

IDEAL CHARACTERIZATIONS OF SEMIGROUPS OF INDUCTIVE TERMS

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ABSTRACT. *With more general definition of a superposition S^n , an inductive composition is an operation on the set of all n -ary terms of type τ and it also induces an inductive product, a binary operation which is less specific than a term product. The more general the operation, the stricter the base set pair with the operation to become a semigroup. This paper focuses on semigroups with this inductive product as their fundamental operation. We consider their left, right, and two-sided ideals as well as those of principal ideals. Maximality and minimality of the mentioned ideals are also characterized. Relations of positional equivalence and root equivalence are introduced as tools in characterizing the mentioned structures.*

Keywords: Inductive composition of terms, Semigroups of inductive terms, Equivalence of terms, Ideals of inductive terms, Principal ideals of inductive terms

1. Introduction. Terms are fundamental concepts in universal-algebra studies. Terms are inductively defined based on *variables*, elements from the set $X_n = \{x_1, \dots, x_n\}$ for some positive integer n or the set $X = \{x_1, x_2, x_3, \dots\}$, and to generate even more number of terms, non-nullary operation symbols from the set $\{f_i \mid i \in I\}$ where I is an index set can be combined with variables or any already obtained term to get new terms. For each $i \in I$, the *type* $\tau = (n_i)_{i \in I}$ indicates that the operation symbol f_i is n_i -ary. The n -ary terms of type τ are formally defined as follows:

- (i) Every variable $x_i \in X_n$ is an n -ary term;
- (ii) If t_1, \dots, t_{n_i} are n_i -ary terms and f_i is an n_i -ary operation symbol, then $f_i(t_1, \dots, t_{n_i})$ is an n -ary term.

The set of all n -ary terms of type τ is denoted by $W_\tau(X_n)$, and analogously, $W_\tau(X)$ denotes the set of all terms of type τ . Terms are important in both algebraic and theoretical structures. In algebraic advantages, terms are used as identities for generating a variety of algebras in which every algebra satisfies the designated identities (see [1, 10, 11] for more details). In theoretical structures, terms are applied in computer science and linguistics (see [1, 11, 13] for more details). We also would like to refer to [16, 18, 20, 21, 26] for current trends in term studies.

The sets $W_\tau(X_n)$ and $W_\tau(X)$ are suitable to be set as base sets under superpositions of terms to obtain algebras of terms. A superposition is an action of variable replacement in a term by other terms to obtain a new term with the same type, so it is certainly closed

over $W_\tau(X_n)$ and $W_\tau(X)$; more importantly, it satisfies the following superassociative law:

$$S_m^p(z, S_m^n(y_1, x_1, \dots, x_n), \dots, S_m^n(y_p, x_1, \dots, x_n)) \approx S_m^n(S_n^p(z, y_1, \dots, y_p), x_1, \dots, x_n)$$

where m, n, p are nonzero natural numbers, $x_1, \dots, x_n, y_1, \dots, y_p$ and z are terms, and S_m^p, S_m^n , and S_n^p are superpositions.

Superpositions of terms have been studied throughout these past years. At the beginning, the superposition S_m^n was defined for n -ary and m -ary terms of the same type. Then Denecke and Leeratanavalee [7] defined the superposition S_g^n on $W_\tau(X)$, the set of all terms of type τ . Moreover, several researchers adapted these superpositions to dealing with various base sets of terms, we refer the readers to [3, 5, 26] for more forms of superpositions of terms.

Since superpositions of terms S_m^n and S_g^n are superassociative, they undoubtedly induce associative binary operations on certain sets of terms. Denecke and Jampachon [6] introduced four different associative binary operations: $+$, $*$, $+_g$, and $*_g$. The ones with no subscript g were defined using the superposition S_n^n which is an operation over $W_\tau(X_n)$ and the ones with subscript g were defined from the superposition S_g^n over $W_{\tau_n}(X)$ by which τ_n means that all operation symbols are n -ary. The superposition S_g^n was also used in defining a term product \cdot_{x_i} on $W_{\tau_n}(X)$ which was emphasized in [16]; it is a special case of a tree-language product from [8].

In 2007, Shtrakov [24] defined the *inductive composition* which is an extended concept of a superposition because of more selection of terms to be replaced, speaking of which, a superposition only considers variables as replacable terms while the inductive composition regards substitution of any term, possibly a variable, acting as a subterm. For a term $t \in W_\tau(X_n)$, the set $sub(t)$ of all of its subterms is inductively defined as follows (see e.g., [13, 23, 24]):

- (i) If $t \in X_n$, then $sub(t) = \{t\}$;
- (ii) If $t = f_i(t_1, \dots, t_{n_i})$, then $sub(t) = \{t\} \cup sub(t_1) \cup \dots \cup sub(t_{n_i})$.

The inductive composition is defined as follows (see e.g., [23, 24]): Let $r, s, t \in W_\tau(X_n)$ be any n -ary terms of type τ . The inductive composition $t(r \leftarrow s)$ is the term inductively defined by

- (i) $t(r \leftarrow s) = t$ if $r \notin sub(t)$;
- (ii) $t(r \leftarrow s) = s$ if $t = r$;
- (iii) $t(r \leftarrow s) = f_i(t_1(r \leftarrow s), \dots, t_{n_i}(r \leftarrow s))$ if $t = f_i(t_1, \dots, t_{n_i})$, $r \in sub(t)$, and $t \neq r$.

The more kinds of replacable terms come with an unexpected result – no associativity satisfaction. Kitpratyakul and Pibaljommee [18] gave a simple example to illustrate that the inductive composition is not associative and defined a binary operation called *inductive product* based on fixing a term to be replaced in the inductive composition as follows.

Let $r \in W_\tau(X_n)$ be fixed and $s, t \in W_\tau(X_n)$. An *r-inductive product*, denoted by \cdot_r , is an operation on $W_\tau(X_n)$ defined by

$$t \cdot_r s := t(r \leftarrow s).$$

For example, let $\tau = (2, 3)$ with a binary operation symbol f and a ternary operation symbol g . Let $r = f(x_1, x_2)$, $s = f(x_3, x_3)$, and $t = g(f(x_2, x_3), x_2, f(x_3, f(x_1, x_2)))$ be ternary (3-ary) terms of type τ . Then we have

$$\begin{aligned} t \cdot_r s &= t(r \leftarrow s) \\ &= g(f(x_2, x_3), x_2, f(x_3, f(x_1, x_2)))(f(x_1, x_2) \leftarrow f(x_3, x_3)) \\ &= g(f(x_2, x_3)(f(x_1, x_2) \leftarrow f(x_3, x_3)), x_2(f(x_1, x_2) \leftarrow f(x_3, x_3)), \\ &\quad f(x_3, f(x_1, x_2))(f(x_1, x_2) \leftarrow f(x_3, x_3))) \end{aligned}$$

$$\begin{aligned}
 &= g(f(x_2, x_3), x_2, f(x_3(f(x_1, x_2) \leftarrow f(x_3, x_3)), f(x_1, x_2)(f(x_1, x_2) \leftarrow f(x_3, x_3)))))) \\
 &= g(f(x_2, x_3), x_2, f(x_3, f(x_3, x_3))).
 \end{aligned}$$

If we set a fixed term r to be a variable, an inductive product actually becomes a term product. This means that an inductive product \cdot_r is a generalization of a term product \cdot_{x_i} .

