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ANTI-HYBRID PURE IDEALS IN ORDERED SEMIGROUPS

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Abstract. *In this paper, the notions of anti-hybrid pure ideals in ordered semigroups are introduced, and some algebraic properties of anti-hybrid pure ideals are studied. We obtain some characterizations of weakly regular ordered semigroups in terms of anti-hybrid pure ideals. Finally, we introduce the concepts of anti-hybrid weakly pure ideals and prove that the anti-hybrid ideals are anti-hybrid weakly pure ideals if such anti-hybrid ideals satisfy idempotent property.*

Keywords: Ordered semigroup, Weakly regular ordered semigroup, Hybrid structure, Anti-hybrid pure ideal, Anti-hybrid weakly pure ideal

1. **Introduction.** The theory of fuzzy sets, introduced by Zadeh [1] in 1965, is the most appropriate theory for dealing with uncertainties at that time. After the introduction, several researchers conducted researches on generalizations of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine, graph theory, logic, operations research and many branches of pure and applied mathematics. For example, Xie et al. applied fuzzy set theory to switching method [2]. The concept of fuzzy sets can be also applied to studying the properties of algebras. Kuroki [3, 4] applied fuzzy set theory to semigroups. In 2007, Kehayopulu and Tsingelis [5] applied fuzzy set theory to ordered semigroups.

In 1999, Molodtsov [6] introduced the concept of soft sets as a new mathematical tool for dealing with uncertainties that are free from the difficulties that have troubled the usual theoretical approaches. Song et al. [7] initiated the study of int-soft semigroups, int-soft left (resp., right) ideals, and int-soft products. In [8], Dudek and Jun introduced and characterized the concept of a soft interior ideal of semigroups. Muhiuddin and Mahboob [9] gave notions and introduced int-soft left (resp., right) ideals, int-soft interior ideals, and int-soft bi-ideals of ordered semigroups over the soft sets.

As a parallel circuit of fuzzy sets and soft sets (or, hesitant fuzzy set), Jun et al. [10] introduced the notion of hybrid structure in a set of parameters over an initial universe set and applied it to BCK/BCI-algebras and linear spaces. They introduced the concepts of hybrid subalgebras, hybrid fields, and hybrid linear spaces.

The concept of hybrid structures can be applied in many areas, including mathematics, statistics, computer science, electrical instruments, industrial operations, business, engineering, social decisions, etc. Anis et al. [11] applied the notion of hybrid structures to semigroups. They introduced the notions of hybrid subsemigroups and hybrid left (resp., right) ideals in semigroups and investigated several properties. Using these notions, they

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considered characterizations of subsemigroups and left (resp., right) ideals. They also introduced the concept of hybrid product and discussed characterizations of hybrid subsemigroups and hybrid left (resp., right) ideals by using the notion of such hybrid product. They provided relations between the hybrid intersection and hybrid product. Elavarasan et al. [12] introduced the notion of hybrid generalized bi-ideals of semigroups and characterizations of regular and left quasiregular semigroups in terms of hybrid generalized bi-ideals. Elavarasan and Jun [13] established some equivalent conditions for semigroups to be regular and intra-regular in terms of hybrid ideals and hybrid bi-ideals. They also characterized left and right simple semigroups, and completely regular semigroups using hybrid ideals and bi-ideals. Not only semigroups can be studied through hybrid structures, but also some other algebraic structures. Many authors have extensively studied the algebraic theory through hybrid structures. In [14], Mekwian and Lekkoksung first applied hybrid structures to ordered semigroups. They characterized regular ordered semigroups by using their hybrid ideals.

Since a hybrid structure is a combination of a soft set and a fuzzy set, we can define a product of two hybrid structures in ordered semigroups that differ from Mekwian and Lekkoksung. Sarasit et al. [15] defined such a new product. They introduced anti-hybrid ideals in ordered semigroups and studied some algebraic properties of anti-hybrid ideals. Several researchers are interested in classifying ordered semigroups in terms of their anti-hybrid ideals, such as Linesawat et al. [16], Linesawat and Lekkoksung [17] and Linesawat et al. [18].

Ahsan and Takahashi [19] introduced the notions of pure ideals and purely prime ideals in semigroups without order. Bashir et al. [20] defined the concepts of pure ideals, weakly purely ideals, and purely prime ideals in ternary semigroups. In [21], Changphas and Sanborisoot introduced the concepts of pure ideals, weakly pure ideals, and purely prime ideals in ordered semigroups. Siribute and Sanborisoot [22] applied fuzzy theory to semigroup theory. They introduced the concepts of pure fuzzy and weakly pure fuzzy ideals in ordered semigroups and characterized weakly regular ordered semigroups by pure fuzzy ideals.

Based on the concept of the purity of fuzzy ideals considered by Siribute and Sanborisoot [22], we apply the concept of the purity to anti-hybrid ideals in ordered semigroups. Firstly, we introduce the concepts of anti-hybrid pure ideals in ordered semigroups and study some algebraic properties of anti-hybrid pure ideals. We characterize weakly regular ordered semigroups in terms of anti-hybrid pure ideals. Finally, the concept of anti-hybrid weakly pure ideals is introduced. We prove that the anti-hybrid ideals are anti-hybrid weakly pure ideals if such anti-hybrid ideals satisfy the idempotent property.

2. **Preliminary.** This section will recall the basic terms and definitions from the ordered semigroup theory and the hybrid structure theory that we will use later in this paper.

A *groupoid* is an algebra $(S;\cdot)$ consisting of a nonempty set S together with a (binary) operation $\cdot$ on S. A *semigroup* $(S;\cdot)$ is a groupoid in which the operation $\cdot$ is associative, that is, $x\cdot(y\cdot z)=(x\cdot y)\cdot z$ for all $x,y,z\in S$.

Definition 2.1 ([5])**.** *The structure $(S;\cdot,\leq)$ is called an ordered semigroup if the following conditions are satisfied:*

(1) $(S;\cdot)$ *is a semigroup;*
(2) $(S;\leq)$ *is a partially ordered set;*
(3) *for every $a,b,c\in S$ if $a\leq b$, then $a\cdot c\leq b\cdot c$ and $c\cdot a\leq c\cdot b$.*

For simplicity, we denoted an ordered semigroup $(S;\cdot,\leq)$ by its carrier set as a bold letter $\mathbf{S}$, and we designate the product $a\cdot b$ of elements a and b of S by ab.

For $K \subseteq S$, we denote

$$(K] := \{a \in S \mid a \le k \text{ for some } k \in K\}.$$

A nonempty subset A of S is called a *subsemigroup* of $\mathbf{S}$ if $(A; \cdot|_{A\times A}, \le_{A\times A})$ is an ordered semigroup.

Let A and B be two nonempty subsets of S. Then we define

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

Definition 2.2 ([5])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. A nonempty subset* A *of* S *is called a* left (resp., right) ideal *of* $\mathbf{S}$ *if*

(1) $SA \subseteq A$ *(resp.,* $AS \subseteq A$*);*
(2) *for* $a \in S$ *and* $b \in A$*, if* $a \le b$*, then* $a \in A$*.*

A nonempty subset I of S is called a *two-side ideal* (*or ideal*) of $\mathbf{S}$ if it is both a left and a right ideal of $\mathbf{S}$.

