}}(a)\}\} \\ &= \max\{\eta_{\mathcal{R}}(a), \eta_{\mathcal{G}}(a), \eta_{\mathcal{L}}(a)\} \\ &= (\eta_{\mathcal{R}} \cup \eta_{\mathcal{G}} \cup \eta_{\mathcal{L}})(a).\end{aligned}$$

By a similar proof of the case above, we can show that $(\nu_{\mathcal{R}}\circ\nu_{\mathcal{G}}\circ\nu_{\mathcal{L}})(a) \leq (\nu_{\mathcal{R}}\cup\nu_{\mathcal{G}}\cup\nu_{\mathcal{L}})(a)$. Therefore, $\mathcal{R}\cap\mathcal{G}\cap\mathcal{L} \subseteq \mathcal{R}\circ\mathcal{G}\circ\mathcal{L}$.

$(iv) \Rightarrow (v)$ Since every picture fuzzy bi-ideal is a picture fuzzy generalized bi-ideal of S, it follows that (v) holds.

$(v) \Rightarrow (i)$ Let $\mathcal{L}$ and $\mathcal{R}$ be a picture fuzzy left ideal and a picture fuzzy right ideal of S, respectively. Since $\mathcal{S}$ itself is a picture fuzzy bi-ideal of S and by hypothesis, we get that $\mathcal{R}\cap\mathcal{L} = \mathcal{R}\cap\mathcal{S}\cap\mathcal{L} \subseteq (\mathcal{R}\circ\mathcal{S})\circ\mathcal{L} \subseteq \mathcal{R}\circ\mathcal{L}$. Otherwise, $\mathcal{R}\circ\mathcal{L} \subseteq \mathcal{R}\cap\mathcal{L}$. Hence, $\mathcal{R}\cap\mathcal{L} = \mathcal{R}\circ\mathcal{L}$. Therefore, S is regular by Lemma 3.1. □

4. **Characterizations of Regular and Intra-Regular Semigroups.** In this section, we study some characterizations of a semigroup, which is both regular and intra-regular, using the concepts of picture fuzzy left (right) ideals, picture fuzzy quasi-ideals and picture fuzzy bi-ideals of semigroups.

A semigroup S is called *intra-regular* [19] if for every element a of S, there exist $x, y \in S$ such that $a = xa^2y$.

Lemma 4.1. [37] *For a semigroup S, the following conditions are equivalent:*

(i) S in intra-regular;

(ii) $\mathcal{L}\cap\mathcal{R} \subseteq \mathcal{L}\circ\mathcal{R}$, for every picture fuzzy left ideal $\mathcal{L}$ and every picture fuzzy right ideal $\mathcal{R}$ of S.

Now, we give some characterizations of intra-regular semigroups by their picture fuzzy left (right) ideals and picture fuzzy (generalized) bi-ideals.

Theorem 4.1. *Let S be a semigroup. Then, the following conditions are equivalent:*

(*i*) *S is intra-regular;*

(*ii*) *$\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{S}$, for every picture fuzzy generalized bi-ideal $\mathcal{G}$ and every picture fuzzy left ideal $\mathcal{L}$ of S;*

(*iii*) *$\mathcal{L} \cap \mathcal{B} \subseteq \mathcal{L} \circ \mathcal{B} \circ \mathcal{S}$, for every picture fuzzy bi-ideal $\mathcal{B}$ and every picture fuzzy left ideal $\mathcal{L}$ of S.*

Proof: (*i*) $\Rightarrow$ (*ii*) Let $\mathcal{G}$ and $\mathcal{L}$ be a picture fuzzy generalized bi-ideal and a picture fuzzy left ideal of S, respectively. Let $a \in S$. Since S is intra-regular, there exist $x, y \in S$ such that $a = xa^2y$. Also,

$$a = xa^2y = x\left(xa^2y\right)ay = \left(x^2a\right)(ayay).$$

Then,

$$\begin{aligned}
(\mu_{\mathcal{L}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{L}}(p), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(q)\}] \\
&\geq \min\left\{\mu_{\mathcal{L}}\left(x^2a\right), (\mu_{\mathcal{G}} \circ \mu_{\mathcal{S}})(ayay)\right\} \\
&= \min\left\{\mu_{\mathcal{L}}\left(x^2a\right), \sup_{ayay=mn}[\min\{\mu_{\mathcal{G}}(m), \mu_{\mathcal{S}}(n)\}]\right\} \\
&\geq \min\{\mu_{\mathcal{L}}(a), \min\{\mu_{\mathcal{G}}(aya), \mu_{\mathcal{S}}(y)\}\} \\
&= \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(aya)\} \\
&\geq \min\{\mu_{\mathcal{L}}(a), \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}\} \\
&= \min\{\mu_{\mathcal{L}}(a), \mu_{\mathcal{G}}(a)\} \\
&= (\mu_{\mathcal{L}} \cap \mu_{\mathcal{G}})(a)
\end{aligned}$$

