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ROBUST MODEL PREDICTIVE CONTROL BASED ON SIGN-PERTURBED SUMS STRATEGY

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Abstract. *This paper studies the problem of robust model predictive control, whose state variable in its quadratic penalty function is related with an unknown parameter vector. After formulating the open loop system and closed loop system into their corresponding linear regressor forms, sign-perturbed sums is proposed to construct one guaranteed confidence region for that unknown parameter vector, while relaxing the strict limitation of the external noise, i.e., no probabilistic density function. Only each noise has a symmetric probability distribution about zero. Consider the controller design and confidence region for the parameter vector in model predictive control simultaneously, then robust model predictive control corresponding to one min-max optimization problem, whose max operation means the worst case performance. Generally, robustness is from the uncertainty of the parameter vector, as here this parameter vector is not a scalar vector, but one guaranteed confidence region, which includes the parameter vector with one chosen probability level. Finally, one simulation example is used to prove our considered theories.*

Keywords: Robust model predictive control, Sign-perturbed sums, Linear regressor, Probability level

1. **Introduction.** There exists the growing research about data driven method, applied optimization and the classical consideration of engineering mathematics and mathematical physics. Data driven discovery is currently revolutionizing how we model, predict and control complex systems. The most pressing scientific and engineering problems in the modern era are not amenable to empirical models or derivations based on first principles. Therefore, researchers are studying data driven approach for a diverse range of complex systems, such as turbulence, the brain, climate, finance, robotics and autonomy. These systems are typically nonlinear, dynamic, multi scale in space and time, high dimensional, with dominant underlying patters that would be characterized and modeled for the eventual goal of sensing, prediction, estimation and control. With modern mathematical methods, enabled by unprecedented availability of data and computational resources, we are now able to handle some unattainable challenge problems. Driving modern data science is the availability of vast and increasing quantities of data, enabled by remarkable innovations in low cost sensors, orders of magnitudes increases in computational power, such vast quantities of data are affording engineers and scientists across all disciplines new opportunities or data driven discovery, which has been referred to as the fourth paradigm of scientific discovery.

Due to the wide consideration of direct data driven control and the combination with other different control strategies, direct data driven model predictive control was proposed

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to design two degrees of freedom controllers for an unknown plant based on input-output measurement [1]. To relax the strict probabilistic description on disturbance in analyzing stability and robustness, the dissipativity properties were analyzed in the constrained optimal control [2], where a special computational approach was proposed to achieve the robustness guarantees. A data driven method to design reference tracking controllers was introduced for nonlinear system in [3], where it delivered directly a time varying state feedback controller by combining an online and an off-line scheme. In [4], a robust data driven model predictive control was proposed to control linear time invariant systems and behavioral systems theory or past measured trajectories were used as implicit model description. A quasi-infinite horizon nonlinear predictive control was presented for tracking of generic reference trajectories [5], and similarly a nonlinear robust model predictive control was considered for general state and input dependent disturbance [6], which used an online constructed tube to tighten the nominal constraints. As a consequence, based on [7], one linear state space form with control input and input-output noises was considered, not the linear rational function. In case of these measurement noises, described as set membership uncertainty, the idea of dynamic programming is introduced to achieve the goal of state estimation. During this year, our contributions about this direct data driven control are formulated here. [8] combines system identification, direct data driven control and optimal algorithm in designing two controllers for one cascade control system, i.e., the inner controller and outer controller without any knowledge of the unknown plant. Then iterative operation is introduced with direct data driven control to yield the so called iterative data driven control [9], which is benefit for stability analysis in one model matching problem. Data driven model predictive control is considered to adjust the varying coupling conditions among the different parts of the system, and then through adding the inequality constraint to the constructed model predictive control, one persistently exciting data driven model predictive control is obtained in [10]. Generally, our considered model predictive control is one special control strategy for the optimal control problem, while considering state and control input constraints, then it corresponds to one numerical optimization problem with equity and inequity constraints. Consider the solving process for this constraint optimization problem, lots of existing theories can be applied from the point of optimization theory, such as optimization algorithm, dynamic programming strategy, convergence analysis, and stability analysis. To the best of our knowledge, model predictive control provides another efficient way to study the optimal control problem with constraint conditions, as these constraints cannot be dealt with well in classical optimal control problem. The above is also the reason about why model predictive control is now widely studied in research field and practical engineering. Robustness means some uncertainties are considered, for example, deterministic uncertainty or probabilistic uncertainty. Whatever deterministic or probabilistic uncertainty, it will lead to one uncertain model, i.e., the obtained model is included in with one guaranteed probability level. This paper combines model predictive control strategy and robust property to consider the robust model predictive control; furthermore, the uncertain actor is identified by sign-perturbed sums. Sign-perturbed sums constructed non-asymptotic confidence regions under mild statistical assumptions. [11] constructs the finite sample quasi distribution free confidence region, being related with statistical parameter estimation approach. More recently, this method was considered around the EIV system, and asymptotic properties [12]. In practice, due to the fact that sign-perturbed sums is within the finite number of measured data or limited priori information about the considered noise, [13] analyzed this finite sample property for direct data driven control, and constructed one confidence region of the unknown parameter in closed loop system. From our previous contribution [14], we see this sign-perturbed sums can establish the guaranteed, non-asymptotic confidence regions around the well known least squares estimate.