Definition 2.3 ([21])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. An ideal* I *of* $\mathbf{S}$ *is called a* left (resp., right) pure ideal *in* $\mathbf{S}$ *if for each* $a \in I$ *there exists* $x \in I$ *such that* $a \le xa$ *(resp.,* $a \le ax$*).*

An ideal I of $\mathbf{S}$ is called a *pure ideal* of $\mathbf{S}$ if it is both a left and a right pure ideal of $\mathbf{S}$.

In what follows, let I be the unit interval, i.e., $I = [0, 1]$, S a set of parameters and $\mathcal{P}(U)$ denote the power set of an initial universe set U.

Definition 2.4 ([11])**.** *A hybrid structure in* S *over* U *is defined to be a mapping*

$$\widetilde{f}_\lambda := \left(\widetilde{f}, \lambda\right) : S \to \mathcal{P}(U) \times I, \quad x \mapsto \left(\widetilde{f}(x), \lambda(x)\right),$$

where $\widetilde{f} : S \to \mathcal{P}(U)$ *and* $\lambda : S \to I$ *are mappings.*

We denote by $H(S)$ the set of all hybrid structures in S over U. We define an order $\ll$ on $H(S)$ as follows. For all $\widetilde{f}_\lambda, \widetilde{g}_\gamma \in H(S)$, $\widetilde{f}_\lambda \ll \widetilde{g}_\gamma \Leftrightarrow \widetilde{f} \sqsubseteq \widetilde{g}$ and $\lambda \succeq \gamma$, where $\widetilde{f} \sqsubseteq \widetilde{g}$ means that $\widetilde{f}(x) \subseteq \widetilde{g}(x)$ and $\lambda \succeq \gamma$ means that $\lambda(x) \ge \gamma(x)$ for all $x \in E$. Moreover, $\widetilde{f}_\lambda = \widetilde{g}_\gamma$ if $\widetilde{f}_\lambda \ll \widetilde{g}_\gamma$ and $\widetilde{g}_\gamma \ll \widetilde{f}_\lambda$.

Definition 2.5 ([11])**.** *Let* $\widetilde{f}_\lambda$ *and* $\widetilde{g}_\gamma$ *be hybrid structures in* S *over* U*. Then*

(1) *the hybrid* union *of* $\widetilde{f}_\lambda$ *and* $\widetilde{g}_\gamma$ *is denoted by* $\widetilde{f}_\lambda \Cup \widetilde{g}_\gamma$ *and is defined to be a hybrid structure*

$$\widetilde{f}_\lambda \Cup \widetilde{g}_\gamma : S \to \mathcal{P}(U) \times I, \quad x \mapsto \left(\left(\widetilde{f} \cup \widetilde{g}\right)(x), (\lambda \wedge \gamma)(x)\right),$$

where $\left(\widetilde{f} \cup \widetilde{g}\right)(x) := \widetilde{f}(x) \cup \widetilde{g}(x)$ *and* $(\lambda \wedge \gamma)(x) := \min\{\lambda(x), \gamma(x)\}$*;*

(2) *the hybrid* intersection *of* $\widetilde{f}_\lambda$ *and* $\widetilde{g}_\gamma$ *is denoted by* $\widetilde{f}_\lambda \Cap \widetilde{g}_\gamma$ *and is defined to be a hybrid structure*

$$\widetilde{f}_\lambda \Cap \widetilde{g}_\gamma : S \to \mathcal{P}(U) \times I, \quad x \mapsto \left(\left(\widetilde{f} \cap \widetilde{g}\right)(x), (\lambda \vee \gamma)(x)\right),$$

where $\left(\widetilde{f} \cap \widetilde{g}\right)(x) := \widetilde{f}(x) \cap \widetilde{g}(x)$ *and* $(\lambda \vee \gamma)(x) := \max\{\lambda(x), \gamma(x)\}$*.*

Remark 2.1. *We can extend Definition 2.5 as follows. Let* $\left\{\widetilde{f}_{i,\lambda_i} : i \in I\right\}$ *be a collection of hybrid structure in* S *over* U*. Then*

(1) $\Cup_{i\in I}\widetilde{f}_{i,\lambda_i}$ *is a hybrid structure*

$$\Cup_{i\in I}\widetilde{f}_{i,\lambda_i}\colon S\to\mathcal{P}(U),\quad x\mapsto\left(\left(\bigcup_{i\in I}\widetilde{f}_i\right),\bigwedge_{i\in I}\lambda_i\right),$$

where $\bigcup$ *and* $\bigwedge$ *are usual union operation and infimum operation, respectively;*

(2) $\Cap_{i\in I}\widetilde{f}_{i,\lambda_i}$ *is a hybrid structure*

$$\Cap_{i\in I}\widetilde{f}_{i,\lambda_i}\colon S\to\mathcal{P}(U),\quad x\mapsto\left(\left(\bigcap_{i\in I}\widetilde{f}_i\right),\bigvee_{i\in I}\lambda_i\right),$$

where $\bigcap$ *and* $\bigvee$ *are usual intersection operation and supremum operation, respectively.*

We denote $\widetilde{S}_S$ the hybrid structure in S over U and it is defined as follows:

$$\widetilde{S}_S\colon S\longrightarrow\mathcal{P}(U)\times I,\quad x\mapsto\left(\widetilde{S}(x),S(x)\right),$$

where $\widetilde{S}(x):=\emptyset$ and $S(x):=1$ for all $x\in S$.

From now on, we apply the concept of hybrid structures to an ordered semigroup $\mathbf{S}$. We call a hybrid structure in S over U by a hybrid structure in $\mathbf{S}$ over U. That is, $H(S)=H(\mathbf{S})$.

Let $a\in S$. Then, we set

$$\mathbf{S}_a:=\{(x,y)\in S\times S\mid a\le xy\}.$$

Definition 2.6 ([15])**.** *Let* $\widetilde{f}_\lambda$ *and* $\widetilde{g}_\gamma$ *be hybrid structures in* $\mathbf{S}$ *over* U*. Then the hybrid* product *of* $\widetilde{f}_\lambda$ *and* $\widetilde{g}_\gamma$ *is denoted by* $\widetilde{f}_\lambda\otimes\widetilde{g}_\gamma$ *and is defined to be a hybrid structure*

$$\widetilde{f}_\lambda\otimes\widetilde{g}_\gamma\colon S\to\mathcal{P}(U)\times I,\quad x\mapsto\left(\left(\widetilde{f}\odot\widetilde{g}\right)(x),(\lambda\circ\gamma)(x)\right),$$

where

$$\left(\widetilde{f}\odot\widetilde{g}\right)(x):=\begin{cases}\bigcap\limits_{(a,b)\in\mathbf{S}_x}\left(\widetilde{f}(a)\cup\widetilde{f}(b)\right) & \text{if } \mathbf{S}_x\neq\emptyset,\\ U & \text{otherwise,}\end{cases}$$

and

$$(\lambda\circ\gamma)(x):=\begin{cases}\bigvee\limits_{(a,b)\in\mathbf{S}_x}\{\min\{\lambda(a),\gamma(b)\}\} & \text{if } \mathbf{S}_x\neq\emptyset,\\ 0 & \text{otherwise.}\end{cases}$$

It is not difficult to verify that $(H(\mathbf{S});\otimes,\ll)$ is an ordered semigroup.