Semigroups are fundamental algebras with a nonempty base set and an associative binary operation. Due to their simplicity, semigroups are usually recognized as conditions for other particular kinds of algebras such as monoids, groups, and rings. Algebraic properties of semigroups can be founded in several textbooks; we recommend [12, 14]. Semigroups also have a closed relation with wording and automata theory. In automata theory, words are elements which are constructed from picking elements in $X_n = \{x_1, \dots, x_n\}$ and optionally applying the binary operation of juxtaposition; both actions can only be done finite times. Since the binary operation of juxtaposition is associative, we actually obtain a semigroup of words. Sometimes, the empty word is added to the base set of this semigroup to make it a monoid. For more information of wording and automata theory, we refer the readers to [1, 11, 13]. There is also a study on gender parity issues by using ARIMA (autoregressive integrated moving average), see [2] for more details.

Ideals are traditionally explored not only of semigroups but of other algebras as well. Ideals have their usage in various topics of semigroup studies such as Green's relations, many kinds of regularity characterization, and Rees congruence which leads to a Rees factor semigroup. For more backgrounds and examples, we refer the readers to [12, 14, 15, 22, 25].

Semigroups of terms have been studied recently. In 2006, Denecke and Jampachon [6] investigated the semigroup of $W_\tau(X_n)$ together with four distinct binary operations: $+$, $*$, $+_g$, and $*_g$. Many researchers modified such semigroups to shift to other types of terms (see e.g., [20, 21]) or even sets of terms (see e.g., [4, 9]). In addition, a term product stated in [16] can also induce a semigroup of terms but that semigroup is not directly investigated; more precisely, there is a study in this direction with sets of terms instead of terms which absorbs such semigroup of terms into its special case (see e.g., [8, 17, 19]). Currently working on this trend, Kitpratyakul and Pibaljomme [18] examined the semigroups concerning the inductive product and the base set of some restriction on $W_\tau(X_n)$.

This paper is the sequel of [18] whose findings concern the characterizations of idempotent and regular elements of the semigroup of inductive terms as well as its Green's relations. In this paper, we continue studying this semigroup and characterizing three types of ideals: left, right, and two-sided for both ideals and principal ideals of the introduced semigroup. Maximality and minimality of those ideals are investigated as well. Following this section, vital propositions regarding inductive products will be provided. Then the definitions and properties of positional equivalence and root equivalence of terms will be represented as the tools in the main theorems, characterizations of ideals and principal ideals of the semigroups of inductive terms. Finally, the conclusion of the work will be proposed together with possible directions of future research.

2. Preliminaries. This section devotes to the review of important findings involving inductive products which can be found in [18] to be used in later sections.

An algebra $(W_\tau(X_n), \cdot_r)$ is not necessary to be a semigroup. One discovered condition to make it a semigroup is to restrict the base set by the following condition.

Lemma 2.1. *Let $r \in W_\tau(X_n)$ be fixed and $s, t, u \in W_\tau(X_n)$. If for any $x, y \in W_\tau(X_n)$, $x \cdot_r y \neq r$ whenever $x \neq r$ or $y \neq r$, then $(t \cdot_r s) \cdot_r u = t \cdot_r (s \cdot_r u)$.*

A maximal subset of $W_\tau(X_n)$ corresponding to the above condition and being closed under the operation \cdot_r is established.

Theorem 2.1. *Let $r \in W_\tau(X_n)$ be fixed. Then the set $W_\tau^r(X_n) := W_\tau(X_n) \setminus (sub(r) \setminus \{r\})$ is a maximal subset of $W_\tau(X_n)$ which is closed under the operation \cdot_r and satisfies the condition: for any $s, t \in W_\tau^r(X_n)$, $t \cdot_r s \neq r$ whenever $s \neq r$ or $t \neq r$.*

By combining the above lemma and theorem, we eventually obtain the (maximal) semigroup $(W_\tau^r(X_n), \cdot_r)$. Note that if $r \in X_n$, then the semigroup arising in this way becomes $(W_\tau(X_n), \cdot_r)$. Moreover, $(W_\tau^r(X_n), \cdot_r)$ is in fact a monoid such that $r \in W_\tau^r(X_n)$ is its identity. For the rest of this paper, we will omit writing $t \notin sub(r) \setminus \{r\}$ in any lemma and theorem, but the readers have to be aware of such condition even if it is not mentioned.

At the end of this section, we emphasize a beneficial lemma of an inductive product on the set $W_\tau^r(X_n)$.

Lemma 2.2. *Let $r \in W_\tau(X_n)$ be fixed and $s, t \in W_\tau^r(X_n)$. Then*

- (i) *If $r \in sub(t)$, then $sub(s) \subseteq sub(t \cdot_r s)$;*
- (ii) *If $r \notin sub(t)$, then $t \cdot_r s = t$;*
- (iii) *$r \in sub(t)$ and $r \in sub(s)$ if and only if $r \in sub(t \cdot_r s)$;*
- (iv) *$t = r$ and $s = r$ if and only if $t \cdot_r s = r$.*

3. Positional Equivalence and Root Equivalence of Terms. In order to characterize ideals and principal ideals of $(W_\tau^r(X_n), \cdot_r)$, we need more details on words and positions of a term. We denote \mathbb{N}^* the set of all finite words over \mathbb{N} and ϵ the empty word. As usual, pq is the concatenation of the words $p, q \in \mathbb{N}^*$. A word $p \in \mathbb{N}^*$ is said to be a *prefix* of a word $q \in \mathbb{N}^*$ if $q = pu$ for some $u \in \mathbb{N}^*$.

For any term $t \in W_\tau(X_n)$, the set of all positions $pos(t)$ of t is inductively defined as follows:

- (i) If $t \in X_n$, then $pos(t) = \{\epsilon\}$;
- (ii) If $t = f_i(t_1, \dots, t_{n_i})$, then $pos(t) = \{\epsilon\} \cup \bigcup_{1 \leq j \leq n_i} (jpos(t_j))$ where $jpos(t_j) = \{jq \mid q \in pos(t_j)\}$.

We denote $l(p)$ to be the length of a position $p \in pos(t)$. The mapping $sub_t : pos(t) \rightarrow sub(t)$ is defined as an interpretation of each position in a term t to the subterm of t whose root node occurs at that position. For a fixed $r \in W_\tau(X_n)$, the mapping $pos^r : W_\tau(X_n) \rightarrow P(\mathbb{N}^*)$ assigns each term to the set of positions of itself where a subterm r is located. It is formally defined by the following inductive ways:

- (i) If $r \notin sub(t)$, then $pos^r(t) = \emptyset$;
- (ii) If $t = r$, then $pos^r(t) = \{\epsilon\}$;
- (iii) If $t = f_i(t_1, \dots, t_{n_i}) \neq r$ and $r \in sub(t)$, then $pos^r(t) = \bigcup_{1 \leq j \leq n_i} (jpos^r(t_j))$.