Here in this paper, our contribution is to introduce robust property into the framework of model predictive control, whose state variable in its quadratic cost function is not a scalar value, but one confidence region. It tells us that the state variable is included in this guaranteed confidence region with an exact probabilistic level. According to this confidence region, sign-perturbed sums is applied to constructing it without any strict limitation on external noise. The only assumption on noise is that noise is a sequence of independent random variables and each one has a symmetric probability distribution about zero, then no detailed probability density function is needed in this sign-perturbed sums. Furthermore, we observe that this sign-perturbed sums depends on linear regressor form, so other forms, being not linear regressor form are needed to change or reformulate as their corresponding linear regressor forms. To show the tedious calculation, reformulation of closed loop system is given to certify the immediate modification with our previous proposed tailor made parameterization. Moreover, the main works of this short note are formulated as follows. 1) Firstly, one commonly used linear discrete time parameter dependent dynamics with an additive disturbance is considered, and then the sign-perturbed sums is used to identify the unknown parameter based on our derived linear regressor form. 2) Secondly, we show this sign-perturbed sums can be applied to closed loop system. 3) Thirdly, our interesting robust model predictive control is established, and the idea of dynamic programming is introduced to solve one min-max optimization problem, i.e., robust model predictive control corresponds to one min-max optimization problem. Generally, this paper combines system identification, model predictive control and numerical optimization, which belong to three different research fields. It means that here model predictive control is considered from the different points of system identification and numerical optimization, being related with the direct data driven model predictive control.

This paper is organized as follows. In Section 2, one linear discrete time parameter dependent dynamics with an additive disturbance is considered, and some preliminaries on noise are formulated. In Section 3, reformulation of closed loop system is studied to change as one linear regressor form. Based on this final linear regressor form for that linear dynamic and closed loop system respectively, in Section 4 sign-perturbed sums is proposed to construct the guaranteed confidence region for the unknown parameter. Due to the fact that the confidence region for the unknown parameter will also bring the confidence region for that state variable, then in Section 5, one min-max optimization problem is established to formulate our considered robust model predictive control, whose state variable is not one scalar value, but one confidence region for state estimation. In Section 6, one simulation example is shown of the effectiveness of this paper. Section 7 gives a final conclusion.

2. **System Formulation.**

Consider the following linear discrete parameter dependent dynamics or systems with an additive disturbance

$$x(t+1) = A(\theta)x(t) + B(\theta)u(t) + w(t) \tag{1}$$

where in Equation (1), $x(t) \in R^n$, $u(t) \in R^m$, $w(t) \in R^n$ are the state, excitation input and external disturbance of the system at the discrete time instant t. One uncertain parameter $\theta \in R^p$, whose true value is defined as $\theta = \theta^*$, and the system matrices $\{A(\theta), B(\theta)\}$ depend affine on that parameter vector $\theta \in R^p$, such that

$$[A(\theta), B(\theta)] = [A_0, B_0] + \sum_{i=1}^{p} [A_i, B_i][\theta]_i \tag{2}$$

i.e.,

$$A(\theta) = A_0 + \sum_{i=1}^{p} A_i\theta_i; \quad B(\theta) = B_0 + \sum_{i=1}^{p} B_i\theta_i \tag{3}$$

where θ_i is the ith element in that parameter vector θ, all orders $\{m, n, p\}$ are known beforehand. A_0, B_0, $\{A_i\}_{i=1}^{p}$, $\{B_i\}_{i=1}^{p}$ are the basic vectors, which are chosen in priori.

After substituting Equation (3) into (1), we get

$$\begin{aligned} x(t+1) &= A(\theta)x(t) + B(\theta)u(t) + w(t) \\ &= \left[A_0 + \sum_{i=1}^{p} A_i\theta_i\right] x(t) + \left[B_0 + \sum_{i=1}^{p} B_i\theta_i\right] + w(t) \\ &= A_0x(t) + A_1x(t)\theta_1 + A_2x(t)\theta_2 + \cdots + A_px(t)\theta_p \\ &\quad + B_0u(t) + B_1u(t)\theta_1 + B_2u(t)\theta_2 + \cdots + B_pu(t)\theta_p + w(t) \end{aligned} \tag{4}$$

where Equation (3) is substituted in Equation (1), and then expanding it.

Continuing to consider the above equation, it holds that

$$\begin{aligned} & x(t+1) \\ =& [A_0x(t) + B_0u(t)] + [A_1x(t) + B_1u(t)]\theta_1 \\ & + [A_2x(t) + B_2u(t)]\theta_2 + \cdots + [A_px(t) + B_pu(t)]\theta_p + w(t) \\ =& \begin{bmatrix} A_0x(t) + B_0u(t) & A_1x(t) + B_1u(t) & \cdots & A_px(t) + B_pu(t) \end{bmatrix} \begin{bmatrix} 1 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_p \end{bmatrix} + w(t) \end{aligned} \tag{5}$$

or

$$\begin{aligned} & [x(t+1) - A_0x(t) - B_0u(t)] \\ =& \begin{bmatrix} A_1x(t) + B_1u(t) & \cdots & A_px(t) + B_pu(t) \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_p \end{bmatrix} + w(t) \end{aligned} \tag{6}$$

Define the following variables as

$$\begin{aligned} y(t+1) &= x(t+1) - A_0x(t) + B_0u(t) \\ \varphi^T(t) &= \begin{bmatrix} A_1x(t) + B_1u(t) & \cdots & A_px(t) + B_pu(t) \end{bmatrix} \end{aligned}$$

Then Equation (6) can be reduced to

$$y(t+1) = \varphi^T(t)\theta + w(t) \tag{7}$$

Equation (7) is one linear regressor form, which depends on that unknown parameter vector θ affine.

