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ELASTIC-PLASTIC LIMIT ANALYSIS OF THE COMBINED THICK-WALLED CYLINDER BASED ON THE UNIFIED STRENGTH THEORY

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Abstract. *This paper adopts the unified strength theory (UST) to deduce the unified solution of the elastic limit, plastic limit and stability limit internal pressures of the thick-walled cylinder and the elastic limit and plastic limit internal pressures of double-layered thick-walled cylinder, and uses the simplified unified solution to draw a conclusion consistent with other literature, demonstrating the correctness of the derivation in this paper. In this paper, examples are used to analyze the impact of diameter ratio and UST parameters on the elastic limit, plastic limit and stability limit of the thick-walled cylinder. The results show that the unified solution increases with the increase in diameter ratio and influence factor of intermediate stress and reduces with the increase in the compressive and tensile strength of materials; as the diameter ratio increases, its impact on the improvement of the elastic limit internal pressure is significantly diminished, while its impact on the increase of the plastic limit internal pressure is significant; the prestressing force between internal and external cylinders can significantly increase the elastic limit internal pressure of the combined thick-walled cylinder, but it has hardly no impact on the plastic limit internal pressure of the combined thick-walled cylinder. This study can provide the theoretical basis of strength design for the engineering application of the thick-walled cylinder structures, e.g., culvert pipes, composite supports, extrusion dies, cylindrical pressure vessels, air cylinders and gun barrels.*
Keywords: Unified strength theory, Thick-walled cylinder, Elastic limit internal pressure, Plastic limit internal pressure, Stability limit internal pressure

1. **Introduction.** The thick-walled cylinder under internal pressure is a structure widely used in actual projects, e.g., female dies extruded, barrels used in military equipment, culvert pipes and supports used in buildings, high-pressure vessels used in chemical equipment, and pipes for conveying high-pressure and high-temperature fluids used in the energy industry [1-9]. Li et al. [4] analyzed the conditions of plastic deformation of long thick-walled cylinder models based on the concept of elastic-plastic constraints, with the aim to fully understand the root cause of brittle fracture of thick-walled cylinders under

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internal pressure and provide the theoretical basis of analysis and computation for the design of resistance to brittle fracture of oil pipelines. Zou and Zhu [6] employed the UST in the analysis of self-enhancement of thick-walled cylinders and derived the best radius of elastic-plastic interface of the self-enhanced cylinder and the diameter ratio without reverse yielding during the self-enhancement after considering the impact of tension-compression (T-C) anisotropy and intermediate principal stress. Where the internal working pressure is high and the size is limited, the combined thick-walled cylinder [10-15] structure was typically used to fully utilize the strength potential of various layers of cylinder body materials and the overall strength of self-enhanced structures. Zhao et al. [15] derived the basic solution of optimal layer radius, surface pressure of the cylinder after covering the external layer on the internal layer and magnitude of interference on the covering of the combined thick-walled cylinder and the unified solution of elastic limit and plastic limit internal pressures of the multi-layered thick-walled cylinder on the basis of the UST, providing the theoretical basis for the strength analysis and design of high-pressure vessels.

At present, the research on the elastic limit, plastic limit and stability limit of the thick-walled cylinder is mainly based on the traditional strength theory, without considering the influence of tension-compression (T-C) anisotropy and intermediate principal stress, which is not conducive to the full exertion of structural strength potential. Based on the unified strength theory and considering the influence of tension-compression (T-C) anisotropy and intermediate principal stress, the calculation formulas of elastic limit, plastic limit and stability limit of the thick-walled cylinder are derived in this paper, which has a certain reference role in engineering application.

2. **Yield Conditions Based on the UST.** The structures like female dies, culvert pipes and composite supports, high-pressure vessels and gun barrels can be regarded as thick-walled cylinders or combined thick-walled cylinders for strength analysis, and typically, these structures are made of materials having significant T-C anisotropy, e.g., hard alloy, high-strength alloy steel, cast iron, aluminium alloy, composite materials and concrete; the impact of the T-C anisotropy and intermediate principal stress of materials on structural yielding can be taken in account in the application of the UST [16,17]. The UST is generally expressed as

$$f = \sigma_1 - \frac{\alpha}{1+b}(b\sigma_2 + \sigma_3) = \sigma_t, \text{ when } \sigma_2 \leq \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \tag{1}$$

$$f' = \frac{1}{1+b}(\sigma_1 + b\sigma_2) - \alpha\sigma_3 = \sigma_t, \text{ when } \sigma_2 \geq \frac{\sigma_1 + \alpha\sigma_3}{1+\alpha} \tag{2}$$

where σ_1, σ_2, σ_3 are the first, second and third principal stresses respectively; the parameter $\alpha = \frac{\sigma_t}{\sigma_C}$ denotes the ratio of the tensile strength limit of materials (σ_t) and the compressive strength limit (σ_C), and its value is not greater than 1; the parameter b denotes the intermediate stress influence coefficient, and it can be expressed by the material strength index as $b = \frac{(1+\alpha)\tau_b - \sigma_t}{\sigma_t - \tau_b}$, and its value is not greater than 1. When the thick-walled cylinder is under internal pressure, it is subject to radial compressive stress (σ_ρ), tangential tensile stress (σ_θ) and axial stress (σ_Z), and the value of σ_Z is affected by the state of stress.

