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USING 2-2-STEP TYPE FUZZY NUMBERS TO APPROXIMATE GENERAL FUZZY NUMBERS BASED ON WEIGHTED METRIC

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Abstract. *In this paper, taking the weighted metric as the measure of approximation, we study the problem of approximating general fuzzy numbers by using 2-2-step type fuzzy numbers. Two kinds of approximation (I-nearest approximation and II-nearest approximation) of using 2-2-step type fuzzy numbers to approximate general fuzzy numbers are defined. Then a conclusion about the relationship between the I-nearest approximation and II-nearest approximation is obtained, and the methods of solving these two nearest approximations are established, which are the most important results in this paper. At last, an example is given to show the effectiveness and usability of methods set up by us.*
Keywords: Fuzzy number, Step type fuzzy number, Membership function, Approximation of fuzzy number

1. **Introduction.** Fuzzy number is a special kind of fuzzy set on real number field, which can be used to express uncertain or imprecise digital information [1, 2]. Approximating general fuzzy numbers by using the fuzzy numbers with simple structure is not only an important research content of fuzzy set theory, but also has a strong application background. There are many researchers engaged in the theory and application of fuzzy number approximation, and many related results have been received. For example, in 2002, Grzegorewski proposed a method of approximating general fuzzy numbers with interval numbers in [3]; in 2012, Jian et al. studied the method of approximating general fuzzy numbers with triangular fuzzy in [4], and Veeramani and Duraisamy introduced an approximate solution method to complete fully fuzzy linear programming problems in which all the parameters and variables are triangular fuzzy numbers based on the research of the nearest symmetric triangular fuzzy number approximation concept in [5]; in 2014, Grzegorzewski and Pasternak-Winiarska studied two fuzzy number approximation problems of trapezoidal fuzzy numbers with expectation interval and discussed the relationships between the two methods in [6], and Ban and Coroianu proved the uniqueness of the nearest trapezoidal fuzzy number with the property that there is at least one trapezoidal fuzzy number with fixed parameters for any given fuzzy number in [7]; in 2015, Dinagar and Jivagan compared the difference of some existing fuzzy number approximation methods in [8]; in 2016, Ban et al. further studied the problem of weighted L-R approximation of fuzzy numbers and disussed the metric properties of this approximation in [9]; in 2017, Huang et al. proposed a convolution method to construct a set of fuzzy number sequences which approximate fuzzy numbers with certain properties, and pointed out that the approximation is differentiable and Lipschitz if the proper smoother is selected in [10], and in the same year, Wang and Li proposed the method of approximating general

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fuzzy numbers by using simple fuzzy numbers (i.e., step-type fuzzy numbers) in [11]; in 2018, Yeganehmanesh and Amirfakhrian proposed the method of approximating general fuzzy numbers by using polynomial fuzzy numbers in [12]; in 2019, Coroianu et al. gave algorithms of approximating general fuzzy numbers by using piecewise linear approximation and further illustrations were given via the computer simulation study in [13], and Gonzalez et al. studied the problem of fuzzy smooth bicubic spline approximation on the three-dimensional fuzzy set in [14]; in 2020, Wang et al. gave the specific calculation formula of approximating general fuzzy numbers by using multi-knots piecewise linear fuzzy numbers in [15], and Hai et al. studied the problem of weighted pseudometric approximation of 2-dimensional fuzzy numbers by using fuzzy 2-cell prismoid numbers preserving the centroid in [16]; in 2021, Coroianu gave a quadratic programming model which can be used to the trapezoidal approximation of general fuzzy number based on the weighted L-2 distance [17], and Lakshmana et al. set up a method of approximating hexagonal fuzzy numbers with nonlinear fuzzy numbers based on nonlinear programming model in [18].

In the study of using simple fuzzy numbers to approximate general fuzzy numbers, so far, the existing methods for solving the nearest (or best) approximation established by predecessors are carried out when the level values corresponding to the nodes to be solved are known (for example, see recent papers [10-18]). In other words, these existing methods are only to optimize the abscissas (first coordinates, in the case of one-dimensional fuzzy numbers [10-15,17,18]) or abscissas and ordinates (first coordinates and second coordinates, in the case of two-dimensional fuzzy numbers [16]) of the nodes to be solved when the level values corresponding to these nodes, i.e., the ordinates (second coordinates, in the case of one-dimensional fuzzy numbers) or the vertical coordinates (third coordinates, in the case of two-dimensional fuzzy numbers) of these nodes are given. In addition, in the study of fuzzy number approximation, some metrics or measures are needed. The existing research on fuzzy number approximation is basically based on the unweighted metric $d(u,v) = \sqrt{\int_0^1 \left(\underline{u}(r) - \underline{v}(r)\right)^2 dr + \int_0^1 \left(\overline{u}(r) - \overline{v}(r)\right)^2 dr}$ so that the method of solving the approximation of a fuzzy number is simple (for example, see [3,4,6,7,11-13,15,18]). However, for a fuzzy set, the greater the membership degree of an element belonging to the fuzzy set, the greater the contribution of the element to the fuzzy set (that is, the more important the element is to the fuzzy set). So, the weighted metric $d(u,v) = \sqrt{\int_0^1 2r\left(\underline{u}(r) - \underline{v}(r)\right)^2 dr + \int_0^1 2r\left(\overline{u}(r) - \overline{v}(r)\right)^2 dr}$ can characterize the difference between two fuzzy numbers u and v more objectively and reasonably than the unweighted metric. That is to say, the approximation of fuzzy numbers obtained by using weighted metric as the measurement scale is more objective and reasonable than that obtained by using unweighted metric. In this paper, it is under the condition that the level values corresponding to nodes to be solved are unknown that we establish a method for solving the nearest (or best) approximation based on weighted metric. That is to say, based on the weighted metric, we are going to establish a method to optimize all coordinates of nodes of the nearest (or best) approximation, which is different from the works done in [3,4,6-18]. Compared with the existing methods for solving the nearest (or best) approximation set up by predecessors (for example, [3,4,6-18]), the method we are going to establish should have some advantages.