Here, $jpos^r(t_j) = \{jq \mid q \in pos^r(t_j)\}$, but if it happens that $pos^r(t_j) = \emptyset$, then we define $jpos^r(t_j) = \emptyset$.

We now give the definition of positional equivalence. For any $s, t \in W_\tau(X_n)$, we say that s is *positional equivalent* to t at the position p if the vertices at the position p of both s and t are labelled by the same variable or operation symbol. Generally, s is *positional equivalent* to t at a nonempty set $P \subseteq pos(s)$ if s is positional equivalent to t at the position p for every $p \in P$. For any $A \subseteq P(\mathbb{N}^*)$, we denote P_A to be the relation on $W_\tau(X_n)$ by which

$$P_A := \begin{cases} (W_\tau(X_n))^2 & \text{if } A = \emptyset \\ \{(u, v) \in (W_\tau(X_n))^2 \mid u \text{ and } v \text{ are positional equivalent at } A\} & \text{if } A \neq \emptyset. \end{cases}$$

Positional equivalent terms can sometimes be very different in complexity. For example, the terms $g(x_1)$ and $f(x_1, f(x_1, x_4))$ are terms in $W_{(2,1)}(X_4)$ where f and g are binary and unary operation symbols, respectively. We see that $(g(x_1), f(x_1, f(x_1, x_4))) \in P_{\{1\}}$ despite how they look so different in terms of variables, operation symbols, and term complexity. For more accuracy of term equivalence, we define a more specialized equivalence called *root equivalence*. For any terms $s, t \in W_\tau(X_n)$, a term t is said to be *root equivalent to s at the position $p \in \text{pos}(t)$* if t and s are positional equivalent at $\text{pre}(p) := \{q \in \mathbb{N}^* \mid q \text{ is a prefix of } p\}$. Also, t and s are root equivalent at a nonempty set $R \subseteq \text{pos}(t)$ if they are root equivalent at every position from R .

Example 3.1. Let $t_1 = h(f(x_1, x_4), g(x_2), x_3)$, $t_2 = h(f(g(x_1), x_5), g(x_2), x_3)$ be terms in $W_{(2,1,3)}(X_5)$ where f, g and h are binary, unary, and ternary operation symbols, respectively. Then t_1 and t_2 are root equivalent at the position 21 as they are positional equivalent at $\text{pre}(21) = \{\epsilon, 2, 21\}$.

Similar to what we define for positional equivalence, for $A \subseteq P(\mathbb{N}^*)$ we denote R_A to be the relation on $W_\tau(X_n)$ by which

$$R_A := \begin{cases} (W_\tau(X_n))^2 & \text{if } A = \emptyset \\ \{(u, v) \in (W_\tau(X_n))^2 \mid u \text{ and } v \text{ are root equivalent at } A\} & \text{if } A \neq \emptyset. \end{cases}$$

From Example 3.1 above, we see that $(t_1, t_2) \in R_{\{1,21,3\}}$ and also $(t_1, t_2) \in R_{\{\epsilon,1,2,21,3\}}$.

Remark 3.1. There are several remarks involving root equivalence of terms.

- (i) Positional equivalence is a generalization of root equivalence and some positional equivalence is not root equivalence. For example, the terms $g(x_1)$ and $f(x_1, f(x_1, x_4))$ are positional equivalent at the position 1; nevertheless, they are not root equivalent at the same position since they are not positional equivalent at $\epsilon \in \text{pre}(1)$.
- (ii) If t and s are root equivalent at $p \in \text{pos}(t)$, then they are also root equivalent at any position from $\text{pre}(p)$.
- (iii) For any terms $s, t \in W_\tau(X_n)$, $(t, s) \in R_A$ does not mean that t and s are not root equivalent at any position besides those from A . Referring to Example 3.1, we can tell that $(t_1, t_2) \in R_{\{3\}}$ even though they are also root equivalent at other positions such as 1 and 21.

The following supporting lemma concerning root equivalence will be useful in ideal characterization of $(W_\tau^r(X_n), \cdot_r)$.

Lemma 3.1. Let $s, t \in W_\tau(X_n)$ and $A, B \in P(\mathbb{N}^*)$. Then

- (i) If $A \subseteq B$, then $R_B \subseteq R_A$;
- (ii) $(t, s) \in R_{A \cup B}$ if and only if $(t, s) \in R_A$ and $(t, s) \in R_B$;
- (iii) $(t, s) \in R_{\text{pos}(t)}$ if and only if $t = s$;
- (iv) Let $A \neq \emptyset$, $(t, s) \in R_A$ and $s', t' \in W_\tau(X_n)$ be n -ary terms of type τ such that $\text{sub}_{t'}(j) = t$ and $\text{sub}_{s'}(j) = s$ for some $j \in \text{pos}(s') \cap \text{pos}(t')$ and let $j_0 \in \text{pre}(j)$ be such that $l(j_0) + 1 = l(j)$. Then $(t', s') \in R_{\{j_0\}}$ if and only if $(t', s') \in R_{j_0}$.

Proof: (i) The result is immediate if $A = \emptyset$. Assume that $\emptyset \neq A \subseteq B$. Let $(t, s) \in R_B$. Then t and s are root equivalent at B ; more precisely, at A since $A \subseteq B$. Hence, $(t, s) \in R_A$. Therefore, $R_B \subseteq R_A$.

(ii) The statement is clearly valid whenever A or B is empty. Next, we consider the situation where $A, B \neq \emptyset$. Assume first that $(t, s) \in R_{A \cup B}$. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, by (i), we get $R_{A \cup B} \subseteq R_A$ and $R_{A \cup B} \subseteq R_B$ which imply that $(t, s) \in R_A$ and $(t, s) \in R_B$. On the other hand, assume that $(t, s) \in R_A$ and $(t, s) \in R_B$. Then t

and s are root equivalent at A and B , so they are also root equivalent at $A \cup B$. Hence, $(t, s) \in R_{A \cup B}$.

(iii) Assume that $(t, s) \in R_{pos(t)}$. If $pos(s) \subsetneq pos(t)$, then we would get a contradiction to the assumption. If $pos(t) \subsetneq pos(s)$, then there is $p \in pos(s) \setminus pos(t)$ such that $q \in pre(p)$ and $l(q) = l(p) - 1$ for some $q \in pos(t)$. Note that $q \in pos(s)$. Then the arity of the operation symbol at the position $q \in pos(t)$ is at least 1 less than the arity of that of $q \in pos(s)$, but this would contradict $(t, s) \in R_{\{q\}}$. Even if $sub_t(q) \in X_n$, there would be no such p . Therefore, $pos(t) = pos(s)$. Altogether with the assumption, we obtain $t = s$. The converse is clear.