Let A be a subset of S. We denote by $\chi_{A^c}\left(\widetilde{S}_S\right)$ the characteristic hybrid structure of complement of A in $\mathbf{S}$ over U and it is defined to be a hybrid structure

$$\chi_{A^c}\left(\widetilde{S}_S\right):S\to\mathcal{P}(U)\times I,\quad x\mapsto\left(\chi_{A^c}\left(\widetilde{S}\right)(x),\chi_{A^c}(S)(x)\right),$$

where

$$\chi_{A^c}\left(\widetilde{S}\right)(x):=\begin{cases}\emptyset & \text{if } x\in A,\\ U & \text{otherwise,}\end{cases}\quad\text{and }\chi_{A^c}(S)(x):=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{otherwise.}\end{cases}$$

We observe that $\chi_{A^c}\left(\widetilde{S}_S\right):=\widetilde{S}_S$ if $A^c=\emptyset$.

3. **Main Results.** In this central section, we introduce the concepts of anti-hybrid pure ideals in ordered semigroups and study some algebraic properties of such anti-hybrid pure ideals. We also characterize weakly regular ordered semigroups in terms of anti-hybrid pure ideals. Finally, we introduce the concept of anti-hybrid weakly pure ideals and prove that any anti-hybrid ideal is an anti-hybrid weakly pure ideal whenever it is idempotent.

Definition 3.1 ([16])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. A hybrid structure* $\widetilde{f}_\lambda$ *in* $\mathbf{S}$ *over* U *is called an* anti-hybrid left (resp., right) ideal *in* $\mathbf{S}$ *over* U *if for every* $x, y \in S$*:*

(1) $\widetilde{f}(xy) \subseteq \widetilde{f}(y)$ *(resp.,* $\widetilde{f}(xy) \subseteq \widetilde{f}(x)$*);*
(2) $\lambda(xy) \geq \lambda(y)$ *(resp.,* $\lambda(xy) \geq \lambda(x)$*);*
(3) $x \leq y$ *implies* $\widetilde{f}(x) \subseteq \widetilde{f}(y)$ *and* $\lambda(x) \geq \lambda(y)$*.*

A hybrid structure in $\mathbf{S}$ over U is called an *anti-hybrid ideal* in $\mathbf{S}$ over U if it is both an anti-hybrid left and an anti-hybrid right ideal in $\mathbf{S}$ over U.

Example 3.1. *Let* $S = \{a, b, c, d\}$*. Define a binary operation* $\cdot$ *on* S *and a binary relation on* S *as follows.*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

and $\leq := \Delta_S \cup \{(a, b)\}$*, where* $\Delta_S := \{(x, x) \mid x \in S\}$*. Then,* $\mathbf{S} := (S; \cdot, \leq)$ *is an ordered semigroup. Let* $U = \{1, 2, 3, 4, 5\}$*. We define a hybrid structure* $\widetilde{f}_\lambda$ *in* $\mathbf{S}$ *over* U *as follows.*

S	$\widetilde{f}$	λ
a	$\{5\}$	0.8
b	$\{1, 5\}$	0.5
c	$\{1, 2, 5\}$	0.1
d	$\{1, 2, 4, 5\}$	0.3

By careful calculation, we see that $\widetilde{f}_\lambda$ *is an anti-hybrid ideal in* $\mathbf{S}$ *over* U*. However, it is not difficult to calculate that* $\widetilde{f}_\lambda$ *is not a hybrid ideal in* $\mathbf{S}$ *over* U *defined by Mekwian and Lekkoksung* [14].

Definition 3.2. *Let* $\mathbf{S}$ *be an ordered semigroup. An anti-hybrid ideal* $\widetilde{f}_\lambda$ *in* $\mathbf{S}$ *over* U *is*

(1) left pure *if* $\widetilde{f}_\lambda \Cup \widetilde{g}_\beta = \widetilde{f}_\lambda \otimes \widetilde{g}_\beta$ *for every anti-hybrid left ideal* $\widetilde{g}_\beta$ *in* $\mathbf{S}$ *over* U*;*
(2) right pure *if* $\widetilde{g}_\beta \Cup \widetilde{f}_\lambda = \widetilde{g}_\beta \otimes \widetilde{f}_\lambda$ *for every anti-hybrid right ideal* $\widetilde{g}_\beta$ *in* $\mathbf{S}$ *over* U*.*

The following remark is a useful tool in calculating the purity of anti-hybrid ideals.

Remark 3.1. *Let* $\widetilde{f}_\lambda$ *be an anti-hybrid left (resp., right) pure ideal in* $\mathbf{S}$ *over* U*. Then for any anti-hybrid left (resp., right) ideal* $\widetilde{g}_\beta$ *in* $\mathbf{S}$ *over* U*, we have*

(1) $\widetilde{f} \cup \widetilde{g} = \widetilde{f} \odot \widetilde{g}$ *(resp.,* $\widetilde{g} \cup \widetilde{f} = \widetilde{g} \odot \widetilde{f}$*);*
(2) $\lambda \wedge \beta = \lambda \circ \beta$ *(resp.,* $\beta \wedge \lambda = \beta \circ \lambda$*).*

An anti-hybrid ideal in $\mathbf{S}$ over U is called an *anti-hybrid pure ideal* in $\mathbf{S}$ over U if it is both an anti-hybrid right and an anti-hybrid left pure ideal in $\mathbf{S}$ over U.

Example 3.2. *Let* $S = \{a, b, c, d\}$*. Define a binary operation* $\cdot$ *on* S *and a binary relation on* S *as follows.*

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	b	c	c
d	a	b	c	c

and $\leq := \Delta_S \cup \{(a,b)\}$, *where* $\Delta_S := \{(x,x) \mid x \in S\}$. *Then,* $\mathbf{S} := (S;\cdot,\leq)$ *is an ordered semigroup. Let* $U = \{1,2,3\}$. *We denote* $L(\mathbf{S})$ *as the set of all hybrid structure* $\widetilde{f}_\lambda$ *in* $\mathbf{S}$ *over* U *such that* $\widetilde{f}_\lambda(a) \ll \widetilde{f}_\lambda(b)$, $\widetilde{f}_\lambda(a) \ll \widetilde{f}_\lambda(c)$ *and* $\widetilde{f}_\lambda(c) \ll \widetilde{f}_\lambda(d)$, *and* $R(\mathbf{S})$ *as the set of all hybrid structure* $\widetilde{f}_\lambda$ *in* $\mathbf{S}$ *over* U *such that* $\widetilde{f}_\lambda(a) \ll \widetilde{f}_\lambda(b)$, $\widetilde{f}_\lambda(b) \ll \widetilde{f}_\lambda(c)$ *and* $\widetilde{f}_\lambda(c) \ll \widetilde{f}_\lambda(d)$. *Then, we can carefully calculate that* $L(\mathbf{S})$ *and* $R(\mathbf{S})$ *are the set of all anti-hybrid left ideals and the set of all anti-hybrid right ideals in* $\mathbf{S}$ *over* U, *respectively. We define a hybrid structure* $\widetilde{f}_\lambda$ *in* $\mathbf{S}$ *over* U *as follows.*