The specific arrangement of this paper is as follows: In Section 2, we briefly review some basic notations, definitions and results about fuzzy numbers; in Section 3, we introduce two concepts of I-nearest 2-2-step type fuzzy number approximation and II-nearest 2-2-step type fuzzy number approximation of general fuzzy number, study the relationship between the two kinds of nearest approximation, and establish the solving methods of the I-nearest 2-2-step type fuzzy number approximation and II-nearest 2-2-step type fuzzy number approximation of general fuzzy number. In Section 4, an example is given to show the effectiveness and usability of methods set up by us. In Section 5, we make a conclusion for this paper.

2. **Basic Definitions and Notations.** A fuzzy subset (in short, a fuzzy set) of the real line R is a function $u : R \to [0,1]$. For each such fuzzy set u, we denote by $[u]^r = \{x \in R : u(x) \geq r\}$ for any $r \in (0,1]$, its r-level set. By supp$u$ we denote the support of u, i.e., the $\{x \in R : u(x) > 0\}$. By $[u]^0$ we denote the closure of the supp$u$, i.e., $[u]^0 = \overline{\{x \in R : u(x) > 0\}}$.

If u is a normal and fuzzy convex fuzzy set of R, $u(x)$ is upper semi-continuous, and $[u]^0$ is compact, then we call u a fuzzy number, and denote the collection of all fuzzy numbers by E.

It is known that if $u \in E$, then for each $r \in [0,1]$, $[u]^r$ is a convex compact set in R, i.e., a closed interval. For $u \in E$, we denote the closed interval as $[u]^r = [\underline{u}(r), \overline{u}(r)]$ for any $r \in [0,1]$.

For any $u, v \in E$, define $u \leq v$ if and only if $\underline{u}(r) \leq \underline{v}(r)$ and $\overline{u}(r) \leq \overline{v}(r)$ for any $r \in [0,1]$.

If mapping d: $E \times E \to R$ and d^*: $E \times E \to R$ are respectively defined by

$$d(u,v) = \sqrt{\int_0^1 \left(\underline{u}(r) - \underline{v}(r)\right)^2 dr + \int_0^1 \left(\overline{u}(r) - \overline{v}(r)\right)^2 dr}, \quad \forall (u,v) \in E \times E$$

and

$$d^*(u,v) = \sqrt{\int_0^1 2r\left(\underline{u}(r) - \underline{v}(r)\right)^2 dr + \int_0^1 2r\left(\overline{u}(r) - \overline{v}(r)\right)^2 dr}, \quad \forall (u,v) \in E \times E$$

then d and d^* are all metrics on E. We say d and d^* to be unweighted metric and weighted metric on E, respectively.

Definition 2.1. [11] *If $u \in E$ is defined as*

$$u(x) = \begin{cases} \alpha, & x \in [a_1, a_2) \\ 1, & x \in [a_2, b_2] \\ \beta, & x \in (b_2, b_1] \\ 0, & x \notin [a_1, b_1] \end{cases}$$

then we call u a 2-2-step type fuzzy number (or 2-2-simple fuzzy number), and denote it as

$$u = \left[\begin{pmatrix} a_1 & \alpha \\ a_2 & 1 \end{pmatrix}, \begin{pmatrix} b_1 & \beta \\ b_2 & 1 \end{pmatrix} \right]$$

where $\alpha, \beta \in (0,1)$ and $a_1, a_2, b_1, b_2 \in R$ with $a_1 \leq a_2 \leq b_2 \leq b_1$. And we denote ${}_2S_2(E) = \{u \in E : u$ is 2-2-step type fuzzy number$\}$.

Let $u = \left[\begin{pmatrix} a_1 & \alpha \\ a_2 & 1 \end{pmatrix}, \begin{pmatrix} b_1 & \beta \\ b_2 & 1 \end{pmatrix} \right] \in {}_2S_2(E)$, then we have that

$$\underline{u}(r) = \begin{cases} a_1, & r \in [0, \alpha] \\ a_2, & r \in (\alpha, 1] \end{cases}$$

and

$$\overline{u}(r) = \begin{cases} b_1, & r \in [0, \beta] \\ b_2, & r \in (\beta, 1] \end{cases}$$

3. **Approximations of Fuzzy Numbers Based on the Weighted Metric.** In this section, based on the weighted distance d^*, we are going to study the problem of approximating general fuzzy numbers with 2-2-step type fuzzy numbers when the level value of the nearest approximation required to be solved is not given. Theoretically speaking (as analyzed in the Introduction), for a given general fuzzy number, solved nearest approximation by using the method to be established by us is closer (i.e., better) to the general fuzzy number than the obtained nearest approximation by using the existing method of using a step type fuzzy number to approximate the general fuzzy number when the level values are known.