Comment: Due to the fact that state vector $x(t)$ exists in the quadratic penalty function, which corresponds to one model predictive control strategy, the important step for the latter controller design is to estimate unknown parameter vector θ through some statistical algorithm, for example, least squares estimation, and maximal likelihood estimation. However, during these statistical identification algorithms, the probabilistic density function of that disturbance or noise $w(t)$ must be known. As this condition is very strict in practical engineering, to relax the prior condition of probabilistic density function on disturbance $w(t)$, this paper proposes to apply sign-perturbed sums to constructing one guaranteed confidence region for those unknown parameter vector. A special case is to choose the center of our constructed guaranteed confidence region as the final parameter estimate. After considering the guaranteed confidence region for those unknown parameter vector θ and state variable $x(t)$ in the quadratic penalty function for model predictive control, our robust model predictive control is yielded.

3. **Reformulation of Closed Loop System.** Observing Equation (7) again, that linear regressor form is benefit for the latter sign-perturbed sums, described in Section 4. As the considered system in Section 2 is one open loop system, here in this section we show the equation for one closed loop system is also modified to the linear regressor form. Then our sign-perturbed sums can be applied for the open loop system and closed loop system simultaneously.

Consider the following closed loop system in Figure 1, where $r(t)$ is external input, $u(t)$ is control input, and $v(t)$ is external disturbance or noise. Output $y(t)$ is returned back to the control input. $\{G(q), C(q)\}$ are plant and feedback controller respectively, and q is the shift operator.

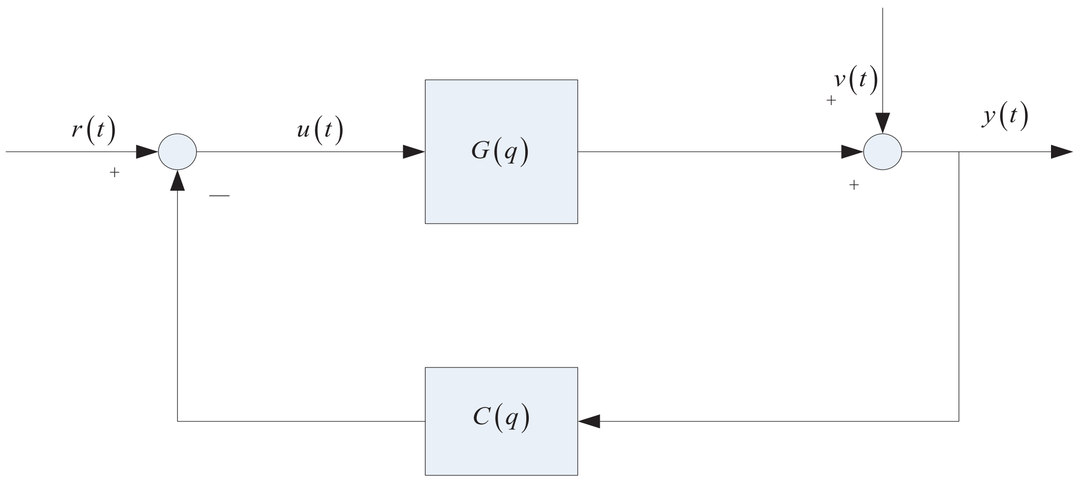

FIGURE 1. Closed loop system

Taking account of this closed loop system structure, we easily have

$$y(t) = G(q)r(t) - G(q)C(q)y(t) + v(t) \tag{8}$$

i.e.,

$$\begin{aligned} y(t) &= \frac{G(q)}{1 + G(q)C(q)} r(t) + \frac{1}{1 + G(q)C(q)} v(t) \\ u(t) &= \frac{1}{1 + G(q)C(q)} r(t) - \frac{C(q)}{1 + G(q)C(q)} v(t) \end{aligned} \tag{9}$$

Introducing one unknown parameter vector θ into parameterizing the plant $G(q, \theta)$, then it holds that

$$y(t) = \frac{G(q, \theta)}{1 + G(q, \theta)C(q)} r(t) + \frac{1}{1 + G(q, \theta)C(q)} v(t) \tag{10}$$

To obtain the linear regressor form for closed loop system, we parameterize the plant $G(q, \theta)$ as one polynomial.

$$G(q, \theta) = \frac{B(q, \theta)}{A(q, \theta)} = \frac{b_1 q^{-1} + \cdots + b_{n_b} q^{-n_b}}{1 + a_1 q^{-1} + \cdots + a_{n_a} q^{-n_a}}$$

$$\theta = \begin{bmatrix} a_1 & \cdots & a_{n_a} & b_1 & \cdots & b_{n_b} \end{bmatrix}^T \tag{11}$$