and

$$\begin{aligned}
(\eta_{\mathcal{L}} \circ \eta_{\mathcal{G}} \circ \eta_{\mathcal{S}})(a) &= \inf_{a=pq}[\max\{\eta_{\mathcal{L}}(p), (\eta_{\mathcal{G}} \circ \eta_{\mathcal{S}})(q)\}] \\
&\leq \max\left\{\eta_{\mathcal{L}}\left(x^2a\right), (\eta_{\mathcal{G}} \circ \eta_{\mathcal{S}})(ayay)\right\} \\
&= \max\left\{\eta_{\mathcal{L}}\left(x^2a\right), \inf_{ayay=mn}[\max\{\eta_{\mathcal{G}}(m), \eta_{\mathcal{S}}(n)\}]\right\} \\
&\leq \max\{\eta_{\mathcal{L}}(a), \max\{\eta_{\mathcal{G}}(aya), \eta_{\mathcal{S}}(y)\}\} \\
&= \max\{\eta_{\mathcal{L}}(a), \eta_{\mathcal{G}}(aya)\} \\
&\leq \max\{\eta_{\mathcal{L}}(a), \max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{G}}(a)\}\} \\
&= \max\{\eta_{\mathcal{L}}(a), \eta_{\mathcal{G}}(a)\} \\
&= (\eta_{\mathcal{L}} \cup \eta_{\mathcal{G}})(a).
\end{aligned}$$

Similarly, we can show that $(\nu_{\mathcal{L}} \circ \nu_{\mathcal{G}} \circ \nu_{\mathcal{S}})(a) \leq (\nu_{\mathcal{L}} \cup \nu_{\mathcal{G}})(a)$. This shows that $\mathcal{L} \cap \mathcal{G} \subseteq \mathcal{L} \circ \mathcal{G} \circ \mathcal{S}$.

(*ii*) $\Rightarrow$ (*iii*) Since every picture fuzzy bi-ideal of S is a picture fuzzy generalized bi-ideal, it implies that (*iii*) holds.

(*iii*) $\Rightarrow$ (*i*) Let $\mathcal{L}$ and $\mathcal{R}$ be a picture fuzzy left ideal and a picture fuzzy right ideal of S, respectively. Since every picture fuzzy right ideal of S is a picture fuzzy bi-ideal, we get that $\mathcal{R}$ is also a picture fuzzy bi-ideal of S. By assumption, $\mathcal{L} \cap \mathcal{R} \subseteq \mathcal{L} \circ (\mathcal{R} \circ \mathcal{S}) \subseteq \mathcal{L} \circ \mathcal{R}$. By Lemma 4.1, S is intra-regular. □

Theorem 4.2. *Let S be a semigroup. Then, the following conditions are equivalent:*

(*i*) *S is intra-regular;*

(*ii*) *$\mathcal{G} \cap \mathcal{R} \subseteq \mathcal{S} \circ \mathcal{G} \circ \mathcal{R}$, for every picture fuzzy generalized bi-ideal $\mathcal{G}$ and every picture fuzzy right ideal $\mathcal{R}$ of S;*

(*iii*) $\mathcal{B} \cap \mathcal{R} \subseteq \mathcal{S} \circ \mathcal{B} \circ \mathcal{R}$, *for every picture fuzzy bi-ideal* $\mathcal{B}$ *and every picture fuzzy right ideal* $\mathcal{R}$ *of* S.

Proof: $(i) \Rightarrow (ii)$ Let $\mathcal{G}$ and $\mathcal{R}$ be a picture fuzzy generalized bi-ideal and a picture fuzzy right ideal of S, respectively. Let $a \in S$. Since S is intra-regular, there exist $x, y \in S$ such that $a = xa^2y$. Thus, we have

$$a = xa^2y = xa\left(xa^2y\right)y = (xaxa)\left(ay^2\right).$$

Then,

$$\begin{aligned}(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{R}})(a) &= \sup_{a=pq}[\min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}})(p), \mu_{\mathcal{R}}(q)\}] \\ &\geq \min\left\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}})(xaxa), \mu_{\mathcal{R}}\left(ay^2\right)\right\} \\ &= \min\left\{\sup_{xaxa=kl}[\min\{\mu_{\mathcal{S}}(k), \mu_{\mathcal{G}}(l)\}], \mu_{\mathcal{R}}\left(ay^2\right)\right\} \\ &\geq \min\{\min\{\mu_{\mathcal{S}}(x), \mu_{\mathcal{G}}(axa)\}, \mu_{\mathcal{R}}(a)\} \\ &= \min\{\mu_{\mathcal{G}}(axa), \mu_{\mathcal{R}}(a)\} \\ &\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \mu_{\mathcal{R}}(a)\} \\ &= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{R}}(a)\} \\ &= (\mu_{\mathcal{G}} \cap \mu_{\mathcal{R}})(a)\end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{S}} \circ \eta_{\mathcal{G}} \circ \eta_{\mathcal{R}})(a) &= \inf_{a=pq}[\max\{(\eta_{\mathcal{S}} \circ \eta_{\mathcal{G}})(p), \eta_{\mathcal{R}}(q)\}] \\ &\leq \max\left\{(\eta_{\mathcal{S}} \circ \eta_{\mathcal{G}})(xaxa), \eta_{\mathcal{R}}\left(ay^2\right)\right\} \\ &= \max\left\{\inf_{xaxa=kl}[\max\{\eta_{\mathcal{S}}(k), \eta_{\mathcal{G}}(l)\}], \eta_{\mathcal{R}}\left(ay^2\right)\right\} \\ &\leq \max\{\max\{\eta_{\mathcal{S}}(x), \eta_{\mathcal{G}}(axa)\}, \eta_{\mathcal{R}}(a)\} \\ &= \max\{\eta_{\mathcal{G}}(axa), \eta_{\mathcal{R}}(a)\} \\ &\leq \max\{\max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{G}}(a)\}, \eta_{\mathcal{R}}(a)\} \\ &= \max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{R}}(a)\} \\ &= (\eta_{\mathcal{G}} \cup \eta_{\mathcal{R}})(a).\end{aligned}$$