Definition 3.1. *Let $u \in E$. If $\hat{u}_I \in {}_2S_2(E)$ with $\underline{\hat{u}_I}(1) = \underline{u}(1)$, $\overline{\hat{u}_I}(1) = \overline{u}(1)$ and $d^*(u, \hat{u}_I) = \min\{d^*(u,v) : v \in {}_2S_2(E)$ with $\underline{v}(1) = \underline{u}(1), \overline{v}(1) = \overline{u}(1)\}$, then $\hat{u}_I$ is called the I-nearest 2-2-step type fuzzy number approximation (in short, I-nearest 2-2-STFN approximation) of u.*

Definition 3.2. *Let $u \in E$. If $\hat{u}_{II} \in {}_2S_2$ and $d^*(u, \hat{u}) = \min\{d^*(u,v) : v \in {}_2S_2(E)\}$, then $\hat{u}_{II}$ is called the II-nearest 2-2-step type fuzzy number approximation (in short, II-nearest 2-2-STFN approximation) of u.*

Theorem 3.1. *Let $u \in E$. If $\hat{u}_I$ is the I-nearest 2-2-step type fuzzy number approximation of u and $\hat{u}_{II}$ is the II-nearest 2-2-step type fuzzy number approximation of u, then $d^*(u, \hat{u}_{II}) \le d^*(u, \hat{u}_I)$.*

Proof: The proof of the theorem can be directly completed from

$$\left\{d^*(u,v) : v \in {}_2S_2(E) \text{ with } \underline{\hat{u}}(1) = \underline{u}(1), \overline{\hat{u}}(1) = \overline{u}(1)\right\} \subset \{d^*(u,v) : v \in {}_2S_2(E)\}$$

Remark 3.1. *Although the I-nearest 2-2-step type fuzzy number approximation still has the same kernel and support set as the approximated fuzzy number, the theorem tells us that the II-nearest 2-2-step type fuzzy number approximation is closer to the approximated fuzzy number u than the I-nearest 2-2-step type fuzzy number approximation, that is, the II-nearest 2-2-step type fuzzy number approximation is better than the I-nearest 2-2-step type fuzzy number approximation.*

Theorem 3.2. *Let $u \in E$. If*

$$\begin{pmatrix} 2\alpha^2 & 4a\alpha - 4\alpha\underline{u}(\alpha) \\ 4a\alpha - 4\alpha\underline{u}(\alpha) & 2a^2 + 4\left(\underline{u}(1) - a\right)\left[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)\right] - 2\underline{u}(1)^2 \end{pmatrix}$$

and

$$\begin{pmatrix} 2\beta^2 & 4b\beta - 4\beta\overline{u}(\beta) \\ 4b\beta - 4\beta\underline{u}(\beta) & 2b^2 + 4\left(\overline{u}(1) - b\right)\left[\overline{u}(\beta) + \beta\overline{u}'(\alpha)\right] - 2\overline{u}(1)^2 \end{pmatrix}$$

are positive definite, then

$$\hat{u}_I = \left[\begin{pmatrix} a & \alpha \\ \underline{u}(1) & 1 \end{pmatrix}, \begin{pmatrix} b & \beta \\ \overline{u}(1) & 1 \end{pmatrix}\right]$$

is the I-nearest 2-2-step type fuzzy number approximation of u, where $r = \alpha$ and $s = \beta$ are respectively the unique solutions of equations

$$r^2\left[\underline{u}(1) - 2\underline{u}(r)\right] + 2\int_0^r t\underline{u}(t)dt = 0 \tag{1}$$

$$s^2\left[\overline{u}(1) - 2\overline{u}(s)\right] + 2\int_0^s t\overline{u}(t)dt = 0 \tag{2}$$

of r and s on $(0,1)$, and $a = \frac{2\int_0^\alpha t\underline{u}(t)dt}{\alpha^2}$, $b = \frac{2\int_0^\beta t\overline{u}(t)dt}{\beta^2}$.

Proof: Let

$$\hat{u}_I = \left[\begin{pmatrix} x & r \\ \underline{u}(1) & 1 \end{pmatrix}, \begin{pmatrix} y & s \\ \overline{u}(1) & 1 \end{pmatrix}\right]$$

be a 2-2-step type fuzzy numbers with $\underline{\hat{u}_I}(1) = \underline{u}(1)$, $\overline{\hat{u}_I}(1) = \overline{u}(1)$. Then we have that

$$\underline{\hat{u}_I}(t) = \begin{cases} x, & t \in [0, r] \\ \underline{\hat{u}_I}(1), & t \in (r, 1] \end{cases}$$

and

$$\overline{\hat{u}_I}(t) = \begin{cases} y, & t \in [0, s] \\ \overline{\hat{u}_I}(1), & t \in (s, 1] \end{cases}$$

Denoting $D(x, r, y, s) = [d^*(u, \hat{u}_I)]^2$, we have that

$$D(x, r, y, s) = 2\left(\int_0^r t(x - \underline{u}(t))^2 dt + \int_r^1 t(\underline{u}(1) - \underline{u}(t))^2 dt + \int_0^s t(y - \overline{u}(t))^2 dt + \int_s^1 t(\overline{u}(1) - \overline{u}(t))^2 dt\right)$$

We can obtain that

$$\frac{\partial D}{\partial x} = 2xr^2 - 4\int_0^r t\underline{u}(t)dt$$

$$\frac{\partial D}{\partial r} = 2rx^2 - 4xr\underline{u}(r) - 2r\underline{u}(1)^2 + 4r\underline{u}(t)\underline{u}(1)$$

$$\frac{\partial D}{\partial y} = 2ys^2 - 4\int_0^s t\overline{u}(t)dt$$

$$\frac{\partial D}{\partial s} = 2sy^2 - 4ys\overline{u}(s) - 2s\overline{u}(1)^2 + 4s\overline{u}(t)\overline{u}(1)$$

Let $\frac{\partial D}{\partial x}, \frac{\partial D}{\partial r}, \frac{\partial D}{\partial y}, \frac{\partial D}{\partial s} = 0$. We have that

$$\begin{cases} xr^2 - 2\int_0^r t\underline{u}(t)dt = 0 \\ rx^2 - 2xr\underline{u}(r) - r\underline{u}(1)^2 + 2r\underline{u}(t)\underline{u}(1) = 0 \\ ys^2 - 2\int_0^s t\overline{u}(t)dt = 0 \\ sy^2 - 2ys\overline{u}(s) - s\overline{u}(1)^2 + 2s\overline{u}(t)\overline{u}(1) = 0 \end{cases} \tag{3}$$