Similarly the parameterized controller is

$$C(q,\theta) = \frac{N_c(q)}{D_c(q)} = \frac{n_0 + n_1 q^{-1} + \cdots + n_{n_N} q^{-n_N}}{1 + d_1 q^{-1} + \cdots + d_{n_D} q^{-n_D}} \tag{12}$$

where $N_c(q)$ and $D_c(q)$ are two coprime polynomials, orders $\{n_b, n_a, n_N, n_D\}$ are all known in advance. From these two parameterized plant and controller, then the parameterized output predictor is yielded.

$$\hat{y}(t/t-1,\theta) = \frac{D_c(q)B(q,\theta)}{D_c(q)A(q,\theta) + N_c(q)B(q,\theta)} \tag{13}$$

Observing that denominator, we obtain

$$\begin{aligned} & D_c(q)A(q,\theta) + N_c(q)B(q,\theta) = 1 + \begin{bmatrix} q^{-1} & q^{-2} & \cdots & q^{-n} \end{bmatrix} \theta_{cl} \\ & n = \max(n_a + n_D, n_b + n_N) \\ & \theta_{cl} = S\theta + \rho \end{aligned} \tag{14}$$

where above matrices and vectors are defined as follows.

$$\rho = \begin{bmatrix} d_1 & \cdots & d_{n_D} & 0 & \cdots & 0 \end{bmatrix}^T \in R^n; \quad S = \begin{bmatrix} P_D & P_N \\ 0 & 0 \end{bmatrix}$$

$$P_D = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ d_1 & 1 & \cdots & 0 \\ d_2 & d_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ d_{n_D} & \cdots & \cdots & d_1 \\ 0 & d_{n_D} & \cdots & d_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{n_D} \end{bmatrix}; \quad P_N = \begin{bmatrix} n_0 & 0 & \cdots & 0 \\ n_1 & n_0 & \cdots & 0 \\ n_2 & n_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ n_{n_N} & \cdots & \cdots & n_1 \\ 0 & n_{n_N} & \cdots & n_2 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & n_{n_N} \end{bmatrix}$$

Making use of the coprime polynomials, we have

$$[D_c(q)A(q,\theta) + N_c(q)B(q,\theta)] = [D_c(q)B(q,\theta)]r(t) \tag{15}$$

or

$$\left\{1 + \left[q^{-1}\ q^{-2}\ \cdots\ q^{-n}\right]\left[(P_D\ P_N)\theta + \rho\right]\right\} y(t) = \left[q^{-1}\ q^{-2}\ \cdots\ q^{-n}\right][0\ P_D]\theta r(t) \tag{16}$$

Expand above equation to get

$$\begin{aligned} & y(t) + [y(t-1)\ y(t-2)\ \cdots\ y(t-n)][(P_D\ P_N)\theta + \rho] \\ & = [r(t-1)\ r(t-2)\ \cdots\ r(t-n)][0\ P_D]\theta \end{aligned} \tag{17}$$

It means that

$$\begin{aligned} y(t) = \{ & [r(t-1)\ r(t-2)\ \cdots\ r(t-n)][0\ P_D] \\ & - [y(t-1)\ y(t-2)\ \cdots\ y(t-n)](P_D\ P_N)\}\theta \end{aligned} \tag{18}$$

Define one regressor variable as that

$$\begin{aligned} \varphi^T(t) = \{ & [r(t-1)\ r(t-2)\ \cdots\ r(t-n)][0\ P_D] \\ & - [y(t-1)\ y(t-2)\ \cdots\ y(t-n)](P_D\ P_N)\} \end{aligned}$$

Then the output of the closed loop system is rewritten as

$$y(t) = \varphi^T(t)\theta \tag{19}$$

Equation (19) is also one linear regressor form. Comparing Equation (7) and (19), we see these two outputs of the open loop system and closed loop system respectively have the same linear regressor forms, which are convenient for the latter sign-perturbed sums.

Comment: Section 2 describes the linear discrete parameter dependent dynamics, which is the same as the system with an additive disturbances. The important result in Section 2 is that obtained linear regressor form (7), being used in Section 3. More specifically, consider one closed loop system, the output of that closed loop system can be reformulated into that linear regressor form (7). Generally, whatever for the open loop system or closed loop system, their outputs correspond to our considered linear regressor form (7), i.e., Equation (7) is our mainly considered form for the later sum-perturbed sums and robust model predictive control.

4. **Sign-Perturbed Sums.** Based on above two linear regressor forms (7) and (19) whatever for open loop system or closed loop system, we construct guaranteed confidence region for that unknown parameter vector θ without any probabilistic density function on noise, only each noise or disturbance $w(t)$ has a symmetric probability distribution about zero. The detailed sign-perturbed sums is introduced as follows. Set the prediction error for a given θ as

$$\begin{aligned} &\xi(t,\theta) = y(t) - \hat{y}(t,\theta) = y(t) - \varphi^T(t)\theta \\ &\hat{y}(t,\theta) = \varphi^T(t)\theta \end{aligned} \tag{20}$$

where $\hat{y}(t,\theta)$ is the predictors, being parameterized by the unknown parameter vector θ. $y(t)$ is the measured output, that is collected through some physical devices. N is the total number of measured data.