(iv) Assume that $(t', s') \in R_{\{j_0\}}$. Let $p \in A$. Then $(t, s) \in R_{\{p\}}$. Thus, $(t, s) \in P_{\{q\}}$ for each $q \in pre(p)$. Since $(t', s') \in R_{\{j_0\}}$, $(t', s') \in R_{\{q_0\}}$ for each $q_0 \in pre(j_0)$. Therefore, $(t', s') \in R_{\{jp\}}$. By (ii), we obtain $(t', s') \in R_{jA}$. The converse is immediate as $j_0 \in pre(ja)$ for any $a \in A$. \square

The concepts of words, positions of terms, and root equivalence introduced in this section will be utilized as tools in the characterizations of ideals and principal ideals in the next section.

4. Ideals and Principal Ideals of $(W_\tau^r(X_n), \cdot_r)$. Ideals are traditional studies of algebraic structures of semigroups. We characterize three types of ideals: left, right, and two-sided of the semigroup $(W_\tau^r(X_n), \cdot_r)$. Recall that a nonempty subset I of $W_\tau^r(X_n)$ is a left [right] ideal of $W_\tau^r(X_n)$ if for any $t \in I$ and $s \in W_\tau^r(X_n)$, $s \cdot_r t \in I$ [$t \cdot_r s \in I$], and I is called an ideal or a two-sided ideal if it is both a left and a right ideal.

For any terms $s, t \in W_\tau(X_n)$, we denote

$$B_s^t := \{p \in pos(t) \mid q \text{ is not a prefix of } p \text{ for any } q \in pos^s(t)\}.$$

Before moving to right-ideal characterization, we need the following lemma.

Lemma 4.1. *Let $r \in W_\tau(X_n)$ be fixed and $t = f_i(t_1, \dots, t_{n_i}) \in W_\tau^r(X_n)$ with $t \neq r$ and $r \in sub(t)$. Then $B_r^t \setminus \{\epsilon\} = \bigcup_{j=1}^{n_i} jB_r^{t_j}$.*

Proof: It is easy to see that if $t_j = r$ for all $j \in \{1, \dots, n_i\}$, then $B_r^t \setminus \{\epsilon\} = \emptyset = \bigcup_{j=1}^{n_i} jB_r^{t_j}$. Now, consider the case where $t_j \neq r$ for some $j \in \{1, \dots, n_i\}$ which is a case rendering $B_r^t \setminus \{\epsilon\}$ and $\bigcup_{j=1}^{n_i} jB_r^{t_j}$ nonempty. Let $a \in B_r^t \setminus \{\epsilon\}$. By the definition of $pos(t)$, we get $a \in jpos(t_j)$ for some $j \in \{1, \dots, n_i\}$, i.e., $a = jc$ for some $c \in pos(t_j)$. Since $a \in B_r^t$, there is no $q \in pos^r(t)$ such that q is a prefix of a . If there is a prefix q_0 of c such that $q_0 \in pos^r(t_j)$, then we would have $jq_0 \in pos^r(t)$ which is a prefix of $a = jc$, a contradiction. Hence, $c \in B_r^{t_j}$. Therefore, $a \in jB_r^{t_j}$, i.e., $a \in \bigcup_{j=1}^{n_i} jB_r^{t_j}$. Next, let $b \in \bigcup_{j=1}^{n_i} jB_r^{t_j}$. Clearly, $b \neq \epsilon$. Then $b = jd$ for some $d \in pos(t_j)$ such that there is no prefix q of d such that $q \in pos^r(t_j)$ for some $j \in \{1, \dots, n_i\}$. If $sub_t(j) = r$, then $t_j = r$ which means that $\epsilon \in pos^r(t_j)$. We get a contradiction where $\epsilon \in pos^r(t_j)$ while it is a prefix of d . Therefore, $sub_t(j) \neq r$, that is, $j \notin pos^r(t)$. Note that $\epsilon \notin pos^r(t_j)$. Therefore, $b \in B_r^t \setminus \{\epsilon\}$. Eventually, we obtain $B_r^t \setminus \{\epsilon\} = \bigcup_{j=1}^{n_i} jB_r^{t_j}$. \square

The key lemma for right-ideal characterization of $(W_\tau^r(X_n), \cdot_r)$ is given below.

Lemma 4.2. *Let $r \in W_\tau(X_n)$ be fixed and $s, t, t' \in W_\tau^r(X_n)$. Then $t' = t \cdot_r s$ if and only if $(t', t) \in R_{B_r^t}$ and $sub_{t'}(p') = s$ for any $p' \in pos^r(t)$.*

Proof: Assume that $t' = t \cdot_r s$. The proof tackles on the structure of t . First, if $r \notin sub(t)$, then $B_r^t = pos(t)$ and $t' = t \cdot_r s = t$. Thus, $(t', t) \in R_{B_r^t}$. Second, if $t = r$, then $pos^r(t) = \{\epsilon\}$. This means that $B_r^t = \emptyset$ implying $R_{B_r^t} = R_\emptyset$. It follows that $(t', t) \in R_\emptyset = R_{B_r^t}$. Since $t' = t \cdot_r s = s$, we see that $sub_{t'}(\epsilon) = s$ for any position in