3. **Elastic-Plastic Analysis of Thick-Walled Cylinder.** As the internal pressure increases, various components of stress of the thick-walled cylinder increase, and when the equivalent stress reaches a critical value, the internal wall of the cylinder will start yielding, i.e., in the elastic limit state; as the internal pressure continues to rise, a plastic

region is formed on the internal wall and is increasing, while the elastic region is reduced accordingly, and when the plastic region extends to all areas of the cylinder, it is in the state of plastic limit. When the thick-walled cylinder is in an elastic state, the stress method or displacement method can be employed to find the solution; when some regions of the cylinder are in a plastic state, equilibrium equation, geometric relationships and plastic constitutive relations shall be met in the plastic region, the elastic problem shall be solved in the elastic region, and the continuous conditions of normal displacement and stress shall be met on the elastic-plastic interface; when the cylinder reaches the state of plastic limit, its load-bearing capacity reaches the critical value, i.e., it cannot continue to bear loads, and the elastic-plastic analysis method or the plastic limit analysis method can be used [18]. Typically, the von Mises yield criterion and the Tresca yield criterion are adopted for elastic-plastic analysis, and in this paper, the impact of T-C anisotropy and intermediate principal stress of materials is considered and the UST is used for the elastic-plastic analysis of the thick-walled cylinder under internal pressure.

3.1. **Elastic limit analysis of thick-walled cylinder.** Suppose the inside diameter of a thick-walled cylinder is r and the outside diameter is R. When the internal pressure (p) is relatively small, the thick-walled cylinder is in an elastic state, and according to Lamé Equations [18], the three principal stresses (hoop stress σ_θ, radial stress σ_ρ and axial stress σ_Z) of the thick-walled cylinder can be expresses as

$$\sigma_\rho = \frac{r^2 p}{R^2 - r^2}\left(1 - \frac{R^2}{\rho^2}\right) \tag{3}$$

$$\sigma_\theta = \frac{r^2 p}{R^2 - r^2}\left(1 + \frac{R^2}{\rho^2}\right) \tag{4}$$

$$\sigma_Z = m\left(\sigma_\theta + \sigma_\rho\right) \tag{5}$$

where p is internal pressure, $\sigma_1 = \sigma_\theta$, $\sigma_2 = \sigma_Z$, $\sigma_3 = \sigma_\rho$, and the value of σ_Z shall consider the stress state of the structure: when the structure is in the state of plane stress, the value will be $m = 0$; when the structure is in the state of plane strain and also in an elastic state, the value will be $m = \nu < 0.5$; when the structure is in the plastic deformation region, the value will be $m = 0.5$ on the basis that the structural volume is kept unchanged. According to $\alpha \leq 1$, the following solution can be found

$$\sigma_Z = m(\sigma_\theta + \sigma_\rho) \leq \frac{\sigma_\theta + \alpha\sigma_\rho}{1 + \alpha} \tag{6}$$

Substituting Formula (6) into Formula (1) can derive the yield condition as follows:

$$\frac{1 + b - mb\alpha}{1 + b}\sigma_\theta - \frac{mb\alpha + \alpha}{1 + b}\sigma_\rho = \sigma_S \tag{7}$$

When the thick-walled cylinder reaches its elastic limit and the internal wall just starts yielding, $\sigma_{\theta|\rho=r}$ and $\sigma_{\rho|\rho=r}$ are substituted into Formula (7) to get

$$\left[\frac{(1 + b - mb\alpha)\left(R^2 + r^2\right)}{(1 + b)\left(R^2 - r^2\right)} + \frac{mb\alpha + \alpha}{1 + b}\right] P_e = \sigma_S \tag{8}$$

Find the solution as follows:

$$P_e = \frac{(1 + b)\left(R^2 - r^2\right)\sigma_S}{(1 + b + \alpha)R^2 + (1 + b - 2mb\alpha - \alpha)r^2} \tag{9}$$

When P_e is the elastic limit internal pressure, the thick-walled cylinder starts yielding, and the materials used for internal walls start to undergo plastic deformation, the value (m) can be taken as 0.5 ($m = 0.5$). If $\alpha = 1$, $b = 0$, $P_e = \frac{\left(R^2 - r^2\right)\sigma_S}{2R^2}$ will be obtained, which is exactly equal to the elastic limit pressure obtained by employing the Tresca yield criterion; if $\alpha = 1$, $b = 1$, $P_e = \frac{2\left(R^2-r^2\right)\sigma_S}{3R^2}$ will be obtained in the state of plane stress, which is exactly equal to the elastic limit pressure obtained by using the twin-shear yield criterion in the state of plane strain; if $\alpha = 1$, $b = \frac{1}{1+\sqrt{3}}$, $P_e = \frac{\left(R^2-r^2\right)\sigma_S}{\sqrt{3}R^2}$ will be obtained, which is exactly equal to the elastic limit pressure obtained by using the von Mises yield criterion [18].

3.2. **Plastic limit analysis of thick-walled cylinder.** If $P < P_e$, the thick-walled cylinder will be in an elastic state; if $P > P_e$, a plastic region will occur around the internal wall of the cylinder, and as internal pressure rises, the plastic region expands outwards, while the elastic region still occurs around the external wall. Due to the axial symmetry of σ_θ and σ_ρ, the interface between the plastic region and the elastic region is a cylindrical surface, as shown in Figure 1.

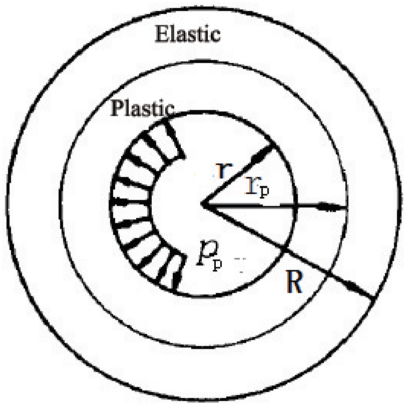

FIGURE 1. Elastic-plastic regions of thick-walled cylinder

As shown in Figure 1, when the cylinder body is in an elastic-plastic state, the internal pressure is P_p and the radius of the elastic-plastic interface is r_p, and two deformation regions are separately considered and can be separately discussed as two thick-walled cylinders. In view of axial symmetry, the radial pressure (q) is separately applied to the external wall of the internal cylinder (in the plastic region) and the internal wall of the external cylinder (in the elastic region). In the plastic region, if $m = 0.5$, the intermediate principal stress is $\sigma_Z = \frac{\sigma_\theta+\sigma_\rho}{2}$ according to the principle that the volume is kept unchanged, and the UST and the simultaneous equations involving equilibrium equation are utilized:

$$\frac{d\sigma_\rho}{d\rho} + \frac{\sigma_\rho - \sigma_\theta}{\rho} = 0 \tag{10}$$

$$\sigma_\theta - \frac{\alpha}{1+b}\left(b\frac{\sigma_\theta + \sigma_\rho}{2} + \sigma_\rho\right) = \sigma_S \tag{11}$$