Solve the normal equation

$$\sum_{t=1}^{N} \varphi(t)\xi(t,\theta) = \sum_{t=1}^{N} \varphi(t)[y(t) - \varphi^T(t)\theta] = 0 \tag{21}$$

i.e.,

$$\sum_{t=1}^{N} \varphi(t)y(t) - \sum_{t=1}^{N} \varphi(t)\varphi^T(t)\theta = 0 \tag{22}$$

Assume the measured output $y(t)$ satisfies

$$y(t) = \varphi^T(t)\theta^* + w(t) \tag{23}$$

where θ^* is the true or real parameter vector. It is used to measure the derivation error between the parameter estimation and true parameter vector. In reality, this true parameter vector is unknown, but we can cluster the iterative parameter estimations, then that cluster point is named as the true parameter vector.

Substitute Equation (23) into (21) to get

$$\begin{aligned} \sum_{t=1}^{N} \varphi(t)[\varphi^T(t)\theta^* + w(t) - \varphi^T(t)\theta] &= \sum_{t=1}^{N} \varphi(t)\varphi^T(t)(\theta^* - \theta) + \sum_{t=1}^{N} \varphi(t)w(t) \\ &= \sum_{t=1}^{N} \varphi(t)\varphi^T(t)\tilde{\theta} + \sum_{t=1}^{N} \varphi(t)w(t) = 0 \\ \tilde{\theta} &= \theta^* - \theta \end{aligned} \tag{24}$$

Sign-perturbed sums exploits the knowledge from the data as much as possible, while needing minimal prior statistical knowledge about the noise. Introduce $m-1$ sign-perturbed sums as

$$\begin{aligned} H_i(\theta) &= \sum_{t=1}^{N} \varphi(t)\alpha_{i,t}\left(y(t) - \varphi^T(t)\theta\right) \\ &= \sum_{t=1}^{N} \alpha_{i,t}\varphi(t)\varphi^T(t)\tilde{\theta} + \sum_{t=1}^{N} \alpha_{i,t}\varphi(t)w(t); \quad i = 1, 2, \ldots, m-1 \end{aligned} \tag{25}$$

where in Equation (25), sequence $\alpha_{i,t}$ are random signs, i.e., they take on the value ± 1 with probability $\frac{1}{2}$.

Set $H_0(\theta)$ as follows

$$H_0(\theta) = \sum_{t=1}^{N} \varphi(t)\left(y(t) - \varphi^T(t)\theta\right) = \sum_{t=1}^{N} \varphi(t)\varphi^T(t)\tilde{\theta} + \sum_{t=1}^{N} \varphi(t)w(t) \tag{26}$$

The goal of sign-perturbed sums is to construct the guaranteed confidence regions based on the ranking of the function $\left\{\|H_i(\theta)\|^2\right\}$ and leave out those θ parameters for which $\left\{\|H_i(\theta)\|^2\right\}$ dominates, $\|.\|$ refers to the 2-norm. Define the p level sign-perturbed sums confidence region as

$$\hat{\theta}_N = \left\{\theta \in R^d : Indicater(\theta) = 1\right\} \tag{27}$$

Set

$$S_i(\theta) = \frac{1}{N} R_N^{-\frac{1}{2}} H_i(\theta); \quad R_N = \frac{1}{N}\sum_{t=1}^{N} \varphi(t)\varphi^T(t) \tag{28}$$

Then denote by $\Gamma(\theta)$ the rank $\left\{\|S_i(\theta)\|^2\right\}$ in the ordering of $\|S_0(\theta)\|, \|S_i(\theta)\|, i = 1, 2, \ldots, m-1$, for example, $\Gamma(\theta) = 1$ means that $\|S_0(\theta)\|$ is the smallest one. Finally, the confidence region from sign-perturbed sums is constructed as

$$\hat{\theta}_N = \left\{\theta \in R^d : \Gamma(\theta) \le m\left(1 - \frac{q}{m}\right)\right\} \tag{29}$$

where q is one chosen integer.

The probability level about the parameter estimator in this above constructed confidence region is reformulated as the following Theorem 4.1.

Theorem 4.1. *Given a rational confidence probability $p \in (0,1)$, set integers $m > q > 0$ such that $p = 1 - \frac{q}{m}$. If $\{w(1), w(2), \ldots, w(N)\}$ is a sequence of independent random variables distributed symmetrically about zero, then it holds that pr $\left\{\theta^* \in \hat{\theta}_N\right\} = p = 1 - \frac{q}{m}$.*

On the basis of the confidence region $\hat{\theta}_N$, an important and generic problem is to estimate the state variable $x(t)$ for the latter robust model predictive control, as the cost function for model predictive control needs the information about the confidence region $\hat{\theta}_N$.

5. **Robust Model Predictive Control.** Model predictive control was initially motivated by the desire to introduce nonlinearities, and control input and state constraints into the linear quadratic framework. Rewrite the cost function per stage as

$$x^T(t)Qx(t) + u^T(t)Ru(t), \quad t = 0, 1, \ldots \tag{30}$$

where Q and R are two positive definite symmetric matrices.