Also, we have that $(\nu_{\mathcal{S}} \circ \nu_{\mathcal{G}} \circ \nu_{\mathcal{R}})(a) \leq (\nu_{\mathcal{G}} \cup \nu_{\mathcal{R}})(a)$. Hence, $\mathcal{G} \cap \mathcal{R} \subseteq \mathcal{S} \circ \mathcal{G} \circ \mathcal{R}$.

$(ii) \Rightarrow (iii)$ Since every picture fuzzy bi-ideal of S is a picture fuzzy generalized bi-ideal, it implies that (iii) holds.

$(iii) \Rightarrow (i)$ Let $\mathcal{L}$ and $\mathcal{R}$ be a picture fuzzy left ideal and a picture fuzzy right ideal of S, respectively. Since every picture fuzzy left ideal of S is a picture fuzzy bi-ideal, then $\mathcal{L}$ is also a picture fuzzy bi-ideal of S. By assumption, $\mathcal{L} \cap \mathcal{R} \subseteq (\mathcal{S} \circ \mathcal{L}) \circ \mathcal{R} \subseteq \mathcal{L} \circ \mathcal{R}$. Therefore, S is intra-regular by Lemma 4.1. □

In the following, we introduce the concept of picture fuzzy quasi-ideals of semigroups and present some example of this concept.

Definition 4.1. *A picture fuzzy set* $\mathcal{Q}$ *on a semigroup* S *is called a picture fuzzy quasi-ideal of* S *if* $(\mathcal{Q} \circ \mathcal{S}) \cap (\mathcal{S} \circ \mathcal{Q}) \subseteq \mathcal{Q}$.

By Lemma 2.1, Lemma 2.3 and Lemma 2.4, we note that every picture fuzzy left (right) ideal of a semigroup S is a picture fuzzy quasi-ideal of S. In general, the converse is not true as shown by the following example.

Example 4.1. *Let $S = \{a, b, c, d\}$. Define the binary operation $\cdot$ on S by the following table:*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	c	c	c
d	a	b	b	b

Then, $(S, \cdot)$ is a semigroup [29]*. Define a picture fuzzy set $\mathcal{A}$ on S by*

$$\begin{array}{llll} \mu_{\mathcal{A}}(a) = 0.8, & \mu_{\mathcal{A}}(b) = 0.3, & \mu_{\mathcal{A}}(c) = 0.1, & \mu_{\mathcal{A}}(d) = 0.1; \\ \eta_{\mathcal{A}}(a) = 0.1, & \eta_{\mathcal{A}}(b) = 0.2, & \eta_{\mathcal{A}}(c) = 0.4, & \eta_{\mathcal{A}}(d) = 0.4; \\ \nu_{\mathcal{A}}(a) = 0.1, & \nu_{\mathcal{A}}(b) = 0.4, & \nu_{\mathcal{A}}(c) = 0.5, & \nu_{\mathcal{A}}(d) = 0.5. \end{array}$$

We can see that $\mathcal{A}$ is a picture fuzzy quasi-ideal of S, but it is not a picture fuzzy left ideal, because

$$\mu_{\mathcal{A}}(cb) = \mu_{\mathcal{A}}(c) = 0.1 < 0.3 = \mu_{\mathcal{A}}(b),$$
$$\eta_{\mathcal{A}}(cb) = \eta_{\mathcal{A}}(c) = 0.4 > 0.2 = \eta_{\mathcal{A}}(b),$$
$$\nu_{\mathcal{A}}(cb) = \nu_{\mathcal{A}}(c) = 0.5 > 0.4 = \nu_{\mathcal{A}}(b).$$

The following lemma can be proved similar to Lemma 2.7.

Lemma 4.2. *Let Q be a nonempty subset of a semigroup S. Then, Q is a quasi-ideal of S if and only if $\mathcal{C}^Q$ is a picture fuzzy quasi-ideal of S.*

Next, we obtain some interesting characterizations of intra-regular semigroups in terms of picture fuzzy (generalized) bi-ideals and picture fuzzy quasi-ideals as follows.