Use solution formulas (10) and (11) to get

$$\sigma_\rho = \frac{1}{1-\alpha}\sigma_S + C\rho^{\left(\frac{2b\alpha+2\alpha-2-2b}{2+2b-b\alpha}\right)} \tag{12}$$

If the boundary condition of σ_ρ is $\sigma_{\rho|\rho=r} = -P_p$, and the parameter of C satisfying that $C = -r^{\left(\frac{2+2b-2b\alpha-2\alpha}{2+2b-b\alpha}\right)}\left[P_p + \frac{\sigma_S}{1-\alpha}\right]$, then Formulas (12) and (11) will be rewritten as

$$\sigma_\rho = \frac{\sigma_S}{1-\alpha} - \left(\frac{r}{\rho}\right)^{\left(\frac{2+2b-2b\alpha-2\alpha}{2+2b-b\alpha}\right)}\left(P_p + \frac{\sigma_S}{1-\alpha}\right) \tag{13}$$

$$\sigma_\theta = \frac{1}{1-\alpha}\sigma_S - \left(\frac{2\alpha + b\alpha}{2+2b-b\alpha}\right)\left(\frac{r}{\rho}\right)^{\left(\frac{2+2b-2b\alpha-2\alpha}{2+2b-b\alpha}\right)}\left(P_p + \frac{\sigma_S}{1-\alpha}\right) \tag{14}$$

In the elastic region, i.e., the external cylinder of $r_p \leq \rho \leq R$, when yielding just starts at the inside diameter of $\rho = r_p$, the external cylinder can be deemed to reach the elastic limit, and use Formula (9) to get

$$q = \frac{(1+b)\left(R^2 - r_p^2\right)\sigma_S}{(1+b+\alpha)R^2 + (1+b-2mb\alpha-\alpha)r_p^2} \tag{15}$$

In the plastic region, i.e., $r \leq \rho \leq r_p$, when $\sigma_{\rho|\rho=r_p} = -q$ occurs at $\rho = r_p$, use Formula (13) to get

$$q = \left(\frac{r}{r_p}\right)^{\left(\frac{2+2b-2b\alpha-2\alpha}{2+2b-b\alpha}\right)}\left(P_p + \frac{\sigma_S}{1-\alpha}\right) - \frac{\sigma_S}{1-\alpha} \tag{16}$$

Based on the continuity of components of stress, the radial stresses (q) are equal in the elastic region and plastic region at $\rho = r_p$, and derive

$$\begin{aligned} P_p = & \frac{(1+b)\left(R^2 - r_p^2\right)\sigma_S}{r_p^2(1+b-b\alpha-\alpha) + R^2(1+b+\alpha)}\left(\frac{r}{r_p}\right)^{\left(\frac{2b\alpha+2\alpha-2-2b}{2+2b-b\alpha}\right)} \\ & + \left(\frac{\sigma_S}{1-\alpha}\right)\left[\left(\frac{r}{r_p}\right)^{\left(\frac{2b\alpha+2\alpha-2-2b}{2+2b-b\alpha}\right)} - 1\right] \end{aligned} \tag{17}$$

Formula (17) is a computational formula used to determine the corresponding internal pressure (P_p) when the elastic-plastic interface radius (r_p) is given. As internal pressure increases, the plastic region is expanding, and if $r_p = R$, the corresponding internal pressure is called the plastic limit pressure, and P_l is expressed as

$$P_l = \left(\frac{\sigma_S}{1-\alpha}\right)\left[\left(\frac{R}{r}\right)^{\left(\frac{2b+2-2\alpha-2b\alpha}{2+2b-b\alpha}\right)} - 1\right] \tag{18}$$

If $\alpha \to 1$, $\lim_{\alpha\to 1} P_l = \lim_{\alpha\to 1}\frac{\sigma_S}{1-\alpha}\left(\frac{2(1+b)(1-\alpha)}{2+2b-b\alpha}\right)\ln\frac{R}{r} = \frac{2+2b}{2+b}\sigma_S\ln\frac{R}{r}$ is obtained, and the following results can be worked out by using different b values: if $b = 0$, $\lim_{\alpha\to 1, b=0} P_l = \sigma_S \ln\frac{R}{r}$ is exactly equal to the plastic limit pressure calculated by using the Tresca yield criterion; if $b = 1$, $\lim_{\alpha\to 1, b=1} P_l = \frac{4}{3}\sigma_S\ln\frac{R}{r}$ is exactly equal to the plastic limit pressure calculated by using the double-shear yield criterion; if $b = \frac{1}{1+\sqrt{3}}$, $\lim_{\alpha\to 1, b=\frac{1}{1+\sqrt{3}}} P_l = \frac{2}{\sqrt{3}}\sigma_S\ln\frac{R}{r}$ is obtained, which is exactly equal to the plastic limit internal pressure [18] calculated by using the von Mises yield criterion.