Impose state and control input constraints

$$x(t) \in X,\, u(t) \in U(x(t)), \quad t = 0, 1, \ldots;\ \theta \in \hat{\theta}_N \tag{31}$$

where X, $U(x(t))$ are two sets.

A stationary feedback controller is derived for that control $\mu(x)$ at stage x, while for all initial state $x(0) \in X$, the state of the closed loop system satisfies

$$\sum_{t=0}^{\infty} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] \le \infty \tag{32}$$

At each stage t and state $x(0) \in X$, it solves N stage deterministic optimal control problem, involving the same quadratic cost and the requirement that the state after N stages be exactly equal to 0.

Consider state set X, control input set U, and parameter set $\hat{\theta}_N$, our proposed robust model predictive control is the following min-max optimization problem

$$\begin{aligned} \min_{u(t)} \max_{\theta \in \hat{\theta}_N} \quad & J(x, u) = \sum_{t=0}^{N} \left[x^T(t)Qx(t) + u^T(t)Ru(t)\right] \\ \text{subjcet to} \quad & x(t+1) = A(\theta)x(t) + B(\theta)u(t) + w(t) \\ & x(t) \in X,\, u(t) \in U(x(t)), \quad t = 0, 1, \ldots;\ \theta \in \hat{\theta}_N \end{aligned} \tag{33}$$

Robust model predictive control applies at stage t the first component of the control policy $\mu(t), \mu(t+1), \ldots, \mu(k+N-1)$, thus obtains $\mu(x(t)) = \mu(t)$ and discards the remaining components. This is the idea of receding horizon. The idea of dynamic programming strategy is applied in solving the above robust model predictive control, and then we formulated the detailed process about dynamic programming strategy.

Set $\{u^*(0), u^*(1), \ldots, u^*(N)\}$ as the corresponding optimal control sequence, choose one sub problem, being started from $u^*(k)$ at the time instant k, we need to minimize one cost-to-go form from the time instant k to the time horizon level N.

$$\sum_{m=k+1}^{N} \left[x^T(m)Qx(m) + u^T(m)Ru(m)\right] + \left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] \tag{34}$$

over the sub-sequence $\{u(k), u(k+1), \ldots, u(N)\}$, $m = k, \ldots, N$.

From the idea of dynamic programming, one truncated optimal sequence $\{u^*(k), u^*(k+1), \ldots, u^*(N)\}$ is optimal for this above subproblem. Then the main process of dynamic programming is to establish the optimal cost functions.

$$J_N^*(u(N)), J_{N-1}^*(u(N-1)), \ldots, J_0^*(u(0)) \tag{35}$$

Iteratively beginning from $J_N^*(u(N))$ and proceeding backwards to $J_{N-1}^*(u(N-1)), \ldots,$ $J_0^*(u(0))$, i.e., begin from $J_N^*(u(N)) = \left[x^T(N)Qx(N) + u^T(N)Ru(N)\right]$ and for $k = 0, 1, \ldots, N$. Set

$$\begin{aligned} J_k^*(u(k)) = & \min_{u(k), \ldots, u(N)} \left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] \\ & + \sum_{m=k+1}^{N} \left[x^T(m)Qx(m) + u^T(m)Ru(m)\right] \\ = & \min_{u(k)} \sum_{m=k}^{N} \left[x^T(m)Qx(m) + u^T(m)Ru(m)\right] \\ = & \min_{u(k)} \left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] \end{aligned}$$

$$+ \min_{u(k+1),\ldots,u(N)} \sum_{m=k+1}^{N} \left[x^T(m)Qx(m) + u^T(m)Ru(m)\right]$$
$$= \min_{u(k)} \left[x^T(k)Qx(k) + u^T(k)Ru(k)\right] + J^*_{k+1}(u(k+1)) \quad (36)$$

where during above mathematical derivations, the principle of optimality is applied.

Consider above description about dynamic programming strategy, when given every initial control input $u(0)$, the value $J_0^*(u(0))$ is equal to the optimal cost function. More importantly, after the optimal cost function sequence $J_N^*(u(N)), J_{N-1}^*(u(N-1)), \ldots, J_0^*(u(0))$ are derived, the dynamic programming strategy is used to construct the optimal control sequence $\{u^*(0), u^*(1), \ldots, u^*(N)\}$ and the corresponding state trajectories $\{x^*(1), x^*(2), \ldots, x^*(N)\}$ for the given initial control input value $u(0)$.