Theorem 4.3. *Let S be a semigroup. Then, the following statements are equivalent:*
- (i) *S is intra-regular;*
- (ii) *$\mathcal{G} \cap \mathcal{Q} \subseteq \mathcal{S} \circ \mathcal{G} \circ \mathcal{Q} \circ \mathcal{S}$, for any picture fuzzy generalized bi-ideal $\mathcal{G}$ and any picture fuzzy quasi-ideal $\mathcal{Q}$ of S;*
- (iii) *$\mathcal{B} \cap \mathcal{Q} \subseteq \mathcal{S} \circ \mathcal{B} \circ \mathcal{Q} \circ \mathcal{S}$, for any picture fuzzy bi-ideal $\mathcal{B}$ and any picture fuzzy quasi-ideal $\mathcal{Q}$ of S.*

Proof: $(i) \Rightarrow (ii)$ Let $\mathcal{G}$ and $\mathcal{Q}$ be a picture fuzzy generalized bi-ideal and a picture fuzzy quasi-ideal of S, respectively. Let $a \in S$. Since S is intra-regular, there exist $x, y \in S$ such that $a = xa^2y$. Thus,

$$a = xa^2y = xa\left(xa^2y\right)y = (xaxa)\left(ay^2\right).$$

It follows that

$$\begin{aligned} (\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}} \circ \mu_{\mathcal{Q}} \circ \mu_{\mathcal{S}}) &= \sup_{a=pq}[\min\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}})(p), (\mu_{\mathcal{Q}} \circ \mu_{\mathcal{S}})(q)\}] \\ &\geq \min\left\{(\mu_{\mathcal{S}} \circ \mu_{\mathcal{G}})(xaxa), (\mu_{\mathcal{Q}} \circ \mu_{\mathcal{S}})\left(ay^2\right)\right\} \\ &= \min\left\{\sup_{xaxa=mn}[\min\{\mu_{\mathcal{S}}(m), \mu_{\mathcal{G}}(n)\}], \sup_{ay^2=kl}[\min\{\mu_{\mathcal{Q}}(k), \mu_{\mathcal{S}}(l)\}]\right\} \\ &\geq \min\left\{\min\{\mu_{\mathcal{S}}(x), \mu_{\mathcal{G}}(axa)\}, \min\left\{\mu_{\mathcal{Q}}(a), \mu_{\mathcal{S}}\left(y^2\right)\right\}\right\} \\ &= \min\{\mu_{\mathcal{G}}(axa), \mu_{\mathcal{Q}}(a)\} \\ &\geq \min\{\min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{G}}(a)\}, \mu_{\mathcal{Q}}(a)\} \\ &= \min\{\mu_{\mathcal{G}}(a), \mu_{\mathcal{Q}}(a)\} \\ &= (\mu_{\mathcal{G}} \cap \mu_{\mathcal{Q}})(a) \end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{S}} \circ \eta_{\mathcal{G}} \circ \eta_{\mathcal{Q}} \circ \eta_{\mathcal{S}}) &= \inf_{a=pq}[\max\{(\eta_{\mathcal{S}} \circ \eta_{\mathcal{G}})(p), (\eta_{\mathcal{Q}} \circ \eta_{\mathcal{S}})(q)\}] \\ &\leq \max\left\{(\eta_{\mathcal{S}} \circ \eta_{\mathcal{G}})(xaxa), (\eta_{\mathcal{Q}} \circ \eta_{\mathcal{S}})\left(ay^2\right)\right\} \\ &= \max\left\{\inf_{xaxa=mn}[\max\{\eta_{\mathcal{S}}(m), \eta_{\mathcal{G}}(n)\}], \inf_{ay^2=kl}[\max\{\eta_{\mathcal{Q}}(k), \eta_{\mathcal{S}}(l)\}]\right\} \\ &\leq \max\left\{\max\{\eta_{\mathcal{S}}(x), \eta_{\mathcal{G}}(axa)\}, \max\left\{\eta_{\mathcal{Q}}(a), \eta_{\mathcal{S}}\left(y^2\right)\right\}\right\} \\ &= \max\{\eta_{\mathcal{G}}(axa), \eta_{\mathcal{Q}}(a)\} \\ &\leq \max\{\max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{G}}(a)\}, \eta_{\mathcal{Q}}(a)\} \\ &= \max\{\eta_{\mathcal{G}}(a), \eta_{\mathcal{Q}}(a)\} \\ &= (\eta_{\mathcal{G}} \cup \eta_{\mathcal{Q}})(a).\end{aligned}$$

Similar to proof of the case above, we have that $(\nu_{\mathcal{S}} \circ \nu_{\mathcal{G}} \circ \nu_{\mathcal{Q}} \circ \nu_{\mathcal{S}}) \leq (\nu_{\mathcal{G}} \cup \nu_{\mathcal{Q}})(a)$. Therefore, $\mathcal{G} \cap \mathcal{Q} \subseteq \mathcal{S} \circ \mathcal{G} \circ \mathcal{Q} \circ \mathcal{S}$.