$pos^r(t) = \{\epsilon\}$. Finally, for $t = f_i(t_1, \dots, t_{n_i}) \neq r$ and $r \in sub(t)$, we inductively assume that $(t_j \cdot_r s, t_j) \in R_{B_r^{t_j}}$ and $sub_{t_j \cdot_r s}(p'_j) = s$ for each $j \in \{1, \dots, n_i\}$, and $p'_j \in pos^r(t_j)$. To show that $(f_i(t_1 \cdot_r s, \dots, t_{n_i} \cdot_r s), f_i(t_1, \dots, t_{n_i})) \in R_{B_r^t}$. Clearly, $(t \cdot_r s, t) \in R_{\{\epsilon\}}$. For each $j \in \{1, \dots, n_i\}$, $sub_{t \cdot_r s}(j) = t_j \cdot_r s$ and $sub_t(j) = t_j$. Applying the assumption $(t_j \cdot_r s, t_j) \in R_{B_r^{t_j}}$ and Lemma 3.1 (iv), we obtain $(t \cdot_r s, t) \in R_{jB_r^{t_j}}$ for each $j \in \{1, \dots, n_i\}$. Lemma 3.1 (ii) yields $(t \cdot_r s, t) \in R_{\bigcup_{j=1}^{n_i} jB_r^{t_j}}$. More precisely, by Lemma 4.1, $(t', t) \in R_{B_r^t}$. Finally, we need to show that $sub_{t'}(p') = s$ for any $p' \in pos^r(t)$. By Lemma 2.2 (iv), we have $t \cdot_r s \neq r$ and hence $pos^r(t \cdot_r s) = \bigcup_{j=1}^{n_i} (jpos^r(t_j \cdot_r s))$. Since $sub_{t_j \cdot_r s}(p'_j) = s$ for each $j \in \{1, \dots, n_i\}$ and $p'_j \in pos^r(t_j)$, we get $sub_{t \cdot_r s}(jp'_j) = s$. As $pos^r(t) = \bigcup_{1 \leq j \leq n_i} (jpos^r(t_j))$, we have that $sub_{t'}(p') = sub_{t \cdot_r s}(p') = s$ for any $p' \in pos^r(t)$. On the other hand, we would like to show that $t' = t \cdot_r s$. The proof again tackles on the structure of t . If $r \notin sub(t)$, then $B_r^t = pos(t)$. Since $(t', t) \in R_{B_r^t} = R_{pos(t)}$, by Lemma 3.1 (iii), we get $t' = t$. That is $t' = t = t \cdot_r s$. For $t = r$, we have $pos^r(t) = \{\epsilon\}$ which leads to $R_{B_r^t} = R_\emptyset$. Then $(t', t) \in R_\emptyset = R_{B_r^t}$ for any $t' \in W_\tau^r(X_n)$. Since $sub_{t'}(\epsilon) = s$ due to $\epsilon \in pos^r(t)$, we see that $t' = s = t \cdot_r s$. For $t = f_i(t_1, \dots, t_{n_i}) \neq r$ and $r \in sub(t)$, we inductively assume that if $(t'_j, t_j) \in R_{B_r^{t_j}}$ and $sub_{t'_j}(p'_j) = s$ for each $p'_j \in pos^r(t_j)$, then $t'_j = t_j \cdot_r s$ for all $j \in \{1, \dots, n_i\}$. It is clear that $\epsilon \in B_r^t$. Since $(t', t) \in R_{B_r^t}$, we particularly have $(t', t) \in R_{\{\epsilon\}}$ and hence t' is of the form $f_i(u_1, \dots, u_{n_i})$ for some $u_1, \dots, u_{n_i} \in W_\tau(X_n)$. Since $B_r^t \setminus \{\epsilon\} = \bigcup_{j=1}^{n_i} jB_r^{t_j}$ from Lemma 4.1, we actually get $(t', t) \in R_{\bigcup_{j=1}^{n_i} jB_r^{t_j}}$. Therefore, $(u_j, t_j) \in R_{B_r^{t_j}}$ for each $j \in \{1, \dots, n_i\}$. Since $pos^r(t) = \bigcup_{j=1}^{n_i} (jpos^r(t_j))$, each $l \in pos^r(t)$ is of the form jl_j for some $j \in \{1, \dots, n_i\}$ and $l_j \in pos^r(t_j)$. By assumption, we have $sub_{t'}(jl_j) = sub_{t'}(l) = s$. Then $sub_{u_j}(l_j) = s$. Since $l = jl_j$ is arbitrary, l_j is also arbitrary. By induction hypothesis, we obtain $u_j = t_j \cdot_r s$ for each $j \in \{1, \dots, n_i\}$. Therefore, $t' = f_i(u_1, \dots, u_{n_i}) = f_i(t_1 \cdot_r s, \dots, t_{n_i} \cdot_r s) = f_i(t_1, \dots, t_{n_i}) \cdot_r s = t \cdot_r s$. \square

Having known Lemma 4.2, right ideals of $(W_\tau^r(X_n), \cdot_r)$ can now be characterized.

Theorem 4.1. *Let $r \in W_\tau(X_n)$ be fixed and $\emptyset \neq I \subseteq W_\tau^r(X_n)$. Then I is a right ideal of $(W_\tau^r(X_n), \cdot_r)$ if and only if I satisfies the following condition: if $t \in I$, then $t' \in I$ for all $t' \in W_\tau^r(X_n)$ such that $(t', t) \in R_{B_r^t}$ and $sub_{t'}(p') = s$ for some $s \in W_\tau^r(X_n)$ and any $p' \in pos^r(t)$.*

Discussing the previous theorem, if $t \in I$ and $r \in sub(t)$, then there are infinite possibilities of the term t' due to the fact that the infinite terms s could be. In contrast, t' can only be t whenever $r \notin sub(t)$. These arguments give the condition of finite right ideals of $(W_\tau^r(X_n), \cdot_r)$.

Corollary 4.1. *Let $r \in W_\tau(X_n)$ be fixed and $\emptyset \neq I \subseteq W_\tau^r(X_n)$. Then I is a finite right ideal of $(W_\tau^r(X_n), \cdot_r)$ if and only if I contains only a finite number of terms $t_1, \dots, t_n \in W_\tau^r(X_n)$ such that $r \notin sub(t_j)$ for each $j \in \{1, \dots, n\}$.*

Left ideals are the next kind of ideals to be characterized. Based on Lemma 4.2, we obtain a beneficial lemma concerning left ideals of $(W_\tau^r(X_n), \cdot_r)$. For any terms $s, t \in W_\tau(X_n)$, we denote

$$A_s^t := \{p \in pos(t) \mid q \text{ is a prefix of } p \text{ for some } q \in pos^s(t)\}.$$

Note that $pos(t) = A_s^t \cup B_s^t$ where \cup represents a disjoint union.

Lemma 4.3. *Let $r \in W_\tau(X_n)$ be fixed and $s, t \in W_\tau^r(X_n)$. Then $t = u \cdot_r s$ for some $u \in W_\tau^r(X_n)$ if and only if $pos^r(t) \subseteq A_s^t$.*

Proof: The result is immediate whenever $r \notin sub(t)$ since $t = t \cdot_r s$ by Lemma 2.2 (ii) and $pos^r(t) = \emptyset$. Therefore, we only need to deal with the situation where $r \in sub(t)$.

Assume that $t = u \cdot_r s$. By Lemma 4.2, $(t, u) \in R_{B_r^u}$ and $sub_t(p') = s$ for each $p' \in pos^r(u)$. Since $t = u \cdot_r s$ and $r \in sub(t)$, Lemma 2.2 (iii) implies that $r \in sub(u)$ and $r \in sub(s)$. As $r \in sub(u)$, by Lemma 2.2 (i), we get $s \in sub(u \cdot_r s) = sub(t)$. By the definition of B_r^u , it is not hard to see that $B_r^u \cap pos^r(u) = \emptyset$. Since $(t, u) \in R_{B_r^u}$ and $B_r^u \cap pos^r(u) = \emptyset$, we obtain $B_r^u \cap pos^r(t) = \emptyset$. Note that $pos(u) = B_r^u \cup A_r^u$. Since $(t, u) \in R_{B_r^u}$ and $sub_t(p') = s$ for each $p' \in pos^r(u)$, we have that $pos(t) = B_r^u \cup A_s^t$. Altogether with $B_r^u \cap pos^r(t) = \emptyset$, we actually get $pos^r(t) \subseteq A_s^t$. On the other hand, assume that $pos^r(t) \subseteq A_s^t$. Since $r \in sub(t)$, $pos^r(t) \neq \emptyset$ and so is A_s^t . It follows that $pos^s(t) \neq \emptyset$. Consider the term $u \in W_\tau^r(X_n)$ such that $(t, u) \in R_{B_s^t}$ and $sub_u(p') = r$ for all $p' \in pos^s(t)$. It follows that $pos^s(t) \subseteq pos^r(u)$. If $t = s$, then u is obviously r and hence $t = s = u \cdot_r s$. Next, suppose that $t \neq s$, $s \in sub(t)$, and there is $a \in pos^r(u) \setminus pos^s(t)$. Then $B_s^t \neq \emptyset$. If $a \in pos(t)$, then either $a \in B_s^t$ or $a \in A_s^t \setminus pos^s(t)$. The case $a \in B_s^t$ would provide, with $a \in pos^r(u) \setminus pos^s(t)$ and $(t, u) \in R_{B_s^t}$, that $sub_t(a) = r$. It follows that $a \in pos^r(t) \subseteq A_s^t$, a contradiction to the disjunction of A_s^t and B_s^t . The case $a \in A_s^t \setminus pos^s(t)$ implies that there is $a_0 \in pre(a) \setminus \{a\}$ such that $a_0 \in pos^s(t)$. Then $a_0 \in pos^r(u)$. Altogether with $a \in pos^r(u) \setminus pos^s(t)$, it follows that r is a proper subterm of r , a contradiction. If $a \notin pos(t)$, then as $(t, u) \in R_{B_s^t}$ and $B_s^t \neq \emptyset$, there is $b \in B_s^t$ with the shortest length such that $b \in pre(a)$. However, this creates an additional path from b to a in the term u but not of the term t , i.e., $(t, u) \notin R_{\{b\}}$. By Lemma 3.1 (i), $(t, u) \notin R_{B_s^t}$, a contradiction. As a consequence, we obtain $pos^s(t) = pos^r(u)$. Since $(t, u) \in R_{B_s^t}$ and $pos^r(u) = pos^s(t)$, it follows that $B_r^u = B_s^t$. Hence, the term u we consider in fact satisfies $(t, u) \in R_{B_r^u}$ and $sub_t(p) = s$ for all $p \in pos^r(u)$. By Lemma 4.2, we finally obtain $t = u \cdot_r s$. \square