$$\widetilde{f}(x) := \begin{cases} \emptyset & \text{if } x = a,b,c, \\ U & \text{otherwise,} \end{cases} \quad \text{and } \lambda(x) := \begin{cases} 1 & \text{if } x = a,b,c, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in S$. *We see that* $\widetilde{f}_\lambda \in L(\mathbf{S}) \cap R(\mathbf{S})$, *that is,* $\widetilde{f}_\lambda$ *is an anti-hybrid ideal in* $\mathbf{S}$ *over* U. *We can calculate that* $\widetilde{f}_\lambda$ *is not right pure, since there exists an anti-hybrid right ideal* $\widetilde{g}_\beta$ *in* $\mathbf{S}$ *over* U *defined by*

$$\widetilde{g}(x) := \begin{cases} \{a\} & \text{if } x = a,b,c, \\ \{a,b\} & \text{otherwise,} \end{cases} \quad \text{and } \beta(x) := \begin{cases} 0.8 & \text{if } x = a,b,c, \\ 0.1 & \text{otherwise,} \end{cases}$$

such that $\widetilde{g}_\beta \Cup \widetilde{f}_\lambda \neq \widetilde{g}_\beta \otimes \widetilde{f}_\lambda$. *In fact, there exists* $c \in S$, *such that* $\left(\widetilde{g} \cup \widetilde{f}\right)(c) \neq \left(\widetilde{g} \odot \widetilde{f}\right)(c)$ *and* $(\beta \wedge \lambda)(c) \neq (\beta \circ \lambda)(c)$. *For any anti-hybrid left ideal* $\widetilde{h}_\gamma$ *in* $\mathbf{S}$ *over* U, *we can calculate that*

(1) $\left(\widetilde{f} \cup \widetilde{h}\right)(x) = \widetilde{h}(x) = \left(\widetilde{f} \odot \widetilde{h}\right)(x)$ *for all* $x = a,b,c$;

(2) $\left(\widetilde{f} \cup \widetilde{h}\right)(d) = U = \left(\widetilde{f} \odot \widetilde{h}\right)(d)$;

(3) $(\lambda \wedge \gamma)(x) = \gamma(x) = (\lambda \circ \gamma)(x)$ *for all* $x = a,b,c$;

(4) $(\lambda \wedge \gamma)(d) = 0 = (\lambda \circ \gamma)(d)$.

Therefore, $\widetilde{f}_\lambda$ *is an anti-hybrid left pure ideal in* $\mathbf{S}$ *over* U.

The following lemmas are important in illustrating our first theorem. Therefore, we recall such useful tools.

Lemma 3.1 ([17])**.** *Let* $\mathbf{S}$ *be an ordered semigroup and* A *a nonempty subset of* S. *Then the following conditions are equivalent:*

(1) A *is a left (resp., right) ideal of* $\mathbf{S}$;

(2) $\chi_{A^c}\left(\widetilde{S}_S\right)$ *is an anti-hybrid left (resp., right) ideal in* $\mathbf{S}$ *over* U.

As a consequence of the above lemma, we have that A is an ideal of $\mathbf{S}$ if and only if $\chi_{A^c}\left(\widetilde{S}_S\right)$ is an anti-hybrid ideal in $\mathbf{S}$ over U.

Lemma 3.2 ([17])**.** *Let* $\mathbf{S}$ *be an ordered semigroup and* A, B *nonempty subsets of* S. *Then the following statements hold:*

(1) $B^c \subseteq A^c$ *if and only if* $\chi_{B^c}\left(\widetilde{S}_S\right) \subseteq \chi_{A^c}\left(\widetilde{S}_S\right)$;

(2) $\chi_{A^c}\left(\widetilde{S}_S\right) \Cup \chi_{B^c}\left(\widetilde{S}_S\right) = \chi_{A^c \cup B^c}\left(\widetilde{S}_S\right) = \chi_{(A \cap B)^c}\left(\widetilde{S}_S\right)$;

(3) $\chi_{A^c}\left(\widetilde{S}_S\right) \otimes \chi_{B^c}\left(\widetilde{S}_S\right) = \chi_{(A^cB^c]}\left(\widetilde{S}_S\right) = \chi_{(AB]^c}\left(\widetilde{S}_S\right)$.

Lemma 3.3 ([21])**.** *Let* $\mathbf{S}$ *be an ordered semigroup and* A *an ideal of* $\mathbf{S}$*. Then the following statements are equivalent:*

(1) *A is a right pure ideal of $\mathbf{S}$;*
(2) *$B \cap A = (BA]$ for every right ideal B of $\mathbf{S}$.*

Similarly, we obtain the following lemma.

Lemma 3.4 ([21])**.** *Let* $\mathbf{S}$ *be an ordered semigroup and* A *an ideal of* $\mathbf{S}$*. Then the following statements are equivalent:*

(1) *A is a left pure ideal of $\mathbf{S}$;*
(2) *$A \cap B = (AB]$ for every left ideal B of $\mathbf{S}$.*

The following theorem provides a characterization of right (resp., left) pure ideals in ordered semigroups using anti-hybrid right (resp., left) pure ideals.

Theorem 3.1. *Let A be an ideal of an ordered semigroup $\mathbf{S}$. Then the following conditions are equivalent:*

(1) *A is a right (resp., left) pure ideal of $\mathbf{S}$;*
(2) *$\chi_{A^c}\left(\widetilde{S}_S\right)$ is an anti-hybrid right (resp., left) pure ideal in $\mathbf{S}$ over U.*

Proof: (1)⇒(2). Suppose that A is a right pure ideal of $\mathbf{S}$. Then, by Lemma 3.1, $\chi_{A^c}\left(\widetilde{S}_S\right)$ is an anti-hybrid ideal in $\mathbf{S}$ over U. Let $\widetilde{f}_\lambda$ be an anti-hybrid right ideal in $\mathbf{S}$ over U. Given $a \in S$. Suppose that $a \notin A$. If $\mathbf{S}_a = \emptyset$, then

$$\left(\widetilde{f} \odot \chi_{A^c}\left(\widetilde{S}\right)\right)(a) = U = \widetilde{f}(a) \cup U = \widetilde{f}(a) \cup \chi_{A^c}\left(\widetilde{S}\right)(a) = \left(\widetilde{f} \cup \chi_{A^c}\left(\widetilde{S}\right)\right)(a),$$

and

$$\big(\lambda \circ \chi_{A^c}(S)\big)(a) = 0 = \min\{\lambda(a), 0\} = \min\{\lambda(a), \chi_{A^c}(S)(a)\} = \big(\lambda \cap \chi_{A^c}(S)\big)(a).$$