$(ii) \Rightarrow (iii)$ Since every picture fuzzy bi-ideal of S is a picture fuzzy generalized bi-ideal, it implies that (iii) holds.

$(iii) \Rightarrow (i)$ Let $\mathcal{L}$ and $\mathcal{R}$ be a picture fuzzy left ideal and a picture fuzzy right ideals of S, respectively. Then, $\mathcal{L}$ is also a picture fuzzy bi-ideal and $\mathcal{R}$ is also a picture fuzzy quasi-ideal of S. By assumption, we have $\mathcal{L} \cap \mathcal{R} \subseteq (\mathcal{S} \circ \mathcal{L}) \circ (\mathcal{R} \circ \mathcal{S}) \subseteq \mathcal{L} \circ \mathcal{R}$. By Lemma 4.1, S is intra-regular. □

The following theorem can be proved similar to Theorem 4.3.

Theorem 4.4. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is intra-regular;*
(ii) *$\mathcal{G} \cap \mathcal{Q} \subseteq \mathcal{S} \circ \mathcal{Q} \circ \mathcal{G} \circ \mathcal{S}$, for any picture fuzzy generalized bi-ideal $\mathcal{G}$ and any picture fuzzy quasi-ideal $\mathcal{Q}$ of S;*
(iii) *$\mathcal{B} \cap \mathcal{Q} \subseteq \mathcal{S} \circ \mathcal{Q} \circ \mathcal{B} \circ \mathcal{S}$, for any picture fuzzy bi-ideal $\mathcal{B}$ and any picture fuzzy quasi-ideal $\mathcal{Q}$ of S.*

Lemma 4.3. [36] *Let S be a semigroup. Then S is regular and intra-regular if and only if $B = B^2$, for every bi-ideal B of S.*

Lemma 4.4. [37] *For a semigroup S, the following conditions are equivalent:*

(i) *S is regular and intra-regular;*
(ii) *$\mathcal{R} \cap \mathcal{L} \subseteq (\mathcal{R} \circ \mathcal{L}) \cap (\mathcal{L} \circ \mathcal{R})$, for every picture fuzzy left ideal $\mathcal{L}$ and every picture fuzzy right ideal $\mathcal{R}$ of S.*

Next, we give some new characterization of both regular and intra-regular semigroups using their picture fuzzy left (right) ideals and picture fuzzy bi-ideals as follows.

Theorem 4.5. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular and intra-regular;*
(ii) *$\mathcal{B} = \mathcal{B} \circ \mathcal{B}$, for every picture fuzzy bi-ideal $\mathcal{B}$ of S;*
(iii) *$\mathcal{A} \cap \mathcal{B} \subseteq (\mathcal{A} \circ \mathcal{B}) \cap (\mathcal{B} \circ \mathcal{A})$, for any picture fuzzy bi-ideals $\mathcal{A}$ and $\mathcal{B}$ of S;*
(iv) *$\mathcal{B} \cap \mathcal{L} \subseteq (\mathcal{B} \circ \mathcal{L}) \cap (\mathcal{L} \circ \mathcal{B})$, for each picture fuzzy bi-ideal $\mathcal{B}$ and each picture fuzzy left ideal $\mathcal{L}$ of S;*
(v) *$\mathcal{R} \cap \mathcal{B} \subseteq (\mathcal{R} \circ \mathcal{B}) \cap (\mathcal{B} \circ \mathcal{R})$, for each picture fuzzy bi-ideal $\mathcal{B}$ and each picture fuzzy right ideal $\mathcal{R}$ of S.*

Proof: $(i) \Rightarrow (iii)$ Let $\mathcal{A}$ and $\mathcal{B}$ be a picture fuzzy bi-ideals of S and $a \in S$. Since S is regular and intra-regular, there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. Thus,

$$a = axa = axaxa = ax\left(ya^2z\right)xa = (axya)(azxa).$$

It follows that

$$\begin{aligned}(\mu_{\mathcal{A}} \circ \mu_{\mathcal{B}})(a) &= \sup_{a=pq}[\min\{\mu_{\mathcal{A}}(p), \mu_{\mathcal{B}}(q)\}] \\ &\geq \min\{\mu_{\mathcal{A}}(axya), \mu_{\mathcal{B}}(azxa)\} \\ &\geq \min\{\min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{A}}(a)\}, \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{B}}(a)\}\} \\ &= \min\{\mu_{\mathcal{A}}(a), \mu_{\mathcal{B}}(a)\} \\ &= (\mu_{\mathcal{A}} \cap \mu_{\mathcal{B}})(a)\end{aligned}$$

and

$$\begin{aligned}(\eta_{\mathcal{A}} \circ \eta_{\mathcal{B}})(a) &= \inf_{a=pq}[\max\{\eta_{\mathcal{A}}(p), \eta_{\mathcal{B}}(q)\}] \\ &\leq \max\{\eta_{\mathcal{A}}(axya), \eta_{\mathcal{B}}(azxa)\} \\ &\leq \max\{\max\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{A}}(a)\}, \max\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{B}}(a)\}\} \\ &= \max\{\eta_{\mathcal{A}}(a), \eta_{\mathcal{B}}(a)\} \\ &= (\eta_{\mathcal{A}} \cup \eta_{\mathcal{B}})(a).\end{aligned}$$

Similarly, we have that $(\nu_{\mathcal{A}} \circ \nu_{\mathcal{B}})(a) \leq (\nu_{\mathcal{A}} \cup \nu_{\mathcal{B}})(a)$. This means that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{A} \circ \mathcal{B}$. In the same way, we can show that $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{B} \circ \mathcal{A}$. Hence, $\mathcal{A} \cap \mathcal{B} \subseteq (\mathcal{A} \circ \mathcal{B}) \cap (\mathcal{B} \circ \mathcal{A})$.