3.3. **Stability analysis of thick-walled cylinder.** As shown in Figure 1, if the internal pressure applied to the thick-walled cylinder exceeds the elastic limit, the plastic region on the internal wall and the elastic region on the external wall will be formed. If the internal pressure is unloaded until it is zero, the elastic region on the external wall will rebound on and compress the plastic region on the internal wall, so the plastic region withstands the residual hoop compressive stress and the elastic region withstands the residual hoop tensile stress. The plastic region stress minus the elastic stress unloaded is the residual stress on the plastic region, i.e., Formulas (13) and (14) minus Formulas (3) and (4) to calculate the residual stress on the plastic region, as follows:

$$\sigma_\rho^r = \frac{\sigma_S}{1-\alpha} - \left(\frac{r}{\rho}\right)^{\left(\frac{2+2b-2b\alpha-2\alpha}{2+2b-b\alpha}\right)}\left(P_p + \frac{\sigma_S}{1-\alpha}\right) - \frac{r^2 P_p}{R^2 - r^2}\left(1 - \frac{R^2}{\rho^2}\right) \tag{19}$$

$$\sigma_\theta^r = \frac{\sigma_S}{1-\alpha} - \left(\frac{2\alpha+b\alpha}{2+2b-b\alpha}\right)\left(\frac{r}{\rho}\right)^{\left(\frac{2+2b-2b\alpha-2\alpha}{2+2b-b\alpha}\right)}\left(P_p + \frac{\sigma_S}{1-\alpha}\right) - \frac{r^2 P_p}{R^2-r^2}\left(1+\frac{R^2}{\rho^2}\right) \quad (20)$$

After unloading, if the residual radial stress is $\sigma_\rho^r = 0$ on the internal wall of the thick-walled cylinder, the hoop stress will be

$$\sigma_\theta^r = \frac{(2+2b)\sigma_S}{2+2b-b\alpha} - \frac{2\alpha+b\alpha}{2+2b-b\alpha}P_p - \frac{R^2+r^2}{R^2-r^2}P_p \quad (21)$$

According to Formula (21), if $\sigma_\theta^r < 0$, $\sigma_1 = \sigma_r^r = 0$, $\sigma_2 = \sigma_Z = \frac{1}{2}\sigma_\theta^r$, $\sigma_3 = \sigma_\theta^r$, satisfying $\sigma_2 \le \frac{\sigma_1+\alpha\sigma_3}{1+\alpha}$. Substitute the residual stress into Formula (1) to derive the limit internal pressure of the thick-walled cylinder under residual stress without yielding:

$$P_t = \frac{2(1+b)(1+\alpha+b)}{2\alpha+b\alpha}\frac{\left(R^2-r^2\right)\sigma_S}{(1+b+\alpha)R^2+(1+b-b\alpha-\alpha)r^2} \quad (22)$$

In case of loading, the internal pressure of stability of the cylinder cannot make the cylinder reach its plastic limit state, and meanwhile, the internal wall of the cylinder cannot be in the state of reverse yielding in case of unloading, so the stability limit internal pressure of the thick-walled cylinder is

$$P_n = \min\{P_l, P_t\} \quad (23)$$

4. **Elastic-Plastic Analysis of Double-Layered Thick-Walled Cylinder.** It can be seen from the analysis of Formulas (9) and (18) that, if the inside diameter (r) is fixed, the elastic limit (P_e) increases as the outside diameter (R) increases, and if the outside diameter is ($R \to \infty$), $P_e \to \frac{2\sigma_S}{3}$ ($\alpha = 1$, $b = 1$), so the effect of improving P_e by increasing the wall thickness is limited; if the inside diameter (r) is fixed, the plastic limit pressure (P_l) can be continuously increased with an increase in the outside diameter (R), so the effect of improving P_l by increasing the wall thickness is significant, but in the actual construction, the outside diameter of the cylinder may be limited, and P_l cannot be improved simply by increasing the wall thickness. If more stringent requirements are set for machining precision, the structure shall be strictly controlled within the range of elastic deformation state, and the adoption of a combined thick-walled cylinder is an available means, and the stress distribution can be adjusted and the strength potential of each layer of thick-walled cylinder materials can be fully utilized to enhance the load-bearing capacity of the structure.

4.1. **Elastic limit analysis of double-layered thick-walled cylinder.** When the internal pressure is applied to the combined thick-walled cylinder, the internal walls of the internal and external cylinder bodies yield at the same time, so that the strength potential of materials can be fully tapped, i.e., theoretically optimal structure, to get

$$\frac{1+b-mb\alpha_1}{1+b}\left[\frac{\left(r_1^2+r^2\right)P_e - 2r_1^2 q}{\left(r_1^2-r^2\right)}\right] + \frac{mb\alpha_1+\alpha_1}{1+b}P_e = \sigma_{S1} \quad (24)$$

$$\frac{1+b-mb\alpha_2}{1+b}\frac{R^2+r_1^2}{R^2-r_1^2}q + \frac{mb\alpha_2+\alpha_2}{1+b}q = \sigma_{S2} \quad (25)$$

In Formulas (24) and (25), R and r refer to the inside and outside diameters of the combined cylinder bodies, respectively, and r_1 is the layer radius. To simplify the expression of derivation process, the following parameters are set:

$$A_1 = 1+b-mb\alpha_1, \quad B_1 = 1+b+\alpha_1, \quad C_1 = 1+b-2mb\alpha_1-\alpha_1$$
$$B_2 = 1+b+\alpha_2, \quad C_2 = 1+b-2mb\alpha_2-\alpha_2$$

Formulas (24) and (25) are used to derive the compressive stress of internal walls of the internal and external cylinder bodies in case of structural yielding, as follows:

$$P_e = \frac{2A_1(1+b)\left(R^2 - r_1^2\right) r_1^2}{\left(B_1 r_1^2 + C_1 r^2\right)\left(B_2 R^2 + C_2 r_1^2\right)} \sigma_{S2} + \frac{(1+b)\left(r_1^2 - r^2\right)}{\left(B_1 r_1^2 + C_1 r^2\right)} \sigma_{S1} \tag{26}$$

$$q = \frac{(1+b)\left(R^2 - r_1^2\right)}{B_2 R^2 + C_2 r_1^2} \sigma_{S2} \tag{27}$$