If $\mathbf{S}_a \neq \emptyset$, then, by the right purity of A, we have that $v \notin A$ for all $(u,v) \in \mathbf{S}_a$. Then $\left(\widetilde{f} \odot \chi_{A^c}\left(\widetilde{S}\right)\right)(a) = U = \left(\widetilde{f} \cup \chi_{A^c}\left(\widetilde{S}\right)\right)(a)$ and $\big(\lambda \circ \chi_{A^c}(S)\big)(a) = 0 = \big(\lambda \cap \chi_{A^c}(S)\big)(a)$. Now, we assume that $a \in A$. By the right purity of A, we have that $\mathbf{S}_a \neq \emptyset$. More precisely, there exists $(a,x) \in \mathbf{S}_a$ such that $x \in A$. Then, by the anti-hybrid right ideality of $\widetilde{f}_\lambda$ and the anti-hybrid left ideality of $\chi_{A^c}\left(\widetilde{S}_S\right)$, we have that

$$\begin{aligned}
\left(\widetilde{f} \cup \chi_{A^c}\left(\widetilde{S}\right)\right)(a) = \widetilde{f}(a) &= \widetilde{f}(a) \cup \chi_{A^c}\left(\widetilde{S}\right)(x) \supseteq \left(\widetilde{f} \odot \chi_{A^c}\left(\widetilde{S}\right)\right)(a) \\
&= \bigcap_{(u,v)\in\mathbf{S}_a} \left(\widetilde{f}(u) \cup \chi_{A^c}\left(\widetilde{S}\right)(v)\right) \supseteq \bigcap_{(u,v)\in\mathbf{S}_a} \left(\widetilde{f}(uv) \cup \chi_{A^c}\left(\widetilde{S}\right)(uv)\right) \\
&\supseteq \bigcap_{(u,v)\in\mathbf{S}_a} \left(\widetilde{f}(a) \cup \chi_{A^c}\left(\widetilde{S}\right)(a)\right) = \widetilde{f}(a) \cup \chi_{A^c}\left(\widetilde{S}\right)(a) \\
&= \left(\widetilde{f} \cup \chi_{A^c}\left(\widetilde{S}\right)\right)(a),
\end{aligned}$$

and

$$\begin{aligned}
\big(\lambda \wedge \chi_{A^c}(S)\big)(a) = \lambda(a) &= \min\{\lambda(a), \chi_{A^c}(S)(x)\} \le \big(\lambda \circ \chi_{A^c}(S)\big)(a) \\
&= \bigvee_{(u,v)\in\mathbf{S}_a} \{\min\{\lambda(u), \chi_{A^c}(S)(v)\}\} \le \bigvee_{(u,v)\in\mathbf{S}_a} \{\min\{\lambda(uv), \chi_{A^c}(S)(uv)\}\}
\end{aligned}$$

$$\leq \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\lambda(a), \chi_{A^c}(S)(a)\}\} = \big(\lambda \circ \chi_{A^c}(S)\big)(a).$$

Altogether, we obtain that $\chi_{A^c}\left(\widetilde{S}_S\right)$ is an anti-hybrid right pure ideal in $\mathbf{S}$ over U.

(2)⇒(1). Let B be a right ideal of $\mathbf{S}$. By Lemma 3.1, we obtain that $\chi_{A^c}\left(\widetilde{S}_S\right)$ and $\chi_{B^c}\left(\widetilde{S}_S\right)$ are an anti-hybrid ideal in $\mathbf{S}$ over U and an anti-hybrid right ideal in $\mathbf{S}$ over U, respectively. Then,

$$\begin{aligned}
\chi_{(B\cap A)^c}\left(\widetilde{S}_S\right) &= \chi_{B^c\cup A^c}\left(\widetilde{S}_S\right) \\
&= \chi_{B^c}\left(\widetilde{S}_S\right) \Cup \chi_{A^c}\left(\widetilde{S}_S\right) && \text{(by Lemma 3.2(2))} \\
&= \chi_{B^c}\left(\widetilde{S}_S\right) \otimes \chi_{A^c}\left(\widetilde{S}_S\right) && \text{(by our presumption)} \\
&= \chi_{(B^cA^c]}\left(\widetilde{S}_S\right) && \text{(by Lemma 3.2(3))} \\
&= \chi_{(BA]^c}\left(\widetilde{S}_S\right).
\end{aligned}$$

By Lemma 3.2, we have $(B \cap A)^c = (BA]^c$, that is, $B \cap A = (BA]$. By Lemma 3.3, A is a right pure ideal of $\mathbf{S}$.

Similarly, we can show that A is left pure if and only if $\chi_{A^c}\left(\widetilde{S}_S\right)$ is left pure. □

By the above theorem, we obtain the following consequence.

Corollary 3.1. *Let A be an ideal of an ordered semigroup $\mathbf{S}$. Then the following conditions are equivalent:*

(1) *A is a pure ideal of $\mathbf{S}$;*

(2) *$\chi_{A^c}\left(\widetilde{S}_S\right)$ is an anti-hybrid pure ideal in $\mathbf{S}$ over U.*

The following results illustrate some properties of anti-hybrid right (resp., left) pure ideals in ordered semigroups.

Theorem 3.2. *Let $\mathbf{S}$ be an ordered semigroup and $\left\{\widetilde{f}_{i,\lambda_i} \,\middle|\, i \in I\right\}$ a family of anti-hybrid right pure ideals in $\mathbf{S}$ over U. Then $\Cup_{i\in I}\widetilde{f}_{i,\lambda_i}$ is an anti-hybrid right pure ideal in $\mathbf{S}$ over U such that*

$$\Cup_{i\in I}\widetilde{f}_{i,\lambda_i} := \left(\bigcup_{i\in I}\widetilde{f}_i, \bigwedge_{i\in I}\lambda_i\right).$$

Proof: Let $\widetilde{g}_\beta$ be an anti-hybrid right ideal in $\mathbf{S}$ over U and $a \in S$. If $\mathbf{S}_a = \emptyset$, then we obtain that $\left(\widetilde{g} \odot \left(\bigcup_{i\in I}\widetilde{f}_i\right)\right)(a) = U = \left(\widetilde{g} \odot \widetilde{f}_i\right)(a) = \left(\widetilde{g} \cup \widetilde{f}_i\right)(a)$ for all $i \in I$. This implies that $\left(\widetilde{g} \odot \left(\bigcup_{i\in I}\widetilde{f}_i\right)\right)(a) = U = \left(\widetilde{g} \cup \left(\bigcup_{i\in I}\widetilde{f}_i\right)\right)(a)$. By a similar methodology, we obtain that $\left(\beta \circ \left(\bigwedge_{i\in I}\lambda_i\right)\right)(a) = 0 = (\beta \circ \lambda_i)(a) = (\beta \wedge \lambda_i)(a)$ for all $i \in I$. This implies that $\left(\beta \circ \left(\bigwedge_{i\in I}\lambda_i\right)\right)(a) = 0 = \left(\beta \cap \left(\bigwedge_{i\in I}\lambda_i\right)\right)(a)$. Suppose that $\mathbf{S}_a \neq \emptyset$. Then,