The previous lemma leads to the very important characterization of left ideals of $(W_\tau^r(X_n), \cdot_r)$.

Theorem 4.2. *Let $r \in W_\tau(X_n)$ be fixed and $\emptyset \neq I \subseteq W_\tau^r(X_n)$. Then I is a left ideal of $(W_\tau^r(X_n), \cdot_r)$ if and only if $s \in I$ implies that $t \in I$ for every $t \in W_\tau^r(X_n)$ such that $pos^r(t) \subseteq A_s^t$.*

Discussing the above theorem, since $I \neq \emptyset$, there is $s \in I$. Assuming that I is a left ideal of $(W_\tau^r(X_n), \cdot_r)$ would provide us $t \in I$ for each $t \in W_\tau^r(X_n)$ such that $r \notin sub(t)$ because such t satisfies $pos^r(t) = \emptyset \subseteq A_s^t$. This illustrates the following essential result.

Corollary 4.2. *Let $r \in W_\tau(X_n)$ be fixed. If I is a left ideal of $(W_\tau^r(X_n), \cdot_r)$, then $\{t \in W_\tau^r(X_n) \mid r \notin sub(t)\} \subseteq I$.*

Unlike some right ideals of $(W_\tau^r(X_n), \cdot_r)$, no left ideals of such semigroup, except some specific cases involving $n = 1$, can be finite since the set $\{t \in W_\tau^r(X_n) \mid r \notin sub(t)\}$ is always infinite.

Characterization of (two-sided) ideals of $(W_\tau^r(X_n), \cdot_r)$ can then be achieved from Theorems 4.1 and 4.2.

Theorem 4.3. *Let $r \in W_\tau(X_n)$ be fixed and $\emptyset \neq I \subseteq W_\tau^r(X_n)$. Then I is an ideal of $(W_\tau^r(X_n), \cdot_r)$ if and only if I satisfies the following two conditions:*

- (i) *if $t \in I$, then $t' \in I$ for all $t' \in W_\tau^r(X_n)$ such that $pos^r(t') \subseteq A_t^t$;*
- (ii) *if $t \in I$, then $t' \in I$ for all $t' \in W_\tau^r(X_n)$ such that $(t', t) \in R_{B^t}$ and $sub_{t'}(p') = s$ for some $s \in W_\tau^r(X_n)$ and any $p' \in pos^r(t)$.*

Given a nonempty subset I of $W_\tau^r(X_n)$, we can now verify whether it is a left, a right, or a two-sided ideal of $(W_\tau^r(X_n), \cdot_r)$. Doing so may practically take much time since each characterization is recursive; however, we can primarily check the set I via Corollaries 4.1 and 4.2 which are consequences of Theorems 4.1 and 4.2, respectively.

Example 4.1. Let $\tau = (2)$ with a binary operation symbol f , $n = 5$, and $r = f(x_1, x_2)$.

(i) Let $I_1 = \{x_4, x_5, f(x_2, x_2)\} \subseteq W_{(2)}^r(X_3)$. Then I_1 is easily seen by Corollary 4.2 to not be a left ideal since $\{t \in W_{(2)}^r(X_3) \mid r \notin \text{sub}(t)\} \not\subseteq I_1$. Meanwhile, I_1 is unquestionably a right ideal by Corollary 4.1. These also imply that I_1 is not an ideal.

(ii) Let $C_2 = \{t \in W_{(2)}^r(X_3) \mid r \notin \text{sub}(t)\}$, $D_2 = \{f(x_1, s) \mid s \in W_{(2)}^r(X_3)\}$, and $I_2 = C_2 \cup D_2$. Firstly, we consider I_2 based on Theorem 4.1. For any term $c \in C_2$, we have that $B_r^c = \text{pos}(c)$ and hence $R_{B_r^c} = \{(c, c)\}$. So, terms satisfying the condition of Theorem 4.1 which relates to $c \in C_2$ are terms in C_2 . Next, consider the term $d \in D_2$ with $r \in \text{sub}(d)$, i.e., $d \in D_2 \setminus C_2$. Starting from $d_1 = f(x_1, r) \in D_2 \setminus C_2$, we see that $B_r^{d_1} = \{\epsilon, 1\}$ and thus $\{t \in W_{(2)}^r(X_3) \mid (t, d_1) \in R_{B_r^{d_1}}\} = D_2$. This means that the terms satisfying the condition of Theorem 4.1 which relates to d_1 belong to D_2 . Note that the other terms $d_2 \in D_2 \setminus C_2$ lead to $B_r^{d_1} \subseteq B_r^{d_2}$. Lemma 3.1 (i) yields $R_{B_r^{d_2}} \subseteq R_{B_r^{d_1}}$ which implies that the terms related to d_2 and satisfying the condition of Theorem 4.1 are contained in $\{t \in W_{(2)}^r(X_3) \mid (t, d_1) \in R_{B_r^{d_1}}\} = D_2$. Consequently, I_2 is a right ideal of $(W_{(2)}^r(X_3), \cdot_r)$. Secondly, we investigate I_2 based on Theorem 4.2. The term $t' = f(f(x_1, r), x_2)$ has $s' = f(x_1, r) \in D_2 \subseteq I_2$ as its subterm. We see that $\text{pos}^r(t') = \{12\} \subseteq \{1, 11, 12\} = A_s^t$, despite $t' \notin I_2$. By Theorem 4.2, we obtain that I_2 is not a left ideal and so it is not an ideal.