From the equation, we can obtain that

$$r^2[\underline{u}(1) - 2\underline{u}(r)] - 2\int_0^r t\underline{u}(t)dt = 0 \tag{4}$$

$$s^2[\overline{u}(1) - 2\overline{u}(r)] - 2\int_0^s t\overline{u}(t)dt = 0 \tag{5}$$

So, if $r = \alpha$ and $s = \beta$ are respectively the unique solutions of Equation (4) and Equation (5) on $(0, 1)$, then we see that

$$\begin{cases} x = \dfrac{2\int_0^\alpha t\underline{u}(t)dt}{\alpha^2} \text{ (denoted by } a) \\ r = \alpha \\ y = \dfrac{2\int_0^\beta t\overline{u}(t)dt}{\beta^2} \text{ (denoted by } b) \\ s = \beta \end{cases}$$

is the unique solution of Equation (3) about (x, r, y, s) on $R \times (0,1) \times R \times (0,1)$.

On the other hand, we can also work out that

$$\frac{\partial^2 D}{\partial x^2} = 2r^2$$

$$\frac{\partial^2 D}{\partial x \partial r} = \frac{\partial^2 D}{\partial r \partial x} = 4r(x - \underline{u}(r))$$

$$\frac{\partial^2 D}{\partial r^2} = 2\left(x^2 - \underline{u}(1) - 2(x + \underline{u}(1))(\underline{u}(r) + r\underline{u}'(r))\right)$$

$$\frac{\partial^2 D}{\partial y^2} = 2s^2$$

$$\frac{\partial^2 D}{\partial y \partial s} = \frac{\partial^2 D}{\partial s \partial y} = 4s(y - \overline{u}(s))$$

$$\frac{\partial^2 D}{\partial s^2} = 2\left(y^2 - \overline{u}(1) - 2(y + \overline{u}(1))(\overline{u}(s) + s\overline{u}'(s))\right)$$

$$\frac{\partial^2 D}{\partial x \partial y} = \frac{\partial^2 D}{\partial y \partial x} = \frac{\partial^2 D}{\partial x \partial s} = \frac{\partial^2 D}{\partial s \partial x} = \frac{\partial^2 D}{\partial r \partial y} = \frac{\partial^2 D}{\partial y \partial r} = \frac{\partial^2 D}{\partial r \partial s} = \frac{\partial^2 D}{\partial s \partial r} = 0$$

Therefore, we can obtain the Hessian matrix M of $M(x, r, y, s)$ at $(x, r, y, s) = (a, \alpha, b, \beta)$ as follows:

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \tag{6}$$

where

$$M_1 = \begin{pmatrix} 2\alpha^2 & 4\alpha(a - \underline{u}(\alpha)) \\ 4\alpha(a - \underline{u}(\alpha)) & 2a^2 + 4(\underline{u}(1) - a)\left(\underline{u}(\alpha) + \underline{u}'(\alpha)\right) - 2\underline{u}(1)^2 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 2\beta^2 & 4\beta(b - \underline{u}(\beta)) \\ 4\beta(b - \underline{u}(\beta)) & 2b^2 + 4(\underline{u}(1) - b)\left(\underline{u}(\beta) + \underline{u}'(\beta)\right) - 2\underline{u}(1)^2 \end{pmatrix}$$

From the positive definiteness of M_1 and M_2, we see that M is also positive definite. So the unique stationary point (a, α, b, β) of function $D(x, r, y, s)$ on $R \times (0,1) \times R \times (0,1)$ is the minimum point of $D(x, r, y, s)$ on $R \times (0,1) \times R \times (0,1)$. Thus,

$$\hat{u}_I = \left[\begin{pmatrix} a & \alpha \\ \underline{u}(1) & 1 \end{pmatrix}, \begin{pmatrix} b & \beta \\ \overline{u}(1) & 1 \end{pmatrix} \right]$$

is the I-nearest 2-2-step type fuzzy number approximation of u.

Theorem 3.3. *Let $u \in E$. If*

$$\begin{pmatrix} 2(1 - \alpha^2) & 4\alpha\underline{u}(\alpha) - 4\alpha a_1 \\ 4\alpha\underline{u}(\alpha) - 4\alpha a_1 & 4(a_1 - a_2)\left[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)\right] - 2\left(a_1^2 - a_2^2\right) \end{pmatrix}$$

$$\begin{pmatrix} 2\alpha^2 & 4a_2\alpha - 4\alpha\underline{u}(\alpha) \\ 4a_2\alpha - 4\alpha\underline{u}(\alpha) & 4(a_1 - a_2)\left[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)\right] - 2\left(a_1^2 - a_2^2\right) \end{pmatrix}$$

$$\begin{pmatrix} 2(1-\beta^2) & 4\beta\overline{u}(\beta) - 4\beta b_1 \\ 4\beta\overline{u}(\beta) - 4\beta b_1 & 4(b_1 - b_2)\left[\overline{u}(\beta) + \beta\overline{u}'(\beta)\right] - 2\left(b_1^2 - b_2^2\right) \end{pmatrix}$$

$$\begin{pmatrix} 2\beta^2 & 4b_2\beta - 4\beta\overline{u}(\beta) \\ 4b_2\beta - 4\beta\overline{u}(\beta) & 4(b_1 - b_2)\left[\overline{u}(\beta) + \beta\overline{u}'(\beta)\right] - 2\left(b_1^2 - b_2^2\right) \end{pmatrix}$$