$$\begin{aligned}
\left(\widetilde{g} \odot \left(\bigcup_{i\in I}\widetilde{f}_i\right)\right)(a) &= \bigcap_{(u,v)\in\mathbf{S}_a}\left(\widetilde{g}(u) \cup \left(\bigcup_{i\in I}\widetilde{f}_i\right)(v)\right) \\
&= \bigcap_{(u,v)\in\mathbf{S}_a}\left(\widetilde{g}(u) \cup \left(\bigcup_{i\in I}\widetilde{f}_i(v)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \bigcup_{i\in I}\left[\bigcap_{(u,v)\in\mathbf{S}_a}\left(\widetilde{g}(u)\cup\widetilde{f}_i(v)\right)\right]\\
&= \bigcup_{i\in I}\left[\left(\widetilde{g}\odot\widetilde{f}_i\right)(a)\right]\\
&= \bigcup_{i\in I}\left[\left(\widetilde{g}\cup\widetilde{f}_i\right)(a)\right]\\
&= \bigcup_{i\in I}\left[\widetilde{g}(a)\cup\widetilde{f}_i(a)\right]\\
&= \widetilde{g}(a)\cup\bigcup_{i\in I}\left(\widetilde{f}_i(a)\right)\\
&= \widetilde{g}(a)\cup\left(\bigcup_{i\in I}\widetilde{f}_i\right)(a)\\
&= \left(\widetilde{g}\cup\left(\bigcup_{i\in I}\widetilde{f}_i\right)\right)(a),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\beta\circ\left(\bigwedge_{i\in I}\lambda_i\right)\right)(a) &= \bigvee_{(u,v)\in\mathbf{S}_a}\left\{\min\left\{\beta(u),\left(\bigwedge_{i\in I}\lambda_i\right)(v)\right\}\right\}\\
&= \bigvee_{(u,v)\in\mathbf{S}_a}\left\{\min\left\{\beta(u),\bigwedge_{i\in I}\lambda_i(v)\right\}\right\}\\
&= \bigwedge_{i\in I}\left[\bigvee_{(u,v)\in\mathbf{S}_a}\{\min\{\beta(u),\lambda_i(v)\}\}\right]\\
&= \bigwedge_{i\in I}(\beta\circ\lambda_i)(a)\\
&= \bigwedge_{i\in I}(\beta\wedge\lambda_i)(a)\\
&= \bigwedge_{i\in I}[\min\{\beta(a),\lambda_i(a)\}]\\
&= \min\left\{\beta(a),\bigwedge_{i\in I}\lambda_i(a)\right\}\\
&= \min\left\{\beta(a),\left(\bigwedge_{i\in I}\lambda_i\right)(a)\right\}\\
&= \left(\beta\cap\left(\bigwedge_{i\in I}\lambda_i\right)\right)(a).
\end{aligned}
$$

Altogether, we have that $\widetilde{g}_\beta \Cup \left(\Cup_{i\in I}\widetilde{f}_{i,\lambda_i}\right) = \widetilde{g}_\beta \otimes \left(\Cup_{i\in I}\widetilde{f}_{i,\lambda_i}\right)$. Therefore, $\Cup_{i\in I}\widetilde{f}_{i,\lambda_i}$ is an anti-hybrid right pure ideal in $\mathbf{S}$ over U. □

By similar method of Theorem 3.2, we have the following theorem.

Theorem 3.3. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\left\{ \widetilde{f}_{i,\lambda_i} \middle| i \in I \right\}$ *a family of anti-hybrid left pure ideals in* $\mathbf{S}$ *over* U*. Then* $\Cup_{i\in I} \widetilde{f}_{i,\lambda_i}$ *is an anti-hybrid left pure ideal in* $\mathbf{S}$ *over* U*.*

Combining Theorems 3.2 and 3.3, we obtain the following result.

Corollary 3.2. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\left\{ \widetilde{f}_{i,\lambda_i} \middle| i \in I \right\}$ *a family of anti-hybrid pure ideals in* $\mathbf{S}$ *over* U*. Then* $\Cup_{i\in I} \widetilde{f}_{i,\lambda_i}$ *is an anti-hybrid pure ideal in* $\mathbf{S}$ *over* U*.*

We show in the next theorem that a finite intersection of anti-hybrid pure ideals is also an anti-hybrid pure ideal.

Theorem 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$*,* $\widetilde{g}_\beta$ *anti-hybrid right (resp., left) pure ideals in* $\mathbf{S}$ *over* U*. Then* $\widetilde{f}_\lambda \Cap \widetilde{g}_\beta$ *is an anti-hybrid right (resp., left) pure ideal in* $\mathbf{S}$ *over* U*.*

Proof: Let $\widetilde{h}_\alpha$ be an anti-hybrid right ideal in $\mathbf{S}$ over U and $a \in S$. If $\mathbf{S}_a = \emptyset$, then

$$\left[\widetilde{h} \odot \left(\widetilde{f} \cap \widetilde{g}\right)\right](a) = \left(\widetilde{h} \odot \widetilde{f}\right)(a) \cap \left(\widetilde{h} \odot \widetilde{g}\right)(a).$$

Since $\widetilde{f}_\lambda$ and $\widetilde{g}_\beta$ are anti-hybrid right pure ideals in $\mathbf{S}$ over U, we obtain that

$$\left(\widetilde{h} \odot \widetilde{f}\right)(a) \cap \left(\widetilde{h} \odot \widetilde{g}\right)(a) = \left[\left(\widetilde{h} \cup \widetilde{f}\right) \cap \left(\widetilde{h} \cup \widetilde{g}\right)\right](a).$$

Therefore, $\left[\widetilde{h} \odot \left(\widetilde{f} \cap \widetilde{g}\right)\right](a) = \left[\widetilde{h} \cup \left(\widetilde{f} \cap \widetilde{g}\right)\right](a)$. Similarly, if $\mathbf{S}_a = \emptyset$, then

$$[\alpha \circ (\lambda \cup \beta)](a) = \max\{(\alpha \circ \lambda)(a), (\alpha \circ \beta)(a)\}.$$

Since $\widetilde{f}_\lambda$ and $\widetilde{g}_\beta$ are anti-hybrid right pure ideals in $\mathbf{S}$ over U, we obtain that

$$\max\{(\alpha \circ \lambda)(a), (\alpha \circ \beta)(a)\} = \max\{(\alpha \cup \lambda)(a), (\alpha \cup \beta)(a)\}.$$

Therefore, $[\alpha \circ (\lambda \cup \beta)](a) = [\alpha \cup (\lambda \cup \beta)](a)$.

Suppose that $\mathbf{S}_a \neq \emptyset$. Then,

$$\begin{aligned}
\left[\widetilde{h} \odot \left(\widetilde{f} \cap \widetilde{g}\right)\right](a) &= \bigcap_{(u,v)\in\mathbf{S}_a} \left[\widetilde{h}(u) \cup \left(\widetilde{f} \cap \widetilde{g}\right)(v)\right] \\
&= \bigcap_{(u,v)\in\mathbf{S}_a} \left[\widetilde{h}(u) \cup \left(\widetilde{f}(v) \cap \widetilde{g}(v)\right)\right] \\
&= \bigcap_{(u,v)\in\mathbf{S}_a} \left[\left(\widetilde{h}(u) \cup \widetilde{f}(v)\right) \cap \left(\widetilde{h}(u) \cup \widetilde{g}(v)\right)\right] \\
&= \bigcap_{(u,v)\in\mathbf{S}_a} \left(\widetilde{h}(u) \cup \widetilde{f}(v)\right) \cap \bigcap_{(u,v)\in\mathbf{S}_a} \left(\widetilde{h}(u) \cup \widetilde{g}(v)\right) \\
&= \left(\widetilde{h} \odot \widetilde{f}\right)(a) \cap \left(\widetilde{h} \odot \widetilde{g}\right)(a) \\
&= \left(\widetilde{h} \cup \widetilde{f}\right)(a) \cap \left(\widetilde{h} \cup \widetilde{g}\right)(a) \\
&= \left[\left(\widetilde{h} \cup \widetilde{f}\right) \cap \left(\widetilde{h} \cup \widetilde{g}\right)\right](a) \\
&= \left[\widetilde{h} \cup \left(\widetilde{f} \cap \widetilde{g}\right)\right](a),
\end{aligned}$$

and

$$[\alpha \circ (\lambda \cup \beta)](a) = \bigvee_{(u,v)\in\mathbf{S}_a} \{\min\{\alpha(u), (\lambda \cup \beta)(v)\}\}$$

$$
\begin{aligned}
&= \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\alpha(u), \max\{\lambda(v), \beta(v)\}\}\} \\
&= \bigvee_{(u,v)\in \mathbf{S}_a} \{\max\{\min\{\alpha(u), \lambda(v)\}, \min\{\alpha(u), \beta(v)\}\}\} \\
&= \max\left\{ \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\alpha(u), \lambda(v)\}\}, \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\alpha(u), \beta(v)\}\} \right\} \\
&= \max\{(\alpha \circ \lambda)(a), (\alpha \circ \beta)(a)\} \\
&= \max\{(\alpha \cup \lambda)(a), (\alpha \cup \beta)(a)\} \\
&= [(\alpha \cup \lambda) \cup (\alpha \cup \beta)](a) \\
&= [\alpha \cup (\lambda \cup \beta)](a).
\end{aligned}
$$