Our next targets will be characterizations of principal left, right, and two-sided ideals of $(W_\tau^r(X_n), \cdot_r)$ on which we mainly focus their maximal and minimal versions. For convenience, let $\langle t \rangle_L$, $\langle t \rangle_R$, and $\langle t \rangle$ be the left, right, and two-sided ideal generated by a term $t \in W_\tau^r(X_n)$, respectively. Recall by the facts for any semigroup that

$$\begin{aligned} \langle t \rangle_L &= \{t\} \cup (W_\tau^r(X_n) \cdot_r \{t\}), \\ \langle t \rangle_R &= \{t\} \cup (\{t\} \cdot_r W_\tau^r(X_n)), \\ \langle t \rangle &= \{t\} \cup (W_\tau^r(X_n) \cdot_r \{t\}) \cup (\{t\} \cdot_r W_\tau^r(X_n)) \cup (W_\tau^r(X_n) \cdot_r \{t\} \cdot_r W_\tau^r(X_n)). \end{aligned}$$

Here, the product of two nonempty sets A and B is defined as

$$A \cdot_r B = \{a \cdot_r b \mid a \in A, b \in B\}.$$

Note that since r is the identity of $(W_\tau^r(X_n), \cdot_r)$, we easily see that all of $\langle r \rangle_L$, $\langle r \rangle_R$, and $\langle r \rangle$ become $W_\tau^r(X_n)$.

A left [right, two-sided] ideal I of a semigroup $(W_\tau^r(X_n), \cdot_r)$ is said to be *maximal* if there is no left [right, two-sided] ideal J with $I \subsetneq J \subsetneq W_\tau^r(X_n)$. A *minimal* aspect can be defined dually. Due to the fact that the identity r generates the whole base set, $W_\tau^r(X_n)$, as the left [right, two-sided] principal ideal, the definition of a maximal and minimal principal left [right, two-sided] ideal can be provided in a similar sense as that of left [right, two-sided] ideal.

We begin with characterizations of minimality of both left ideals and right ideals of the semigroup $(W_\tau^r(X_n), \cdot_r)$. From Corollary 4.2, we know the necessary condition of left ideals of such semigroup which turns to be useful in searching for a minimal left ideal.

Theorem 4.4. Let $r \in W_\tau^r(X_n)$ be fixed. Then the set $\{t \in W_\tau^r(X_n) \mid r \notin \text{sub}(t)\}$ is the only minimal left ideal of $(W_\tau^r(X_n), \cdot_r)$.

Proof: Let $N := \{t \in W_\tau^r(X_n) \mid r \notin \text{sub}(t)\}$, $s \in W_\tau^r(X_n)$, and $t \in N$. Then $r \notin \text{sub}(t)$. Lemma 2.2 (iii) yields $r \notin \text{sub}(s \cdot_r t)$ which implies that $s \cdot_r t \in N$. Therefore, N is a left ideal of $(W_\tau^r(X_n), \cdot_r)$. The minimality and uniqueness come from Corollary 4.2. \square

The application of Lemma 2.2 (iii) and Corollary 4.2 also provides that $\langle t \rangle_L = \{t' \in W_\tau^r(X_n) \mid r \notin \text{sub}(t')\}$ for all $t \in W_\tau^r(X_n)$ such that $r \notin \text{sub}(t)$. It is actually the only minimal principal left ideal of $(W_\tau^r(X_n), \cdot_r)$.

The following lemma is useful in minimal-right-ideal consideration.

Lemma 4.4. *Let $r \in W_\tau(X_n)$ be fixed. If I is a right ideal of $(W_\tau^r(X_n), \cdot_r)$, then I contains at least one element $t \in W_\tau^r(X_n)$ with $r \notin \text{sub}(t)$.*

Proof: Assume that $I \subseteq W_\tau^r(X_n)$ is a right ideal. Let $s \in I$ and $u \in W_\tau^r(X_n)$ with $r \notin \text{sub}(u)$. Then $s \cdot_r u \in I$. Also, by Lemma 2.2 (iii), we have $r \notin \text{sub}(s \cdot_r u)$. \square

Minimal right ideals are characterized in the next theorem. It appears that minimal right ideals are not unique, whereas the minimal left ideal is.

Theorem 4.5. *Let $r \in W_\tau(X_n)$ be fixed. Then minimal right ideals of $(W_\tau^r(X_n), \cdot_r)$ are of the form of singleton sets $\{t\}$ in which $r \notin \text{sub}(t)$.*

Proof: A singleton set of such t is easily seen to be a right ideal of $(W_\tau^r(X_n), \cdot_r)$ by applying Lemma 2.2 (ii) and it is indeed minimal as its cardinality is 1. The guarantee of having no other forms of minimal right ideals is achieved from Lemma 4.4. \square

On principal perspective, we see from Lemma 2.2 (ii) that for any $t \in W_\tau^r(X_n)$ such that $r \notin \text{sub}(t)$, $\langle t \rangle_R = \{t\}$. Altogether with Lemma 4.4, we see that minimal principal right ideals take the same form of those of minimal right ideals.

We continue characterizing maximal ideals of both left ideals and right ideals as well as of those principal aspects. Corollary 4.2 provides $\{t \in W_\tau^r(X_n) \mid r \notin \text{sub}(t)\} \subseteq L$ for any left ideal L of $(W_\tau^r(X_n), \cdot_r)$ and the inclusion would be proper if L contains any term which has r as its subterm. This ensures that $\langle t \rangle_L \subset \langle t' \rangle_L$ for any $t, t' \in W_\tau^r(X_n)$ such that $r \notin \text{sub}(t)$ and $r \in \text{sub}(t')$. This means that in order to find a maximal principal left ideal of $(W_\tau^r(X_n), \cdot_r)$, we need to consider only those $\langle t' \rangle_L$ where $r \in \text{sub}(t')$. The criteria may also be adapted in the case of principal right ideals, analogously implied from Lemma 4.4.

Note that $t = s \cdot_r u$ implies $\langle t \rangle_L \subseteq \langle u \rangle_L$ and $\langle t \rangle_R \subseteq \langle s \rangle_R$. When we want the term t to generate a maximal principal left ideal or right ideal, we can only allow s and u to be t itself or the identity r . More precisely, the only exceptions of product expressions of t must be $t = t \cdot_r r$ and $t = r \cdot_r t$. Fortunately, we have the following remark concerning an optional support from Lemma 4.3.

Remark 4.1. *Let $r \in W_\tau(X_n)$ and $t \in W_\tau^r(X_n)$ with $r \in \text{sub}(t)$. Then the following statements are equivalent:*

- (i) $\langle t \rangle_L [\langle t \rangle_R]$ is a maximal principal left [right] ideal of $(W_\tau^r(X_n), \cdot_r)$;
- (ii) $\text{pos}^r(t) \subseteq A_v^t$ if and only if $v \in \{r, t\}$.