It can be seen from Formula (26) that the internal pressure (P_e) applied to the internal walls of the combined cylinder bodies is the function of the layer radius (r_1), and if $\frac{\partial P_e}{\partial r_1} = 0$, the elastic limit of the combined thick-walled cylinder reaches its maximum value, and the layer radius (r_1) satisfies the following equation:

$$a r_1^4 + b r_1^2 + c = 0 \tag{28}$$

where

$$a = 4A_1\sigma_{S2}\left(B_1B_2R^2 + B_1C_2R^2 + C_1C_2r^2\right) - 2(B_1 + C_1)C_2^2\sigma_{S1}r^2$$
$$b = 8A_1B_2C_1\sigma_{S2}R^2r^2 - 4B_2C_2(B_1 + C_1)\sigma_{S1}R^2r^2$$
$$c = -4A_1B_2C_1\sigma_{S2}R^4r^2 - 2B_2^2(B_1 + C_1)\sigma_{S1}R^4r^2$$

If the impact of T-C anisotropy and intermediate principal stress of materials is not considered, $\alpha_1 = \alpha_2 = 1$ and $b = 0$, and the expression used for deriving the layer radius is

$$r_1 = \sqrt[4]{\frac{\sigma_{S1}}{\sigma_{S2}}} \times \sqrt{Rr} \tag{29}$$

In this case, the UST is degraded to the Tresca yield criterion, and when the same materials are used in the internal and external layers of the cylinder, the layer radius is calculated as follows:

$$r_1 = \sqrt{Rr} \tag{30}$$

It is consistent with the conclusions stated in [15,18,19].

Formulas (26)-(28) are used to derive the computation expressions of the elastic limit internal pressure (P_e) of the combined thick-walled cylinder, the compressive stress (q) between the internal and external layers of the cylinder and the layer radius (r_1), and according to the principle of superposition in stress,

$$q = P_1 + P_{1e} \tag{31}$$

In Formula (31), P_1 is the surface pressure of the cylinder after covering the external layer on the internal layer, and P_{1e} is the compressive stress between the internal and external layers of the cylinder under the internal working pressure (P_e). When the internal working pressure (P_e) functions, the internal layer of the thick-walled cylinder is affected by the internal working pressure (P_e) and the compressive stress of external surface (P_{1e}), and the external layer of the thick-walled cylinder is affected by the internal wall pressure (P_{1e}), and if the deformation of internal and external cylinder bodies is the same at the layer radius, the following equation is satisfied:

$$\begin{aligned} &\frac{1}{E_2}\left[(1+\nu_2)\frac{R^2 r_1 P_{1e}}{R^2 - r_1^2} + (1-\nu_2)\frac{r_1^3 P_{1e}}{R^2 - r_1^2}\right] \\ &= \frac{1}{E_1}\left[-(1+\nu_1)\frac{r^2 r_1\left(P_{1e} - P_e\right)}{r_1^2 - r^2} + (1-\nu_1)\frac{r^2 r_1 P_e - r_1^3 P_{1e}}{r_1^2 - r^2}\right] \end{aligned} \tag{32}$$

where ν_1 and ν_2 refer to the Poisson's ratio of materials used for internal and external cylinder bodies respectively, and Formula (32) is used to derive P_{1e}:

$$P_{1e} = \frac{2E_2\left(R^2 - r_1^2\right) r^2 r_1 P_e}{E_1\left[(1+\nu_2) r_1 R^2 + (1-\nu_2) r_1^3\right]\left(r_1^2 - r^2\right) + E_2\left[(1+\nu_1) r_1 r^2 + (1-\nu_1) r_1^3\right]\left(R^2 - r_1^2\right)} \tag{33}$$

Formulas (27) and (33) are used to separately calculate q and P_{1e}, then Formula (31) is substituted to calculate the surface pressure of the cylinder after covering the external layer on the internal layer (P_1), and finally the magnitude of interference between the internal and external layers of the cylinder can be derived as follows:

$$\delta = \frac{P_1 r_1}{E_1}\left(\frac{r_1^2 + r^2}{r_1^2 - r^2} - \nu_1\right) + \frac{P_1 r_1}{E_2}\left(\frac{R^2 + r_1^2}{R^2 - r_1^2} + \nu_2\right) \tag{34}$$

4.2. **Plastic limit analysis of double-layered thick-walled cylinder.** When the internal pressure is applied to the combined thick-walled cylinder, the internal and external cylinder bodies reach the plastic limit at the same time, so that the strength potential of materials can be maximized, i.e., plastic limit of the structure. For the combined cylinder body, the inside diameter is expressed as r, the outside diameter is expressed as R, and the layer radius is expressed as r_1; the pressure between the internal and external layers of the cylinder is expressed as q, and the internal cylinder body is affected by plastic limit and the pressure of $P_l(q)$, and Formula (12) is used to get

$$-P_l = \frac{\sigma_{S1}}{1-\alpha_1} + C_1 r^{\left(\frac{2b\alpha_1 + 2\alpha_1 - 2 - 2b}{2 + 2b - b\alpha_1}\right)} \tag{35}$$

$$-q = \frac{\sigma_{S1}}{1-\alpha_1} + C_1 r_1^{\left(\frac{2b\alpha_1 + 2\alpha_1 - 2 - 2b}{2 + 2b - b\alpha_1}\right)} \tag{36}$$

And the plastic limit of the external cylinder body is expressed as q, and Formula (18) is used to get

$$q = \frac{\sigma_{S2}}{1-\alpha_2}\left(\left(\frac{r_1}{R}\right)^{\left(\frac{2b\alpha_2 + 2\alpha_2 - 2 - 2b}{2 + 2b - b\alpha_2}\right)} - 1\right) \tag{37}$$