Altogether, we obtain that $\widetilde{h}_\alpha \Cup \left(\widetilde{f}_\lambda \Cap \widetilde{g}_\beta\right) = \widetilde{h}_\alpha \otimes \left(\widetilde{f}_\lambda \Cap \widetilde{g}_\beta\right)$. Therefore, $\widetilde{f}_\lambda \Cap \widetilde{g}_\beta$ is an anti-hybrid right pure ideal in $\mathbf{S}$ over U.

By similar arguments, it is not difficult to illustrate that $\widetilde{f}_\lambda \Cap \widetilde{g}_\beta$ is an anti-hybrid left pure ideal in $\mathbf{S}$ over U whenever $\widetilde{f}_\lambda$ and $\widetilde{g}_\beta$ are anti-hybrid left pure ideal in $\mathbf{S}$ over U. □

By Theorem 3.4, we obtain the following result.

Corollary 3.3. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$, $\widetilde{g}_\beta$ *are anti-hybrid pure ideals in* $\mathbf{S}$ *over* U. *Then* $\widetilde{f}_\lambda \Cap \widetilde{g}_\beta$ *is an anti-hybrid pure ideal in* $\mathbf{S}$ *over* U.

Definition 3.3 ([21])**.** *An ordered semigroup* $\mathbf{S}$ *is said to be* right (resp., left) weakly regular *if for any* $a \in S$ *there exist* $x, y \in S$ *such that* $a \leq axay$ *(resp.,* $a \leq xaya$*).*

An ordered semigroup $\mathbf{S}$ is called *weakly regular* if it is both a right and a left weakly regular ordered semigroup.

Lemma 3.5 ([21])**.** *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is right weakly regular;*
(2) *every ideal of* $\mathbf{S}$ *is a right pure ideal of* $\mathbf{S}$.

We characterize right weakly regular ordered semigroups in terms of anti-hybrid right pure ideals as follows.

Theorem 3.5. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is right weakly regular;*
(2) *every anti-hybrid ideal in* $\mathbf{S}$ *over* U *is right pure.*

Proof: (1)⇒(2). Let $\widetilde{f}_\lambda$ be an anti-hybrid ideal in $\mathbf{S}$ over U and $\widetilde{g}_\beta$ an anti-hybrid right ideal in $\mathbf{S}$ over U. Let $a \in S$. Since $\mathbf{S}$ is right weakly regular, there exist $x, y \in S$ such that $a \leq axay = (ax)(ay)$. This implies that $\mathbf{S}_a \neq \emptyset$. Then,

$$
\begin{aligned}
\left(\widetilde{g} \odot \widetilde{f}\right)(a) &= \bigcap_{(u,v)\in \mathbf{S}_a} \left[\widetilde{g}(u) \cup \widetilde{f}(v)\right] \\
&\supseteq \bigcap_{(u,v)\in \mathbf{S}_a} \left[\widetilde{g}(uv) \cup \widetilde{f}(uv)\right] \\
&\supseteq \bigcap_{(u,v)\in \mathbf{S}_a} \left[\widetilde{g}(a) \cup \widetilde{f}(a)\right]
\end{aligned}
$$

$$\begin{aligned} &= \widetilde{g}(a) \cup \widetilde{f}(a) \\ &= \left(\widetilde{g} \cup \widetilde{f}\right)(a), \end{aligned}$$

and

$$\begin{aligned} \left(\widetilde{g} \odot \widetilde{f}\right)(a) &= \bigcap_{(u,v)\in \mathbf{S}_a} \left[\widetilde{g}(u) \cup \widetilde{f}(v)\right] \\ &\subseteq \left[\widetilde{g}(ax) \cup \widetilde{f}(ay)\right] \\ &\subseteq \widetilde{g}(a) \cup \widetilde{f}(a) \\ &= \left(\widetilde{g} \cup \widetilde{f}\right)(a). \end{aligned}$$

This implies that $\left(\widetilde{g} \odot \widetilde{f}\right)(a) = \left(\widetilde{g} \cup \widetilde{f}\right)(a)$. Similarly, we have

$$\begin{aligned} (\beta \circ \lambda)(a) &= \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\beta(u), \lambda(v)\}\} \\ &\leq \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\beta(uv), \lambda(uv)\}\} \\ &\leq \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\beta(a), \lambda(a)\}\} \\ &= \min\{\beta(a), \lambda(a)\} \\ &= (\beta \cap \lambda)(a), \end{aligned}$$

and

$$\begin{aligned} (\beta \circ \lambda)(a) &= \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\beta(u), \lambda(v)\}\} \\ &\geq \min\{\beta(ax), \lambda(ay)\} \\ &\geq \min\{\beta(a), \lambda(a)\} \\ &= (\beta \cap \lambda)(a). \end{aligned}$$

This implies that $(\beta\circ\lambda)(a) = (\beta \cap \lambda)(a)$. Altogether, we have $\widetilde{g}_\beta \otimes \widetilde{f}_\lambda = \widetilde{g}_\beta \Cup \widetilde{f}_\lambda$. Therefore, $\widetilde{f}_\lambda$ is an anti-hybrid right pure ideal in $\mathbf{S}$ over U.

(2)⇒(1). Let A and B be an ideal and a right ideal of $\mathbf{S}$, respectively. By Lemma 3.1 and Lemma 3.2, we obtain that $\chi_{A^c}\left(\widetilde{S}_S\right)$ and $\chi_{B^c}\left(\widetilde{S}_S\right)$ are an anti-hybrid ideal and an anti-hybrid right ideal in $\mathbf{S}$ over U, respectively. By our assumption, $\chi_{A^c}\left(\widetilde{S}_S\right)$ is an anti-hybrid right pure ideal in $\mathbf{S}$ over U. Then,

$$\begin{aligned} \chi_{(BA]^c}\left(\widetilde{S}_S\right) &= \chi_{B^c}\left(\widetilde{S}_S\right) \otimes \chi_{A^c}\left(\widetilde{S}_S\right) && \text{(by Lemma 3.2(3))} \\ &= \chi_{B^c}\left(\widetilde{S}_S\right) \Cup \chi_{A^c}\left(\widetilde{S}_S\right) && \left(\text{since } \chi_{B^c}\left(\widetilde{S}_S\right) \text{ is right pure}\right) \\ &= \chi_{B^c \cup A^c}\left(\widetilde{S}_S\right) && \text{(by Lemma 3.2(2))} \\ &= \chi_{(B\cap A)^c}\left(\widetilde{S}_S\right). \end{aligned}$$

By Lemma 3.2(1), we obtain that $(BA]^c = (B \cap A)^c$, that is, $(BA] = B \cap A$. This means that A is a right pure ideal of $\mathbf{S}$. Therefore, by Lemma 3.5, $\mathbf{S}$ is right weakly regular. □

Similarly, we obtain the following theorem.