We denote $\text{Pre}_k(\text{pos}^r(t))$ to be the set of all positions of $\text{pos}^r(t)$ but with the deletion of each k -number of strings from the right. Also, we set $\min(\text{pos}^r(t)) := \{p \in \text{pos}^r(t) \mid l(p) \leq l(q) \text{ for all } q \in \text{pos}^r(t)\}$. For example, let $p = 4321$, $q = 256$, and $\text{pos}^r(t) = \{p, q\}$. Then $\text{Pre}_2(\text{pos}^r(t)) = \{43, 2\}$ and $\min(\text{pos}^r(t)) = \{q\}$. Moreover, generalized from the notation $\text{sub}_t(p)$ for a single position p of a term t , we denote for each nonempty set of positions $P \subseteq \text{pos}(t)$, $\text{sub}_t(P) := \{\text{sub}_t(p) \mid p \in P\}$.

Theorem 4.6. *Let $r \in W_\tau(X_n)$ be fixed and $t \in W_\tau^r(X_n)$ with $r \in \text{sub}(t)$. Then $\langle t \rangle_L [\langle t \rangle_R]$ is a maximal principal left [right] ideal of $(W_\tau^r(X_n), \cdot_r)$ if and only if it falls into one of the following conditions:*

- (i) There exists $p \in \text{pos}^r(t)$ such that $l(p) = 1$;
- (ii) All $p \in \text{pos}^r(t)$ are of length $l(p) \geq 2$ and $|\text{sub}_t(\text{Pre}_k(\text{pos}^r(t)))| \geq 2$ for each $k \in \{1, \dots, l(q) - 1\}$ where $q \in \min(\text{pos}^r(t))$.

Proof: Assume that $\langle t \rangle_L$ satisfies the condition (i). Let $p \in \text{pos}^r(t)$ with $l(p) = 1$. Then $\text{pre}(p) = \{\epsilon, p\}$. It follows that the only possibilities of $\text{pos}^r(t) \subseteq A_v^t$ are when $v = t$ and $v = r$ which correspond to the position ϵ and p , respectively. Hence, $\langle t \rangle_L$ is maximal by Remark 4.1. Next, assume that $\langle t \rangle_L$ satisfies the condition (ii). Let $q \in \min(\text{pos}^r(t))$. By assumption, $l(q) \geq 2$. Note that the possible prefixes of q are of length not exceeding $l(q)$. Let $v \in W_\tau^r(X_n)$ with $q \in A_v^t$, $\text{sub}_t(a) = v$, and $a \in \text{pre}(q)$. Then $l(a) \leq l(q)$, i.e., $l(q) - l(a) \geq 0$. Let $m = l(q) - l(a)$. We see that $a \in \text{Pre}_m(\text{pos}^r(t))$ and $0 \leq m \leq l(q)$. Consider $m \in \{1, \dots, l(q) - 1\}$. By assumption, $|\text{sub}_t(\text{Pre}_m(\text{pos}^r(t)))| \geq 2$. This means that there is another term $u \neq v$ with $u \in \text{sub}_t(\text{Pre}_m(\text{pos}^r(t)))$. Since $u, v \in \text{sub}_t(\text{Pre}_m(\text{pos}^r(t)))$, their root nodes are labeled at the m distance from some root nodes labeled as r , and so u and v are not a subterm of the other. Hence, there is $c \in \text{pos}^r(t)$ such that $c \in A_u^t$ but $c \notin A_v^t$. Therefore, there are no terms $w \in \text{sub}_t(\text{Pre}_m(\text{pos}^r(t)))$ such that $\text{pos}^r(t) \subseteq A_w^t$. The only possible prefixes of q left are ϵ and q itself which relate to $m = l(q)$ and $m = 0$, respectively. The former corresponds to the whole term t while the latter corresponds to the term r . Therefore, $\langle t \rangle_L$ is maximal by Remark 4.1. Conversely, suppose that $\langle t \rangle_L$ does not satisfy both the conditions (i) and (ii). It follows that all $p \in \text{pos}^r(t)$ are of length $l(p) \geq 2$ and $|\text{sub}_t(\text{Pre}_k(\text{pos}^r(t)))| = 1$ for some $k \in \{1, \dots, l(q) - 1\}$ where $q \in \min(\text{pos}^r(t))$. Hence, there is the unique term $s \in \text{sub}_t(\text{Pre}_k(\text{pos}^r(t)))$. Therefore, $\text{pos}^r(t) \subseteq A_s^t$. Let $s_q \in \text{pre}(q)$ be such that $\text{sub}_t(s_q) = s$. Then $l(s_q) = l(q) - k \notin \{0, l(q)\}$ which means that the term s is not t or r . Therefore, $\langle t \rangle_L$ is not maximal by Remark 4.1. The right ideal aspect can be followed by a similar approach. \square

Maximal left [right] ideals of $(W_\tau^r(X_n), \cdot_r)$ seem to have the unique form.

Theorem 4.7. *Let $r \in W_\tau(X_n)$ be fixed. Then $W_\tau^r(X_n) \setminus \{r\}$ is the only maximal left [right] ideal of $(W_\tau^r(X_n), \cdot_r)$.*

Proof: Let $t \in W_\tau^r(X_n)$ and $s \in W_\tau^r(X_n) \setminus \{r\}$. By Lemma 2.2 (iv), the only possible product expression giving out r is $r \cdot_r r$. Therefore, $t \cdot_r s \in W_\tau^r(X_n) \setminus \{r\}$ which means that $W_\tau^r(X_n) \setminus \{r\}$ is a left ideal of $(W_\tau^r(X_n), \cdot_r)$ and undoubtedly a maximal one. Let $A \subseteq W_\tau^r(X_n)$ be a left ideal such that $A \not\subseteq W_\tau^r(X_n) \setminus \{r\}$. Then $r \in A$. It follows that $t = t \cdot_r r \in A$ and thus $A = W_\tau^r(X_n)$. Consequently, $W_\tau^r(X_n) \setminus \{r\}$ is the only maximal left ideals of $(W_\tau^r(X_n), \cdot_r)$. We can analogously prove for right ideals. \square

5. Conclusions. The core algebras studied in this article are semigroups of terms with an inductive product as their associative binary operation. The positional equivalence and the root equivalence as well as their properties were presented. These concepts were implemented in our main studies. It was discovered that both left-ideal and right-ideal characterizations of the semigroups have a connection with root equivalence and so does two-sided ideals. Minimal left ideals and minimal right ideals of the semigroup were realized as the set of all terms in the base set with no fixed term r from the inductive product as their subterms and a singleton set of such term, respectively. Their maximal versions were surprisingly discovered as the base set which discards the fixed term r of the inductive product. Minimalities of principal left ideals and principal right ideals take the same classification as those of corresponding ideals. Identified as different classification from that of ideals, maximal principal left ideals and maximal principal right ideals coincide by certain forms of conditions. It is worth mentioning that these results are also valid in the context of the semigroup attached by a term product \cdot_{x_i} which was mentioned in [16] as it is a particular structure of the semigroup in this paper. Possible directions of future works are to study other kinds of ideals such as a quasi-ideal, a bi-ideal, and an interior

ideal, and to investigate behaviour of many kinds of ideals in regularity consideration for the semigroup of inductive terms.

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