are all positive definite, then

$$\hat{u}_{II} = \left[\begin{pmatrix} a_2 & \alpha \\ a_1 & 1 \end{pmatrix}, \begin{pmatrix} b_2 & \beta \\ b_1 & 1 \end{pmatrix}\right]$$

is the II-nearest 2-2-step type fuzzy number approximation of u, where $r = \alpha$ and $s = \beta$ are respectively the unique solutions of equations

$$r^2\left(1 - r^2\right)\underline{u}(r) - r^2\int_r^1 t\underline{u}(t)dt - \left(1 - r^2\right)\int_0^r t\underline{u}(t)dt = 0 \tag{7}$$

$$s^2\left(1 - s^2\right)\overline{u}(s) - s^2\int_s^1 t\overline{u}(t)dt - \left(1 - s^2\right)\int_0^s t\overline{u}(t)dt = 0 \tag{8}$$

of r and s on $(0,1)$, and $a_1 = \frac{2\int_\alpha^1 t\underline{u}(t)dt}{1-\alpha^2}$, $a_2 = \frac{2\int_0^\alpha t\underline{u}(t)dt}{\alpha^2}$, $b_1 = \frac{2\int_\beta^1 t\underline{u}(t)dt}{1-\beta^2}$, $b_2 = \frac{2\int_0^\beta t\underline{u}(t)dt}{\beta^2}$.

Proof: Let

$$\hat{u}_{II} = \left[\begin{pmatrix} x_2 & r \\ x_1 & 1 \end{pmatrix}, \begin{pmatrix} y_2 & s \\ y_1 & 1 \end{pmatrix}\right]$$

be a 2-2-step type fuzzy numbers. Then

$$\underline{\hat{u}_{II}}(t) = \begin{cases} x_2, & t \in [0, r] \\ x_1, & t \in (r, 1] \end{cases}$$

and

$$\overline{\hat{u}_{II}}(t) = \begin{cases} y_2, & t \in [0, s] \\ y_1, & t \in (s, 1] \end{cases}$$

Denoting $D(x_1, x_2, r, y_1, y_2, s) = \left[d^*(u, \hat{u}_{II})\right]^2$, we have that

$$D(x_1, x_2, r, y_1, y_2, s) = \int_0^r 2t(x_2 - \underline{u}(t))^2 dt + \int_r^1 2t(x_1 - \underline{u}(t))^2 dt + \int_0^s 2t(y_2 - \overline{u}(t))^2 dt + \int_s^1 2t(y_1 - \overline{u}(t))^2 dt$$

So, we can obtain that

$$\frac{\partial D}{\partial x_1} = 2\left(1 - r^2\right)x_1 - 4\int_r^1 t\underline{u}(t)dt$$

$$\frac{\partial D}{\partial x_2} = 2r^2 x_2 - 4\int_0^r t\underline{u}(t)dt$$

$$\frac{\partial D}{\partial r} = 4r(x_1 - x_2)\underline{u}(t) - 2r\left(x_1^2 - x_2^2\right)$$

$$\frac{\partial D}{\partial y_1} = 2\left(1 - s^2\right)y_1 - 4\int_s^1 t\overline{u}(t)dt$$

$$\frac{\partial D}{\partial y_2} = 2s^2 y_2 - 4\int_0^s t\overline{u}(t)dt$$

$$\frac{\partial D}{\partial s} = 4s(y_1 - y_2)\overline{u}(s) - 2s\left(y_1^2 - y_2^2\right)$$

Let $\frac{\partial D}{\partial x_1}, \frac{\partial D}{\partial x_2}, \frac{\partial D}{\partial r}, \frac{\partial D}{\partial y_1}, \frac{\partial D}{\partial y_2}, \frac{\partial D}{\partial s} = 0$. We have that

$$\begin{cases} 2(1-r^2)x_1 - 4\int_r^1 t\underline{u}(t)dt = 0 \\ 2r^2 x_2 - 4\int_0^r t\underline{u}(t)dt = 0 \\ 4r(x_1 - x_2)\underline{u}(t) - 2r\left(x_1^2 - x_2^2\right) = 0 \\ 2(1-s^2)y_1 - 4\int_s^1 t\overline{u}(t)dt = 0 \\ 2s^2 y_2 - 4\int_0^s t\overline{u}(t)dt = 0 \\ 4s(y_1 - y_2)\overline{u}(s) - 2s\left(y_1^2 - y_2^2\right) = 0 \end{cases} \tag{9}$$

From Equation (9), we can obtain that

$$r^2\left(1-r^2\right)\underline{u}(r) - r^2\int_r^1 t\underline{u}(t)dt - \left(1-r^2\right)\int_0^r t\underline{u}(t)dt = 0 \tag{10}$$

$$s^2\left(1-s^2\right)\overline{u}(s) - s^2\int_s^1 t\overline{u}(t)dt - \left(1-s^2\right)\int_0^s t\overline{u}(t)dt = 0 \tag{11}$$

Therefore, if $r = \alpha$ and $s = \beta$ are respectively the unique solutions of Equation (10) and Equation (11) on $(0, 1)$, then we know that

$$\begin{cases} x_1 = \dfrac{2\int_\alpha^1 t\underline{u}(t)dt}{1-\alpha^2} \text{ (denoted by } a_1) \\ x_2 = \dfrac{2\int_0^\alpha t\underline{u}(t)dt}{\alpha^2} \text{ (denoted by } a_2) \\ r = \alpha \\ y_1 = \dfrac{2\int_\beta^1 t\overline{u}(t)dt}{1-\beta^2} \text{ (denoted by } b_1) \\ y_2 = \dfrac{2\int_0^\beta t\overline{u}(t)}{\beta^2} \text{ (denoted by } b_2) \\ s = \beta \end{cases}$$

is the unique solution of Equation (9) about $(x_1, x_2, r, y_1, y_2, s)$ on $R^2 \times (0,1) \times R^2 \times (0,1)$.