Use solution formulas (35)-(37) to get

$$P_l = \left[\left(\frac{\sigma_{S2}}{1-\alpha_2}\right)\left(\frac{r_1}{R}\right)^{\left(\frac{2b\alpha_2 + 2\alpha_2 - 2 - 2b}{2 + 2b - b\alpha_2}\right)} + \frac{\sigma_{S1}}{1-\alpha_1} - \frac{\sigma_{S2}}{1-\alpha_2}\right]\left(\frac{r}{r_1}\right)^{\left(\frac{2b\alpha_1 + 2\alpha_1 - 2 - 2b}{2 + 2b - b\alpha_1}\right)} - \frac{\sigma_{S1}}{1-\alpha_1} \tag{38}$$

The plastic limit of the combined cylinder body (P_l) is the function of the layer radius (r_1), and if $\frac{\partial P_l}{\partial r_1} = 0$, the maximum plastic limit of structure (P_l) will be used to get

$$r_1 = \sqrt[m_2]{\frac{m_1\left(\sigma_{S1} + \alpha_1\sigma_{S2} - \sigma_{S2} - \alpha_2\sigma_{S1}\right)}{(1-\alpha_1)(m_2 - m_1)\sigma_{S2}}}R \tag{39}$$

where $m_1 = \frac{2b\alpha_1 + 2\alpha_1 - 2 - 2b}{2 + 2b - b\alpha_1}$ and $m_2 = \frac{2b\alpha_2 + 2\alpha_2 - 2 - 2b}{2 + 2b - b\alpha_2}$, and if the T-C strength anisotropy of materials is not considered and $\alpha_1 \to 1$ and $\alpha_2 \to 1$ are used, the yield limit of the combined cylinder body is as follows:

$$P_l = \frac{2+2b}{2+b}\left(\sigma_{S1}\ln\frac{r_1}{r} + \sigma_{S2}\ln\frac{R}{r_1}\right) \tag{40}$$

In this case, there is not optimal layer radius (r_1), and the plastic limit of the combined cylinder is exactly the sum of plastic limits of the internal and external cylinder bodies [16], and if the same materials are used by the internal and external cylinder bodies,

$$P_l = \frac{2+2b}{2+b}\sigma_S \ln\frac{R}{r} \tag{41}$$

In this case, the layer radius makes no sense, and the plastic limit of the combined cylinder body depends only on the inside and outside diameters and the yield strength of materials, and the surface pressure of the cylinder after covering the external layer on the internal layer has no impact on the plastic limit of structures.

5. **Case Analysis.** A thick-walled cylinder of inner diameter $r = 50$ mm and outer diameter $R = 80$ mm operates under the internal pressure of 650 MPa, and the yield strength of cylinder material is $\sigma_S = 1470$ MPa, the elastic modulus (E) is 210 GPa and Poisson's ratio (ν) is 0.3. The formulas derived in this paper will be used for the strength analysis of this thick-walled cylinder hereunder.

5.1. **Elastic limit, plastic limit and stability limit analysis.** First of all, the von Mises strength theory is used to get the tension-compression strength ratio of materials $\alpha = 1$ and the intermediate stress influence coefficient $b = \frac{1}{1+\sqrt{3}}$, and Formulas (9), (18) and (23) are used to separately calculate the elastic limit ($P_e = 517.2$ MPa), plastic limit ($P_l = 797.8$ MPa) and stability limit ($P_n = 797.8$ MPa) of the cylinder, which are consistent with those stated in [7]. If the diameter ratio of the cylinder is $\frac{R}{r} = 2 \rightarrow 6$, the elastic limit, plastic limit and stability limit of the cylinder are separately calculated, and the results are shown in Table 1.

Table 1. Changes of the elastic limit, plastic limit and stability limit of the cylinder with the diameter ratio (MPa)

Limit internal pressure	$R/r = 2$	$R/r = 3$	$R/r = 4$	$R/r = 5$	$R/r = 6$
P_e	636.5	754.4	795.7	814.8	825.1
P_l	1176.6	1864.8	2353.1	2731.9	3041.3
P_n	1176.6	1508.8	1591.3	1629.5	1650.3

It can be seen from Table 1 that the elastic limit (P_e) is increased as the diameter ratio increases, but the lifting rate is rapidly reduced; the plastic limit (P_l) is increased as the diameter ratio increases, but the lifting rate is slowly reduced; the stability limit (P_n) is increased as the diameter ratio increases, but the lifting rate is rapidly reduced. If the diameter ratio is relatively small, the stability limit will be determined by the plastic limit; as the diameter ratio increases, the stability limit of the cylinder will be determined by P_t which is derived from Formula (22); through calculation, if diameter ratio of the $\frac{R}{r} = 2.218$, the plastic limit of the cylinder (P_l) is exactly equal to P_t, and as $\frac{R}{r}$ increases, the plastic limit (P_l) will be greater than P_t.

If the impact of the T-C anisotropy of materials ($\alpha = 1$) is not considered, the selection of intermediate principal stress (b) will have a significant impact on the elastic limit, plastic limit and stability limit of the cylinder, and the results are shown in Tables 2-4. If b is increased from 0 to 1, each of elastic limit, plastic limit and stability limit will be increased by about $\frac{1}{3}$.

If the T-C anisotropy of cylinder materials is considered, the elastic limit and plastic limit internal pressures will be increased, and if the T-C anisotropy coefficient is $\alpha = 0.85$, the elastic limit and plastic limit internal pressures will be increased, as shown in Figures 2 and 3: the elastic limit internal pressure can be increased by about 8% at most and the plastic limit internal pressure by about 15% at most; the larger the diameter ratio is, the higher the lifting rates of elastic limit and plastic limit are; the smaller b is, the higher the lifting rates of elastic limit and plastic limit are.