Since every picture fuzzy left (right) ideal of S is a picture fuzzy bi-ideal and by Lemma 4.4, we have that $(iii) \Rightarrow (ii)$, $(iii) \Rightarrow (iv)$, $(iii) \Rightarrow (v)$, $(iv) \Rightarrow (i)$ and $(v) \Rightarrow (i)$ hold.

$(ii) \Rightarrow (i)$ Let B be a bi-ideal of S and $a \in B$. By Corollary 2.2, $\mathcal{C}^B$ is a picture fuzzy bi-ideal of S. By assumption, $(\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^B})(a) = \mu_{\mathcal{C}^B}(a) = 1$. Then,

$$\sup_{a=yz}[\min\{\mu_{\mathcal{C}^B}(y), \mu_{\mathcal{C}^B}(z)\}] = (\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^B})(a) = 1.$$

Thus, there exist $b, c \in S$ such that $a = bc$, $\mu_{\mathcal{C}^B}(b) = 1$ and $\mu_{\mathcal{C}^B}(c) = 1$. Also, $a = bc \in BB$. So, $B \subseteq BB$. On the other hand, $BB \subseteq B$. Therefore, $B = BB$. By Lemma 4.3, S is regular and intra-regular. □

Lemma 4.5. [15] *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) *S is regular and intra-regular;*
(ii) *$B \cap Q \subseteq BQB$, for every bi-ideal B and every quasi-ideal Q of S;*
(iii) *$B \cap L \subseteq BLB$, for every bi-ideal B and every left ideal L of S;*
(iv) *$B \cap R \subseteq BRB$, for every bi-ideal B and every right ideal R of S;*
(v) *$B \cap Q \subseteq QBQ$, for every bi-ideal B and every quasi-ideal Q of S;*
(vi) *$L \cap Q \subseteq QLQ$, for every left ideal L and every quasi-ideal Q of S;*
(vii) *$R \cap Q \subseteq QRQ$, for every right ideal R and every quasi-ideal Q of S.*

Finally, we give some characterization of a semigroup, which is both regular and intra-regular, by their picture fuzzy left (right) ideals, picture fuzzy quasi-ideals and picture fuzzy bi-ideals as follows.

Theorem 4.6. *Let S be a semigroup. Then, the following statements are equivalent:*

(i) *S is regular and intra-regular;*
(ii) *$\mathcal{B} \cap \mathcal{Q} \subseteq \mathcal{B} \circ \mathcal{Q} \circ \mathcal{B}$, for every picture fuzzy bi-ideal $\mathcal{B}$ and every picture fuzzy quasi-ideal $\mathcal{Q}$ of S;*

(*iii*) $\mathcal{B} \cap \mathcal{L} \subseteq \mathcal{B} \circ \mathcal{L} \circ \mathcal{B}$, *for each picture fuzzy bi-ideal* $\mathcal{B}$ *and each picture fuzzy left ideal* $\mathcal{L}$ *of* S*;*
(*iv*) $\mathcal{B} \cap \mathcal{R} \subseteq \mathcal{B} \circ \mathcal{R} \circ \mathcal{B}$, *for any picture fuzzy bi-ideal* $\mathcal{B}$ *and any picture fuzzy right ideal* $\mathcal{R}$ *of* S*;*
(*v*) $\mathcal{B} \cap \mathcal{Q} \subseteq \mathcal{Q} \circ \mathcal{B} \circ \mathcal{Q}$, *for every picture fuzzy bi-ideal* $\mathcal{B}$ *and every picture fuzzy quasi-ideal* $\mathcal{Q}$ *of* S*;*
(*vi*) $\mathcal{L} \cap \mathcal{Q} \subseteq \mathcal{Q} \circ \mathcal{L} \circ \mathcal{Q}$, *for each picture fuzzy left ideal* $\mathcal{L}$ *and each picture fuzzy quasi-ideal* $\mathcal{Q}$ *of* S*;*
(*vii*) $\mathcal{R} \cap \mathcal{Q} \subseteq \mathcal{Q} \circ \mathcal{R} \circ \mathcal{Q}$, *for any picture fuzzy right ideal* $\mathcal{R}$ *and any picture fuzzy quasi-ideal* $\mathcal{Q}$ *of* S.