On the other hand, we have that

$$\frac{\partial^2 D}{\partial^2 x_1} = 2\left(1-r^2\right), \quad \frac{\partial^2 D}{\partial x_1 \partial r} = \frac{\partial^2 D}{\partial r \partial x_1} = -4x_1 r + 4r\underline{u}(r)$$

$$\frac{\partial^2 D}{\partial^2 x_2} = 2r^2, \quad \frac{\partial^2 D}{\partial x_2 \partial r} = \frac{\partial^2 D}{\partial r \partial x_2} = 4x_2 r - 4r\underline{u}(r)$$

$$\frac{\partial^2 D}{\partial^2 y_1} = 2\left(1 - s^2\right), \quad \frac{\partial^2 D}{\partial y_1 \partial s} = \frac{\partial D}{\partial s \partial y_1} = -4y_1 s + 4s\overline{u}(s)$$

$$\frac{\partial^2 D}{\partial^2 y_2} = 2s^2, \quad \frac{\partial^2 D}{\partial y_2 \partial s} = \frac{\partial D}{\partial s \partial y_2} = 4y_2 s - 4s\overline{u}(s)$$

$$\frac{\partial^2 D}{\partial^2 r} = 4(x_1 - x_2)\left[\underline{u}(r) + r\underline{u}'(r)\right] - 2\left(x_1^2 - x_2^2\right)$$

$$\frac{\partial^2 D}{\partial^2 s} = 4(y_1 - y_2)\left[\overline{u}(s) + s\overline{u}'(s)\right] - 2\left(y_1^2 - y_2^2\right)$$

$$\begin{aligned}\frac{\partial^2 D}{\partial x_1 \partial x_2} &= \frac{\partial^2 D}{\partial x_2 \partial x_1} = \frac{\partial^2 D}{\partial x_1 \partial y_1} = \frac{\partial^2 D}{\partial y_1 \partial x_1} = \frac{\partial^2 D}{\partial x_1 \partial y_2} = \frac{\partial^2 D}{\partial y_2 \partial x_1} = \frac{\partial^2 D}{\partial x_1 \partial s} = \frac{\partial^2 D}{\partial s \partial x_1} \\ &= \frac{\partial^2 D}{\partial x_2 \partial y_1} = \frac{\partial^2 D}{\partial y_1 \partial x_2} = \frac{\partial^2 D}{\partial x_2 \partial y_2} = \frac{\partial^2 D}{\partial y_2 \partial x_2} = \frac{\partial^2 D}{\partial x_2 \partial s} = \frac{\partial^2 D}{\partial s \partial x_2} = \frac{\partial^2 D}{\partial y_1 \partial r} \\ &= \frac{\partial^2 D}{\partial r \partial y_1} = \frac{\partial^2 D}{\partial y_2 \partial r} = \frac{\partial^2 D}{\partial r \partial y_2} = 0\end{aligned}$$

So, we can see that the Hessian matrix M of the function $D(x_1, x_2, r, y_1, y_2, s)$ at $(x_1, x_2, r, y_1, y_2, s) = (a_1, a_2, \alpha, b_1, b_2, \beta)$ as follows:

$$M = \begin{pmatrix} M_1 & 0 & 0 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & M_3 & 0 \\ 0 & 0 & 0 & M_4 \end{pmatrix} \tag{12}$$

where

$$M_1 = \begin{pmatrix} 2\left(1 - \alpha^2\right) & 4\alpha\underline{u}(\alpha) - 4\alpha a_1 \\ 4\alpha\underline{u}(\alpha) - 4\alpha a_1 & 4(a_1 - a_2)\left[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)\right] - 2\left(a_1^2 - a_2^2\right) \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 2\alpha^2 & 4a_2\alpha - 4\alpha\underline{u}(\alpha) \\ 4a_2\alpha - 4\alpha\underline{u}(\alpha) & 4(a_1 - a_2)\left[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)\right] - 2\left(a_1^2 - a_2^2\right) \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 2\left(1 - \beta^2\right) & 4\beta\overline{u}(\beta) - 4\beta b_1 \\ 4\beta\overline{u}(\beta) - 4\beta b_1 & 4(b_1 - b_2)\left[\overline{u}(\beta) + \beta\overline{u}'(\beta)\right] - 2\left(b_1^2 - b_2^2\right) \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 2\beta^2 & 4b_2\beta - 4\beta\overline{u}(\beta) \\ 4b_2\beta - 4\beta\overline{u}(\beta) & 4(b_1 - b_2)\left[\overline{u}(\beta) + \beta\overline{u}'(\beta)\right] - 2\left(b_1^2 - b_2^2\right) \end{pmatrix}$$

From the positive definiteness of M_1, M_2, M_3 and M_4, we see that M is also positive definite. So the unique stationary point $(a_1, a_2, \alpha, b_1, b_2, \beta)$ of function $D(x_1, x_2, r, y_1, y_2, s)$ on $R^2 \times (0, 1) \times R^2 \times (0, 1)$ is the minimum point of $D(x_1, x_2, r, y_1, y_2, \beta)$ on $R^2 \times (0, 1) \times R^2 \times (0, 1)$. Thus,

$$\hat{u}_{II} = \left[\begin{pmatrix} a_2 & \alpha \\ a_1 & 1 \end{pmatrix}, \begin{pmatrix} b_2 & \beta \\ b_1 & 1 \end{pmatrix}\right]$$

is the II-nearest 2-2-step type fuzzy number approximation of u.

4. **Example.** In the following, a specific example is given to show the application of the proposed methods by us in this paper.