Theorem 3.6. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is left weakly regular;*
(2) *every anti-hybrid ideal in* $\mathbf{S}$ *over* U *is left pure.*

Combining Theorems 3.7 and 3.8, we have the following result.

Corollary 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *is weakly regular;*
(2) *every anti-hybrid ideal in* $\mathbf{S}$ *over* U *is pure.*

Now, we present the concepts of left weakly purity and right weakly purity of anti-hybrid ideals. In our last main result, the coincidence of these two concepts is provided.

Definition 3.4. *An anti-hybrid ideal* $\widetilde{f}_\lambda$ *in* $\mathbf{S}$ *over* U *is called an anti-hybrid* right (resp., left) weakly pure *ideal if* $\widetilde{g}_\beta \otimes \widetilde{f}_\lambda = \widetilde{g}_\beta \Cup \widetilde{f}_\lambda$ *(resp.,* $\widetilde{f}_\lambda \otimes \widetilde{g}_\beta = \widetilde{f}_\lambda \Cup \widetilde{g}_\beta$*) for every anti-hybrid ideal* $\widetilde{g}_\beta$ *in* $\mathbf{S}$ *over* U.

An anti-hybrid ideal is called an *anti-hybrid weakly pure ideal* in $\mathbf{S}$ over U if it is both an anti-hybrid left and an anti-hybrid right weakly pure ideal in $\mathbf{S}$ over U.

A hybrid structure $\widetilde{f}_\lambda$ in $\mathbf{S}$ over U is *idempotent with respect to* $\otimes$ if $\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda = \widetilde{f}_\lambda$.

We recall an auxiliary result which is used in our last theorem.

Lemma 3.6 ([17])**.** *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$, $\widetilde{g}_\beta$ *are anti-hybrid ideals in* $\mathbf{S}$ *over* U. *Then* $\widetilde{f}_\lambda \Cup \widetilde{g}_\beta$ *is an anti-hybrid ideal in* $\mathbf{S}$ *over* U.

Our last result illustrates that the concepts of left weakly purity and right weakly purity of anti-hybrid ideals coincide.

Theorem 3.7. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$ *an anti-hybrid ideal in* $\mathbf{S}$ *over* U. *Then the following statements are equivalent:*

(1) $\widetilde{f}_\lambda$ *is anti-hybrid right weakly pure ideal;*
(2) $\widetilde{f}_\lambda$ *is idempotent with respect to* $\otimes$*;*
(3) $\widetilde{f}_\lambda$ *is anti-hybrid left weakly pure ideal.*

Proof: (1)⇒(2). Let $\widetilde{f}_\lambda$ be an anti-hybrid right weakly pure ideal in $\mathbf{S}$ over U. Then

$$\widetilde{f}_\lambda \otimes \widetilde{f}_\lambda = \widetilde{f}_\lambda \Cup \widetilde{f}_\lambda = \widetilde{f}_\lambda.$$

Therefore, $\widetilde{f}_\lambda$ is idempotent with respect to $\otimes$.

(2)⇒(1). Let $\widetilde{g}_\beta$ be an anti-hybrid ideal in $\mathbf{S}$ over U. By Lemma 3.6, we obtain that $\widetilde{g}_\beta \Cup \widetilde{f}_\lambda$ is an anti-hybrid ideal in $\mathbf{S}$ over U. By our assumption, we have that

$$\widetilde{g}_\beta \Cup \widetilde{f}_\lambda = \left(\widetilde{g}_\beta \Cup \widetilde{f}_\lambda\right) \otimes \left(\widetilde{g}_\beta \Cup \widetilde{f}_\lambda\right) \gg \widetilde{g}_\beta \otimes \widetilde{f}_\lambda.$$

On the other hand, let $a \in S$. If $\mathbf{S}_a = \emptyset$, then $\left(\widetilde{g} \odot \widetilde{f}\right)(a) = U \supseteq \left(\widetilde{g} \cup \widetilde{f}\right)(a)$, and $(\beta \circ \lambda)(a) = 0 \leq (\beta \cap \lambda)(a)$. Suppose that $\mathbf{S}_a \neq \emptyset$. Then,

$$\begin{aligned}\left(\widetilde{g} \odot \widetilde{f}\right)(a) &= \bigcap_{(u,v)\in \mathbf{S}_a} \left[\widetilde{g}(u) \cup \widetilde{f}(v)\right] \\ &\supseteq \bigcap_{(u,v)\in \mathbf{S}_a} \left[\widetilde{g}(uv) \cup \widetilde{f}(uv)\right]\end{aligned}$$

$$
\begin{aligned}
&\supseteq \bigcap_{(u,v)\in \mathbf{S}_a} \left[\widetilde{g}(a) \cup \widetilde{f}(a)\right] \\
&= \widetilde{g}(a) \cup \widetilde{f}(a) \\
&= \left(\widetilde{g} \cup \widetilde{f}\right)(a),
\end{aligned}
$$

and

$$
\begin{aligned}
(\beta \circ \lambda)(a) &= \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\beta(u), \lambda(v)\}\} \\
&\leq \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\beta(uv), \lambda(uv)\}\} \\
&\leq \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{\beta(a), \lambda(a)\}\} \\
&= \min\{\beta(a), \lambda(a)\} \\
&= (\beta \cap \lambda)(a).
\end{aligned}
$$

Thus, we obtain that $\widetilde{g}_\beta \Cup \widetilde{f}_\lambda \ll \widetilde{g}_\beta \otimes \widetilde{f}_\lambda$. Altogether, we have that $\widetilde{g}_\beta \Cup \widetilde{f}_\lambda = \widetilde{g}_\beta \otimes \widetilde{f}_\lambda$. Therefore, $\widetilde{f}_\lambda$ is an anti-hybrid right pure ideal in $\mathbf{S}$ over U.

Illustrating (2)⇔(3) can be done similarly. □

By Theorem 3.7, we obtain the following result.

Corollary 3.5. *Let* $\mathbf{S}$ *be an ordered semigroup and* $\widetilde{f}_\lambda$ *an anti-hybrid ideal in* $\mathbf{S}$ *over* U*. Then the following statements are equivalent:*

(1) $\widetilde{f}_\lambda$ *is idempotent with respect to* $\otimes$*;*

(2) $\widetilde{f}_\lambda$ *is anti-hybrid weakly pure ideal.*

4. **Conclusions.** In this present paper, we introduced the concept of anti-hybrid pure ideals in ordered semigroups. Some related properties of anti-hybrid pure ideals are studied. We characterized weakly regular ordered semigroups in terms of anti-hybrid pure ideals. Finally, we introduced the concept of anti-hybrid weakly pure ideals. We proved that the anti-hybrid ideals are anti-hybrid weakly pure ideals if such anti-hybrid ideals satisfied idempotent property. In our future work, we will apply the notions of anti-hybrid ideals and anti-hybrid pure ideals to the theory of hyperstructures, ordered hyperstructures, semirings, hemirings, groups, BCI/BCK algebras, etc.

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