Proof: $(i) \Rightarrow (ii)$ Let $\mathcal{B}$ and $\mathcal{Q}$ be a picture fuzzy bi-ideal and a picture fuzzy quasi-ideal of S, respectively. Let $a \in S$. Since S is regular and intra-regular, there exist $x, y, z \in S$ such that $a = axa$ and $a = ya^2z$. Then,

$$a = axa = (axa)x(axa) = ax\left(ya^2z\right)x\left(ya^2z\right)xa = (axya)(azxya)(azxa).$$

Consider

$$\begin{aligned}
\mu_{\mathcal{Q}}(azxya) &\geq [(\mu_{\mathcal{Q}} \circ \mu_{\mathcal{S}}) \cap (\mu_{\mathcal{S}} \circ \mu_{\mathcal{Q}})](azxya) \\
&= \min\{(\mu_{\mathcal{Q}} \circ \mu_{\mathcal{S}})(azxya), (\mu_{\mathcal{S}} \circ \mu_{\mathcal{Q}})(azxya)\} \\
&= \min\left\{\sup_{azxya=pq}[\min\{\mu_{\mathcal{Q}}(p), \mu_{\mathcal{S}}(q)\}], \sup_{azxya=uv}[\min\{\mu_{\mathcal{S}}(u), \mu_{\mathcal{Q}}(v)\}]\right\} \\
&\geq \min\{\min\{\mu_{\mathcal{Q}}(a), \mu_{\mathcal{S}}(zxya)\}, \min\{\mu_{\mathcal{S}}(azxy), \mu_{\mathcal{Q}}(a)\}\} \\
&= \min\{\min\{\mu_{\mathcal{Q}}(a), 1\}, \min\{1, \mu_{\mathcal{Q}}(a)\}\} \\
&= \min\{\mu_{\mathcal{Q}}(a), \mu_{\mathcal{Q}}(a)\} \\
&= \mu_{\mathcal{Q}}(a)
\end{aligned}$$

and

$$\begin{aligned}
\eta_{\mathcal{Q}}(azxya) &\leq [(\eta_{\mathcal{Q}} \circ \eta_{\mathcal{S}}) \cap (\eta_{\mathcal{S}} \circ \eta_{\mathcal{Q}})](azxya) \\
&= \max\{(\eta_{\mathcal{Q}} \circ \eta_{\mathcal{S}})(azxya), (\eta_{\mathcal{S}} \circ \eta_{\mathcal{Q}})(azxya)\} \\
&= \max\left\{\inf_{azxya=pq}[\max\{\eta_{\mathcal{Q}}(p), \eta_{\mathcal{S}}(q)\}], \inf_{azxya=uv}[\max\{\eta_{\mathcal{S}}(u), \eta_{\mathcal{Q}}(v)\}]\right\} \\
&\leq \max\{\max\{\eta_{\mathcal{Q}}(a), \eta_{\mathcal{S}}(zxya)\}, \max\{\eta_{\mathcal{S}}(azxy), \eta_{\mathcal{Q}}(a)\}\} \\
&= \max\{\max\{\eta_{\mathcal{Q}}(a), 0\}, \max\{0, \eta_{\mathcal{Q}}(a)\}\} \\
&= \max\{\eta_{\mathcal{Q}}(a), \eta_{\mathcal{Q}}(a)\} \\
&= \eta_{\mathcal{Q}}(a).
\end{aligned}$$

Proving it in the same way as in the previous case, we have that $\nu_{\mathcal{Q}}(azxya) \leq \nu_{\mathcal{Q}}(a)$. Now, we obtain that

$$\begin{aligned}
(\mu_{\mathcal{B}} \circ \mu_{\mathcal{Q}} \circ \mu_{\mathcal{B}})(a) &= \sup_{a=mn}[\min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{Q}})(m), \mu_{\mathcal{B}}(n)\}] \\
&\geq \min\{(\mu_{\mathcal{B}} \circ \mu_{\mathcal{Q}})(axyaazxya), \mu_{\mathcal{B}}(azxa)\} \\
&= \min\left\{\sup_{axyaazxya=st}[\min\{\mu_{\mathcal{B}}(s), \mu_{\mathcal{Q}}(t)\}], \mu_{\mathcal{B}}(azxa)\right\} \\
&\geq \min\{\min\{\mu_{\mathcal{B}}(axya), \mu_{\mathcal{Q}}(azxya)\}, \mu_{\mathcal{B}}(azxa)\} \\
&\geq \min\{\min\{\min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{B}}(a)\}, \mu_{\mathcal{Q}}(a)\}, \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{B}}(a)\}\} \\
&= \min\{\mu_{\mathcal{B}}(a), \mu_{\mathcal{Q}}(a)\}
\end{aligned}$$