Example 4.1. *Suppose the fuzzy number u be defined as the following:*

$$u(x) = \begin{cases} (x-1)^2, & x \in [1,2) \\ 1, & x \in [2,3] \\ \sqrt{4-x}, & x \in (3,4] \\ 0, & x \notin [1,4] \end{cases}$$

Then we have

$$\begin{cases} \underline{u}(t) = 1 + \sqrt{t} \\ \underline{u}'(t) = \dfrac{1}{2\sqrt{t}} & t \in [0,1] \\ \overline{u}(t) = 4 - t^2 \\ \overline{u}'(t) = -2t \end{cases}$$

and $\underline{u}(1) = 2$, $\overline{u}(1) = 3$.

1) Solving Equations (1) and (2), we have that $\alpha = \frac{25}{36}$ *and* $\beta = \frac{\sqrt{6}}{3}$. *Then we can calculate that* $a = \frac{2\int_0^\alpha t\underline{u}(t)dt}{\alpha^2} = \frac{5}{3}$, $b = \frac{2\int_0^\beta t\overline{u}(t)dt}{\beta^2} = \frac{11}{3}$.

On the other hand, we can work out $\underline{u}(\alpha) = \underline{u}\left(\frac{25}{36}\right) = 1.8333$, $\underline{u}'(\alpha) = \underline{u}'\left(\frac{25}{36}\right) = 0.6000$, $\overline{u}(\beta) = \overline{u}\left(\frac{\sqrt{6}}{3}\right) = 3.3333$ *and* $\overline{u}'(\beta) = \overline{u}'\left(\frac{\sqrt{6}}{3}\right) = -1.6330$, *so, we have that*

$$\begin{aligned} M_1 &= \begin{pmatrix} 2\alpha^2 & 4a\alpha - 4\alpha\underline{u}(\alpha) \\ 4a\alpha - 4\alpha\underline{u}(\alpha) & 2a^2 + 4(\underline{u}(1) - a)[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)] - 2\underline{u}(1)^2 \end{pmatrix} \\ &= \begin{pmatrix} 0.97 & -0.46 \\ -0.46 & 0.56 \end{pmatrix} \\ M_2 &= \begin{pmatrix} 2\beta^2 & 4b\beta - 4\beta\overline{u}(\beta) \\ 4b\beta - 4\beta\underline{u}(\beta) & 2b^2 + 4(\overline{u}(1) - b)[\overline{u}(\beta) + \beta\overline{u}'(\alpha)] - 2\overline{u}(1)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1.3333 & 1.0888 \\ 1.0888 & 3.5557 \end{pmatrix} \end{aligned}$$

It is obvious that M_1, M_2 *are all positive definite. By Theorem 3.2, we know that*

$$\hat{u}_I = \left[\begin{pmatrix} \frac{5}{3} & \frac{25}{36} \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} \frac{11}{3} & \frac{\sqrt{6}}{3} \\ 3 & 1 \end{pmatrix} \right]$$

is the I-nearest 2-2-step type fuzzy number approximation of u.

2) Solving Equations (7) and (8), we obtain the unique solution $r = \alpha = 0.549$ *and* $s = \beta = \frac{\sqrt{2}}{2}$. *Then we can work out* $a_1 = \frac{2\int_\alpha^1 t\underline{u}(t)dt}{1-\alpha^2} = 1.889$, $a_2 = \frac{2\int_0^\alpha t\underline{u}(t)dt}{\alpha^2} = 1.593$, $b_1 = \frac{2\int_\beta^1 t\overline{u}(t)dt}{1-\beta^2} = 3.25$, $b_2 = \frac{2\int_0^\beta t\overline{u}(t)dt}{\beta^2} = 3.75$.

On the other hand, we can work out $\underline{u}(\alpha) = \underline{u}(0.549) = 1.7409$, $\underline{u}'(\alpha) = \underline{u}'(0.549) = 0.6748$, $\overline{u}(\beta) = \overline{u}\left(\frac{\sqrt{2}}{2}\right) = 3.5$ *and* $\overline{u}'(\beta) = \overline{u}'\left(\frac{\sqrt{2}}{2}\right) = -1.414$, *so, we have that*

$$\begin{aligned} M_1 &= \begin{pmatrix} 2(1-\alpha^2) & 4\alpha(\underline{u}(\alpha) - x_1) \\ 4\alpha(\underline{u}(\alpha) - x_1) & 4(x_1 - x_2)[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)] - 2(x_1^2 - x_2^2) \end{pmatrix} \\ &= \begin{pmatrix} 1.3972 & -0.33 \\ -0.33 & 2.4999 \end{pmatrix} \end{aligned}$$

$$M_2 = \begin{pmatrix} 2\alpha^2 & 4\alpha(x_2 - \underline{u}(\alpha)) \\ 4\alpha(x_2 - \underline{u}(\alpha)) & 4(x_1 - x_2)\left[\underline{u}(\alpha) + \alpha\underline{u}'(\alpha)\right] - 2\left(x_1^2 - x_2^2\right) \end{pmatrix}$$
$$= \begin{pmatrix} 0.6028 & -0.32 \\ -0.32 & 2.4999 \end{pmatrix}$$
$$M_3 = \begin{pmatrix} 2\left(1 - (\beta)^2\right) & 4\beta\overline{u}(\beta) - 4\beta y_1 \\ 4\beta\overline{u}(\beta) - 4\beta y_1 & 4(y_1 - y_2)\left[\overline{u}(\beta) + \beta\overline{u}'(\beta)\right] - 2\left(y_1^2 - y_2^2\right) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0.7071 \\ 0.7071 & 2 \end{pmatrix}$$
$$M_4 = \begin{pmatrix} 2\beta^2 & 4\beta(y_2 - \overline{u}(\beta)) \\ 4\beta(y_2 - \overline{u}(\beta)) & 4(y_1 - y_2)\left[\overline{u}(\beta) + \beta\overline{u}'(\beta)\right] - 2\left(y_1^2 - y_2^2\right) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0.7071 \\ 0.7071 & 2 \end{pmatrix}$$

It is obvious that M_1, M_2, M_3, M_4 are all positive definite. Thus by Theorem 3.3, we see that

$$\hat{u}_{II} = \left[\begin{pmatrix} 1.593 & 0.549 \\ 1.889 & 1 \end{pmatrix}, \begin{pmatrix} 3.75 & 0.7071 \\ 3.25 & 1 \end{pmatrix}\right]$$

is the II-nearest 2-2-step type fuzzy number approximation of u.