6. **Simulation Example.** Generally, the main process of this paper is to construct one model predictive control, whose one physical variable is included in one established set, not a real value. We call it as the robust model predictive control due to the existence of the parameter set. Consider the system Equation (1), the real system in simulation example is given as follows:

$$P_0(z) = \frac{0.25z^3 + 0.12z^2}{z^4 - 1.6z^3 + 0.8z^2 - 0.64z + 0.65}; \quad H_0(z) = \frac{z^4 + 0.2z^3}{z^4 + 0.5z^3} \quad (37)$$

where the transfer function in Equation (37) is from our previously published work [13], and its parameterized form is given as

$$P(z,\theta) = \frac{a_5z^3 + a_6z^2}{z^4 + a_1z^3 + a_2z^2 + a_3z + a_4}; \quad H_0(z) = \frac{z^4 + b_2z^3}{z^4 + b_1z^3} \quad (38)$$

Noise signal $w(t)$ is considered which passing through the filter $H_0(z)$, and sampled time is chosen as $T_s = 1$ second. To be convenient for the latter analysis, set the true parameter vector θ_0 as

$$\theta_0 = \begin{bmatrix} -1.6 & 0.8 & -0.64 & 0.65 & 0.25 & 0.12 & 0.5 & 0.2 \end{bmatrix}^T \quad (39)$$

The data generating system is operated in closed loop system with a model predictive control based on our commissioning model (P_{init}, H_{init}), where θ_{init} is chosen as

$$\theta_{init} = \begin{bmatrix} -1.7 & 0.7 & -0.4 & 0.8 & 0.15 & 0.1 & 0.4 & 0.1 \end{bmatrix}^T \quad (40)$$

Figure 2 shows the applied input signal, being chosen by the designer. Collect the obtained output signal $y(t)$ with some physical devices, and plot it in Figure 3. From these two figures, we see the input signal is regular, but irregular for the output signal. According to our derived results in this paper, firstly we need to construct one parameter set for one physical variable, i.e., unknown parameter, and then this parameter set is considered in the following model predictive control to form the named robust model predictive control strategy. Finally, dynamic programming is proposed to solve this robust model predictive control, which corresponds to one numerical optimization problem.

Based on the applied input curve and output curve, the input and output measured data are used to construct the parameter set. In order to apply sign-perturbed sums to construct the confidence regions for those eight parameters, we do not assume the noise is a special noise, and its probability densigy function is unknown either. Divide those eight parameters into four pairs, i.e., $(a_1\ a_2) = (-1.6\ 0.8)$, $(a_3\ a_4) = (-0.64\ 0.65)$, $(a_5\ a_6) = (0.25\ 0.12)$, $(b_1\ b_2) = (0.5\ 0.2)$. The estimation results are plotted in Figure 4 and Figure 5, which use a sequence of ellipsoids to approximate the true parameter values.