TABLE 2. Changes of the elastic limit of the cylinder with the parameter (b)

R/r	$b = 0$	$b = 0.25$	$b = 0.5$	$b = 0.75$	$b = 1$
2	551.2	612.5	661.5	701.6	735
3	653.3	725.9	784	831.5	871.1
4	689.1	765.6	826.9	877	918.8
5	705.6	784	846.7	898	940.8
6	714.6	794	857.5	909.5	952.8

TABLE 3. Changes of the plastic limit of the cylinder with the parameter (b)

R/r	$b = 0$	$b = 0.25$	$b = 0.5$	$b = 0.75$	$b = 1$
2	1018.9	1132.1	1222.7	1296.8	1358.6
3	1615	1794.4	1938	2055.4	2153.3
4	2037.9	2264.3	2445.4	2593.6	2717.1
5	2365.8	2628.8	2839.1	3011.1	3154.5
6	2633.9	2926.5	3160.7	3352.2	3511.9

TABLE 4. Changes of the stability limit of the cylinder with the parameter (b)

R/r	$b = 0$	$b = 0.25$	$b = 0.5$	$b = 0.75$	$b = 1$
2	1018.9	1132.1	1222.7	1296.8	1358.6
3	1306.7	1451.9	1568	1663	1742.2
4	1378.1	1531.3	1653.8	1754	1837.5
5	1411.2	1568	1693.4	1796.1	1881.6
6	1429.2	1588	1715	1818.9	1905.6

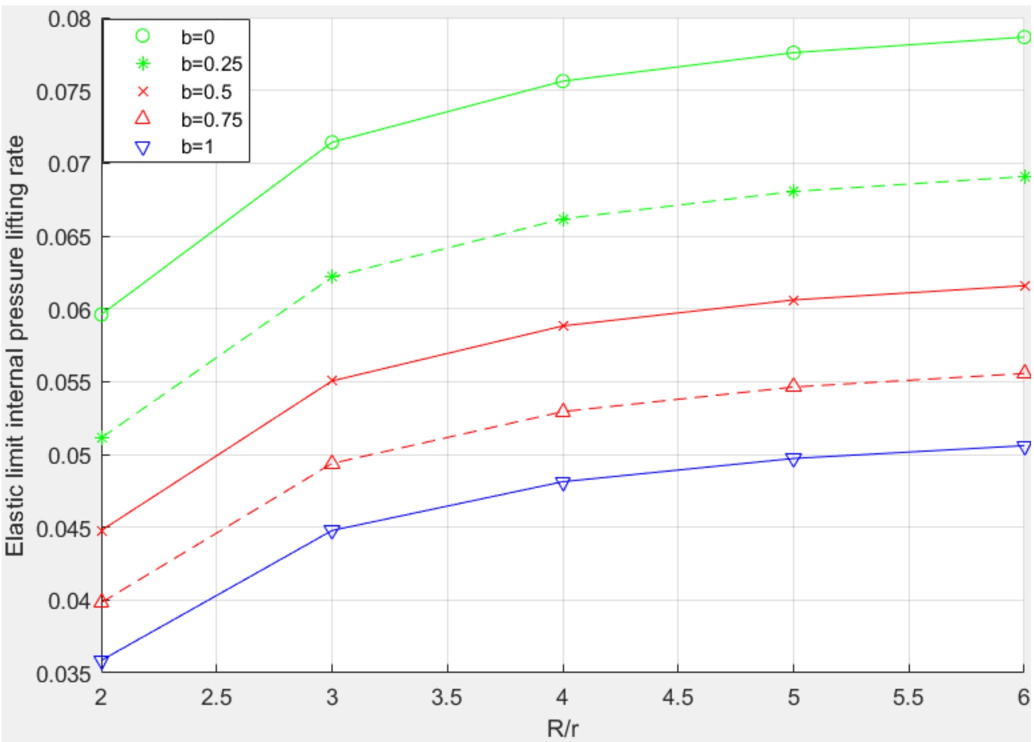

FIGURE 2. Lifting rate of the elastic limit of the thick-walled cylinder in case of $\alpha = 0.85$

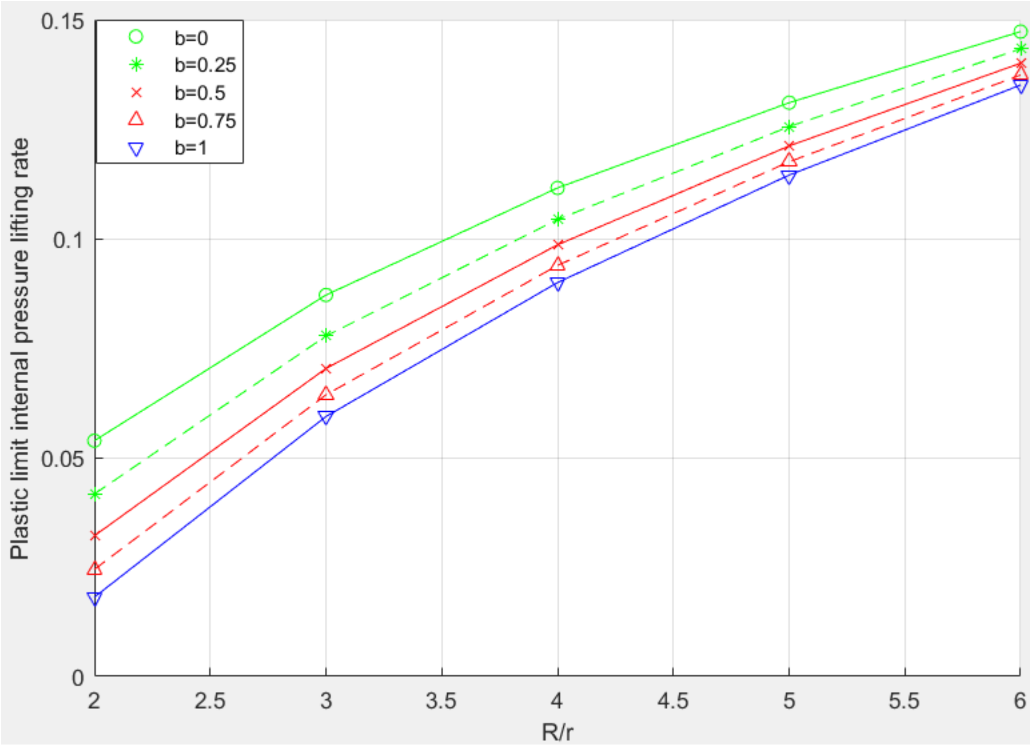

FIGURE 3. Lifting rate of the plastic limit of the thick-walled cylinder in case of $\alpha = 0.85$