In addition, we can work out that $d^*(u, \hat{u}_I) = 1.6204$ and $d^*(u, \hat{u}_{II}) = 0.1702$. Therefore, we see $d^*(u, \hat{u}_I) > d^*(u, \hat{u}_{II})$. So, the specific example also verifies the correctness of Theorem 3.1, i.e., the II-nearest 2-2-step type fuzzy number approximation of u is better than I-nearest 2-2-step type fuzzy number approximation of u.

5. Conclusions.

In this paper, for using 2-2-step type fuzzy number to approximate a given general fuzzy number, we introduced the concepts of I-nearest approximation (Definition 3.1) and II-nearest approximation (Definition 3.2), and pointed out the relationship (Theorem 3.1) between them. Then, for a given general fuzzy number, we established the methods (Theorems 3.2 and 3.3) of solving its I-nearest 2-2-step type fuzzy number approximation and II-nearest 2-2-step type fuzzy number approximation, respectively. At last, we used a specific example (Example 4.1) to show the effectiveness and usability of methods set up in this paper. In the future, we can study the problem of using multi-steps type fuzzy number to approximate a given general fuzzy number.

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REFERENCES

[1] Q. Qi, G. Wang, C. Li and Y. Xu, Ranking multichannel uncertain integer quantity digital information based on discrete quasi-fuzzy number vector, *ICIC Express Letters*, vol.14, no.5, pp.505-514, 2020.

[2] G. Wang, Q. Nan and J. Li, Fuzzy integers and methods of constructing them to represent uncertain or imprecise integer information, *International Journal of Innovative Computing, Information and Control*, vol.11, no.4, pp.1483-1494, 2015.

[3] P. Grzegorewski, Nearest interval approximation of a fuzzy number, *Fuzzy Sets and Systems*, vol.130, pp.321-330, 2002.

[4] L. Jian, X. Zhong and Y. Qi, Triangular approximation preserving the centroid of fuzzy numbers, *International Journal of Computer Mathematics*, vol.89, no.6, pp.810-821, 2012.

[5] C. Veeramani and C. Duraisamy, Solving fuzzy linear programming problem using symmetric fuzzy number approximation, *International Journal of Operational Research*, vol.15, no.3, pp.321-336, 2012.
[6] P. Grzegorzewski and K. Pasternak-Winiarska, Natural trapezoidal approximations of fuzzy numbers, *Fuzzy Sets and Systems*, vol.250, pp.90-109, 2014.
[7] A. Ban and L. Coroianu, Existence, uniqueness and continuity of trapezoidal approximations of fuzzy numbers under a general condition, *Fuzzy Sets and Systems*, vol.257, pp.3-22, 2014.
[8] D. S. Dinagar and K. Jivagan, Comparison of different approximations of fuzzy numbers, *International Journal of Fuzzy Logic Systems*, vol.5, no.4, pp.11-25, 2015.
[9] A. Ban, L. Coroianu and A. Khastan, Conditioned weighted L-R approximations of fuzzy numbers, *Fuzzy Sets and Systems*, vol.283, pp.56-82, 2016.
[10] H. Huang, C. Wu, J. Xie and D. Zhang, Approximation of fuzzy numbers using the convolution method, *Fuzzy Sets and Systems*, vol.310, pp.14-46, 2017.
[11] G. Wang and J. Li, Approximations of fuzzy numbers by step type fuzzy numbers, *Fuzzy Sets and Systems*, vol.310, pp.47-59, 2017.
[12] S. Yeganehmanesh and M. Amirfakhrian, An approximation of fuzzy numbers based on polynomial form fuzzy numbers, *International Journal of Analysis and Applications*, vol.16, no.2, pp.290-305, 2018.
[13] L. Coroianu, M. Gagolewski and P. Grzegorewski, Piecewise linear approximation of fuzzy numbers: Algorithms, arithmetic operations and stability of characteristics, *Soft Comput.*, vol.23, pp.9491-9505, 2019.
[14] P. Gonzalez, H. Idais, M. Pasadas and M. Yasin, 3D fuzzy data approximation by fuzzy smoothing bicubic splines, *Mathematics and Computers in Simulation*, vol.164, pp.94-102, 2019.
[15] G. Wang, C. Shen and Y. Wang, Approximation of fuzzy numbers by using multi-knots piecewise linear fuzzy numbers, *Journal of Intelligent & Fuzzy Systems*, vol.39, pp.3597-3615, 2020.
[16] S. Hai, Z. Gong and Z. Chen, Weighted pseudometric approximation of 2-dimensional fuzzy numbers by fuzzy 2-cell prismoid numbers preserving the centroid, *Fuzzy Sets and Systems*, vol.387, pp.158-173, 2020.
[17] L. Coroianu, Trapezoidal approximations of fuzzy numbers using quadratic programs, *Fuzzy Sets and Systems*, vol.417, pp.71-92, 2021.
[18] V. Lakshmana, G. Nayagam and J. Murugan, Hexagonal fuzzy approximation of fuzzy numbers and its applications in MCDM, *Complex and Intelligent Systems*, vol.7, pp.1459-1487, 2021.

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