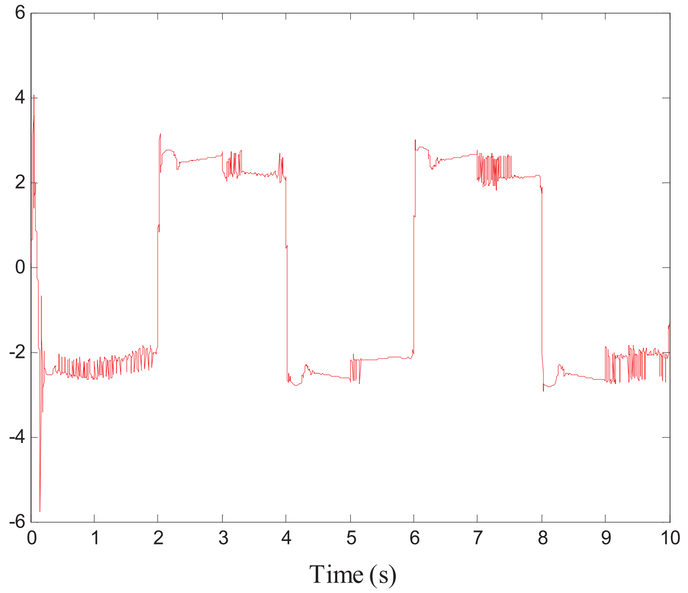

FIGURE 2. The applied input signal

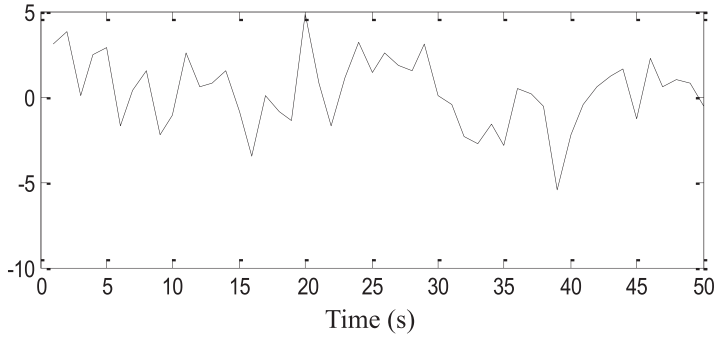

FIGURE 3. The observed output signal

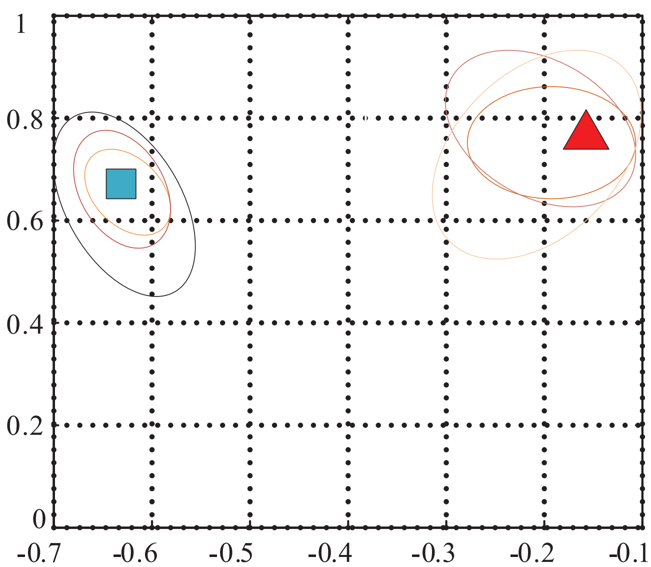

FIGURE 4. Confidence regions for two pairs of parameters

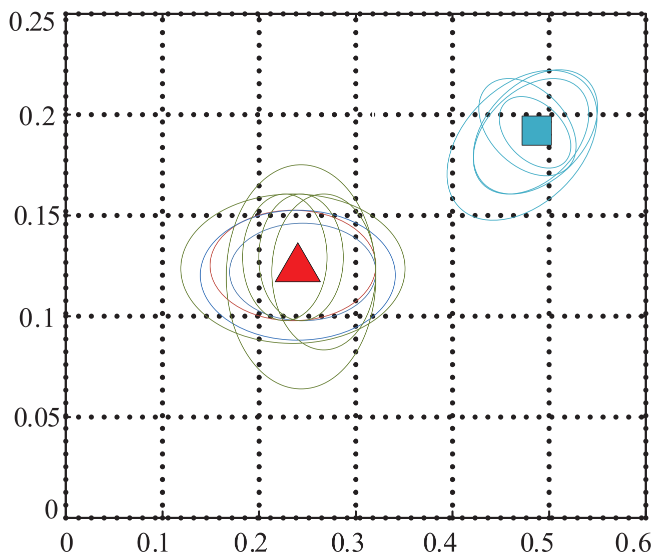

Figure 5. Confidence regions for other two pairs of parameters

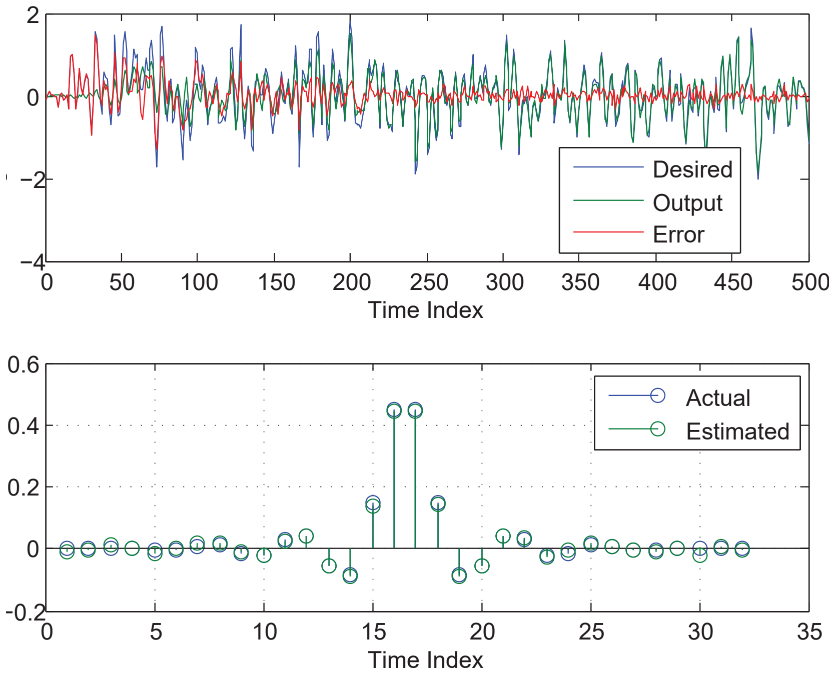

Figure 6. Comparison of true output and estimated output

The ellipsoid approximations correspond to our constructed confidence regions, and the shape of the ellipsoid is dependent of the semi definite matrix $R_N = \frac{1}{N}\sum_{t=1}^{N}\varphi(t)\varphi^T(t)$.

After substituting the above guaranteed confidence regions into our proposed robust model predictive control, i.e., that min-max optimization problem. Based on this priori known input-output trajectory $\{u(k), y(k)\}_{k=0}^{N-1}$ of data length $N = 1000$ from Figure 2 and Figure 3, Equation (33) is proposed to yield the output predictor during the latter time interval. Figure 6 shows the resulting output predictors as well as the true output for comparison. From Figure 6, we see the output predictors are nice, and the errors can be neglected.

7. **Conclusion.** During lots of statistical identification strategies, the external disturbance and noise are all supposed to be white noises and their probabilistic density functions are needed in the latter identification process. To relax the above strict limitation on disturbance and noise, we only assume each noise has a symmetric probability distribution about zero. Then sign-perturbed sums is proposed to construct one guaranteed confidence region for the unknown parameter, which exists in one linear regressor form. Based on this confidence region of the unknown parameter through sign-perturbed sums, robust model predictive control is established to be one min-max optimization problem, which is related with the controller and confidence region of the unknown parameter. The detailed process about applying reinforcement learning to solving the min-max optimization problem is our ongoing work.

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