5.2. **Analysis of elastic limit and plastic limit of the double-layered cylinder.** A double-layered cylinder is used to improve the elastic limit and plastic limit of the cylinder structure, the structural materials used in the internal and external layers of the cylinder are kept unchanged, and the diameter ratio is $\frac{R}{r} = 2 \to 6$. First of all, if the T-C anisotropy is not considered and the coefficient of intermediate principal stress is $b = 0 \to 1$, the elastic limit of the combined cylinder will be significantly increased (as shown in Table 5), while the plastic limit is not changed. It can be seen from Table 5 that the larger the coefficient of intermediate principal stress (b) and diameter ratio ($\frac{R}{r}$), the higher the lifting rate of the combined cylinder structure, with the maximum lifting rate of 0.65 times.

TABLE 5. Lifting rates of plastic limit of the combined cylinder

R/r	$b = 0$	$b = 0.25$	$b = 0.5$	$b = 0.75$	$b = 1$
2	0.26	0.27	0.27	0.28	0.28
3	0.40	0.41	0.42	0.43	0.44
4	0.49	0.50	0.52	0.53	0.54
5	0.55	0.57	0.58	0.59	0.60
6	0.59	0.61	0.63	0.64	0.65

If the T-C anisotropy of the combined cylinder materials is considered, the elastic limit and plastic limit will be increased, and if the T-C anisotropy coefficient is $\alpha = 0.85$, the elastic limit and plastic limit will be increased, as shown in Table 6: the elastic limit can be increased by about 10.2%; the larger the diameter ratio ($\frac{R}{r}$) is or the smaller b is, the higher the lifting rate of elastic limit is. The increase in the plastic limit of the combined cylinder is completely consistent with that of the thick-walled cylinder (as shown in Figure 3).

TABLE 6. Lifting rates of elastic limit of the combined cylinder in case of $\alpha = 0.85$

R/r	$b = 0$	$b = 0.25$	$b = 0.5$	$b = 0.75$	$b = 1$
2	0.059	0.048	0.039	0.032	0.026
3	0.080	0.070	0.062	0.056	0.051
4	0.091	0.082	0.074	0.068	0.063
5	0.098	0.089	0.082	0.076	0.071
6	0.102	0.094	0.087	0.081	0.076

6. **Conclusions.** In this paper, the elastic-plastic analysis of the thick-walled cylinder structure was conducted as per the UST to get the unified solution of the elastic limit, plastic limit and stability limit internal pressures of the thick-walled cylinder and the elastic limit and plastic limit internal pressures of the double-layered cylinder, and the particular values of α and b are used to degrade the unified solution to the particular solution based on the traditional strength theory (e.g., Tresca yield criterion and von Mises yield criterion). Through the above-mentioned case analysis, the impact of three factors of materials (T-C anisotropy coefficient (α), intermediate principal stress coefficient (b) and diameter ratio ($\frac{R}{r}$)) on the elastic limit and plastic limit internal pressures of the cylinder is discussed in this paper as follows.

1) Based on the UST, the unified solution of the elastic limit, plastic limit and stability limit internal pressures of the thick-walled cylinder and the elastic limit and plastic limit internal pressures of the double-layered cylinder is derived in this paper after the impact of T-C anisotropy and intermediate principal stress are considered, and the particular values of α and b are used to degrade the unified solution to the analytical solution based on the traditional strength theory.

2) The elastic limit, plastic limit and stability limit internal pressures of the thick-walled cylinder will increase as the T-C anisotropy coefficient (α) decreases, and the plastic limit internal pressure has a higher lifting rate; the elastic limit and plastic limit internal pressures will be reduced as b increases and the internal pressure will be improved as the diameter ratio increases.

3) The von Mises yield criterion is used, and if $\frac{R}{r} \leq 2.218$, $P_l = P_n$, and if $\frac{R}{r} > 2.218$, $P_l > P_n$; as the T-C anisotropy coefficient (α) of materials is reduced, the maximum diameter ratio ($\frac{R}{r}$) satisfying $P_l = P_n$ will be increased, and the results are shown in Table 7.

TABLE 7. Changes of $\left(\frac{R}{r}\right)_{\max}$ with the parameter (α) in case of $P_l = P_n$

$\alpha = 1$	$\left(\frac{R}{r}\right)_{\max} = 2.218$
$\alpha = 0.95$	$\left(\frac{R}{r}\right)_{\max} = 2.308$
$\alpha = 0.90$	$\left(\frac{R}{r}\right)_{\max} = 2.406$
$\alpha = 0.85$	$\left(\frac{R}{r}\right)_{\max} = 2.516$
$\alpha = 0.80$	$\left(\frac{R}{r}\right)_{\max} = 2.638$
$\alpha = 0.75$	$\left(\frac{R}{r}\right)_{\max} = 2.774$
$\alpha = 0.70$	$\left(\frac{R}{r}\right)_{\max} = 2.928$

4) The surface pre-pressure of the combined cylinder structure after covering the external layer on the internal layer can be used to significantly increase the elastic limit internal pressure, but it does not have a significant impact on the plastic limit internal pressure.

Based on the UST and on the assumption that materials comply with the ideal elastic-plastic model, the unified solution of the limit internal pressure of thick-walled cylinder and double-layered cylinder is derived in this paper, which is the general solution for general materials, and if strain hardening and the Bauschinger effect are to be taken into account, further study will be required on this basis; meanwhile, if the internal wall of the cylinder is prolate or oblate or in the shape similar to a standard circle, further study will also be required for the calculation of its limit internal pressure on the basis of this paper.

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