$$= (\mu_{\mathcal{B}} \cap \mu_{\mathcal{Q}})(a)$$

and

$$\begin{aligned}(\eta_{\mathcal{B}} \circ \eta_{\mathcal{Q}} \circ \eta_{\mathcal{B}})(a) &= \inf_{a=mn}[\max\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{Q}})(m), \eta_{\mathcal{B}}(n)\}] \\ &\leq \max\{(\eta_{\mathcal{B}} \circ \eta_{\mathcal{Q}})(axyaazxya), \eta_{\mathcal{B}}(azxa)\} \\ &= \max\left\{\inf_{axyaazxya=st}[\max\{\eta_{\mathcal{B}}(s), \eta_{\mathcal{Q}}(t)\}], \eta_{\mathcal{B}}(azxa)\right\} \\ &\leq \max\{\max\{\eta_{\mathcal{B}}(axya), \eta_{\mathcal{Q}}(azxya)\}, \eta_{\mathcal{B}}(azxa)\} \\ &\leq \max\{\max\{\max\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{B}}(a)\}, \eta_{\mathcal{Q}}(a)\}, \max\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{B}}(a)\}\} \\ &= \max\{\eta_{\mathcal{B}}(a), \eta_{\mathcal{Q}}(a)\} \\ &= (\eta_{\mathcal{B}} \cup \eta_{\mathcal{Q}})(a).\end{aligned}$$

In the same way, we can show that $(\nu_{\mathcal{B}} \circ \nu_{\mathcal{Q}} \circ \nu_{\mathcal{B}})(a) \leq (\nu_{\mathcal{B}} \cup \nu_{\mathcal{Q}})(a)$. This means that $\mathcal{B} \cap \mathcal{Q} \subseteq \mathcal{B} \circ \mathcal{Q} \circ \mathcal{B}$.

Since every picture fuzzy left (right) ideal of S is a picture fuzzy quasi-ideal, it implies that $(ii) \Rightarrow (iii)$ and $(ii) \Rightarrow (iv)$ are true.

$(iii) \Rightarrow (i)$ Let B and L be a bi-ideal and a left ideal of S, respectively. Let $a \in B \cap L$. By Corollary 2.2 and Lemma 2.6, we have that $\mathcal{C}^B$ and $\mathcal{C}^L$ are a picture fuzzy bi-ideal and a picture fuzzy left ideal of S, respectively. By assumption, $\mathcal{C}^B \cap \mathcal{C}^L \subseteq \mathcal{C}^B \circ \mathcal{C}^L \circ \mathcal{C}^B$. Thus,

$$(\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^L} \circ \mu_{\mathcal{C}^B})(a) \geq (\mu_{\mathcal{C}^B} \cap \mu_{\mathcal{C}^L})(a) = \min\{\mu_{\mathcal{C}^B}(a), \mu_{\mathcal{C}^L}(a)\} = 1.$$

Since $\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^L} \circ \mu_{\mathcal{C}^B}$ is a fuzzy set of S, we get $(\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^L} \circ \mu_{\mathcal{C}^B})(a) \leq 1$. It turns out that

$$\sup_{a=yz}[\min\{(\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^L})(y), \mu_{\mathcal{C}^B}(z)\}] = (\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^L} \circ \mu_{\mathcal{C}^B})(a) = 1$$

which implies that there exist $b, c \in S$ such that $a = bc$, $(\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^L})(b) = 1$ and $\mu_{\mathcal{C}^B}(c) = 1$. So,

$$\sup_{b=pq}[\min\{\mu_{\mathcal{C}^B}(p), \mu_{\mathcal{C}^L}(q)\}] = (\mu_{\mathcal{C}^B} \circ \mu_{\mathcal{C}^L})(b) = 1$$

that is, there exist $d, e \in S$ such that $b = de$, $\mu_{\mathcal{C}^B}(d) = 1$ and $\mu_{\mathcal{C}^L}(e) = 1$. It follows that $a = bc = (de)c \in BLB$. Hence, $B \cap L \subseteq BLB$. By Lemma 4.5, S is regular and intra-regular.

Using a similar proof of the previous case, we can show that $(iv) \Rightarrow (i)$, $(vi) \Rightarrow (i)$ and $(vii) \Rightarrow (i)$ hold.

$(i) \Rightarrow (v)$ The proof is similar to $(i) \Rightarrow (ii)$.

Since every picture fuzzy left (right) ideal of S is a picture fuzzy bi-ideal, it follows that $(v) \Rightarrow (vi)$ and $(v) \Rightarrow (vii)$ hold. This completes the proof. □

5. **Conclusions.** In this paper, we have introduced the concepts of picture fuzzy quasi-ideals and picture fuzzy (generalized) bi-ideals and also discussed by using the notion of picture fuzzy products. Moreover, we studied characterizations of regular semigroups and intra-regular semigroups by their picture fuzzy left (right) ideals, picture fuzzy quasi-ideals and picture fuzzy (generalized) bi-ideals. Finally, we characterized when a semigroup is a both regular and intra-regular semigroup based on picture fuzzy left (right) ideals, picture fuzzy quasi-ideals and picture fuzzy (generalized) bi-ideals. In our future study, we will investigate the concepts of picture fuzzy (m, n)-ideals, picture fuzzy $(m, 0)$-ideals and picture fuzzy $(0, n)$-ideals of semigroups, where m and n are positive integers, which are generalizations of picture fuzzy left (right) ideals and picture fuzzy (generalized) bi-ideals of semigroups.

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