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STRUCTURES OF NORMED PATH-EDGE SPACES OF IRREDUCIBLE FUZZY GRAPHS

Umilkeram Qasim Obaid
Department of Mathematics
College of Science
Mustansiriyah University
PO Box 14022, Palestine Street, Baghdad, Iraq
umilkeram32@gmail.com; haider@uomustansiriyah.edu.iq

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Abstract. *The algebraic structure of fuzzy graphs based on the adjacency matrix of graphs was studied and used to represent these graphs. In this paper, by defining the normed structure of an irreducible fuzzy digraph, we provide a new structure representation of this graph. Therefore, our aim is to develop generalizations of the ideas of length of a path to more general notion represented by the notion of norm of a path in the digraph. The path between two distinct vertices that goes through a certain edge e_k in this digraph is defined by introducing a new concept called sum-ID-path. This notion leads to designing a set of all sum-ID-paths and edge-disjoint unions of these paths, namely the path-edge space and it is proven as a vector space that is then proven as a normed space, namely normed path-edge space of irreducible fuzzy graph. For the applicability of the modern method, the example presented in this paper illustrates the validity of this technique to classify the fuzzy graph as a normed structure and at the same time, the successful technique is able to identify the main sum-ID-paths that travel through edges in the digraph.*
Keywords: Fuzzy graph, Irreducible graph, Sum-ID-length, Path-edge space, Normed path-edge space

1. **Introduction.** Graph theory is one of the beneficial mathematical concepts that is interested in networks of points linked by lines [1]. It has expanded into many fields of mathematical research with implementations such as biology [2,3], chemistry [4,5], computer science [6] and social network analysis [7]. Another mathematical concept that utilizes a set whose points have degrees of membership is called fuzzy set theory. Zadeh [8] was the originator of the theory of fuzzy set. From then on, great applications are made in many real-life situations that can be modeled using fuzzy set [9,10]. The significant progression in this theory has led to the emergence of a new concept called fuzzy graph. The idea of fuzziness application in graph theory was initially prepared by Rosenfeld in 1975 [11]. To deal with the study of fuzzy graph theory, we refer the readers to [12-15].

The concept of an irreducible digraph was investigated in [4] and further properties related to the corresponding adjacency matrix of irreducible graphs can be gotten in [16]. However, the relation between the algebraic properties of a digraph (the structure of its adjacency matrix) and its irreducibility properties of a digraph that has a large or complicated graph is not able to give all information about its characteristics. Through this description one can look at the digraph that has large vertices or edges and say that normed structures can be an effective method of studying digraphs in which one uses some functional techniques to explicate its structures. This motivates us to investigate the

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mathematical structures of a digraph of an irreducible fuzzy graph $ID_{FG}(V, E)$ especially on normed structures without employing the adjacency matrix that is used to represent a digraph.

Therefore, the first part of this paper, namely Sections 3-5, offers the major contributions that explore a new perspective of the structure of an irreducible fuzzy digraph $ID_{FG}(V, E)$ and how to classify its mathematical structure as a normed space concept based on construction of a vector space concept from the set of all sum-ID-paths between vertices that go through a certain edge e_k in this digraph. In the next part of the paper, we show that these concepts can be applied to any example of $ID_{FG}(V, E)$ or any example in different areas of science that is modeled on an irreducible fuzzy graph. The example given on this work reveals the simple and efficiency of the modern technique to classify $ID_{FG}(V, E)$ as a normed structure. Moreover, the successful technique is able to identify the main sum-ID-paths which are the maximizations over all paths of the minimum sum-ID-length of these paths that travel through edges e_k in the digraph.

In this paper, the study of an irreducible fuzzy graph $ID_{FG}(V, E)$ by using the structure of normed space and sum-path of a graph is presented and organized as follows. Section 2 introduces a few definitions and terminologies pertaining to this study. In Section 3, the path between two distinct vertices that travels through a certain edge e_k in $ID_{FG}(V, E)$ is discussed and defined by introducing a new concept called sum-ID-path. This notion drives to create a set of all sum-ID-paths and edge-disjoint unions of these paths in $ID_{FG}(V, E)$, namely the path-edge space of $ID_{FG}(V, E)$ that is proven as a vector space over $\{0, 1\}^{|E|}$ as shown in Section 4. This leads to constructing a modern kind of normed space, namely normed path-edge space of $ID_{FG}(V, E)$ which is proven as normed space and given in Section 5. The application of main results is established in Section 6. Finally, conclusion closes the paper in Section 7.

2. Preliminaries.

A directed graph (digraph) is of vital importance in many real-life applications and defined as follows.

Definition 2.1. [17] *A digraph $D(V, E)$ represents a certain relation $E \subset V \times V$ between points of a set V where V indicates the set of vertices (or nodes or points), E indicates the set of directed edges (or directed links or directed arcs) and each directed edge of E is an ordered pair of distinct vertices.*

A strongly connected graph is an essential concept in the graph $D(V, E)$. It is worth observing that the term of strongly connected graph is commonly used in directed graphs while the term of connected graph is commonly used in undirected graphs. We deal with a digraph that is strongly connected (or irreducible) during this research and it is defined as follows.

Definition 2.2. [17, 18] *A digraph $D(V, E)$ is termed strongly connected (or irreducible) if each node in the digraph has access to every other node.*

This definition means if given any vertices v_i and v_j in $D(V, E)$, there is a directed path from v_i to v_j. Thus, it is common that every graph $D(V, E)$ is irreducible if and only if its adjacency matrix is irreducible [16]. Note that every cycle is an irreducible subgraph but irreducible is not always cycle [19] (see Figure 1).

Some main concepts of fuzziness in graph that are related to a directed graph are also given and initiated by a notion of fuzzy graph.

Definition 2.3. [11] *A fuzzy graph $D_{FG}(\sigma, \mu)$ with a vertex set V as the underlying set is a pair of mappings such that $\sigma\colon V \to [0, 1]$ is a fuzzy subset of V and $\mu\colon V \times V \to [0, 1]$*

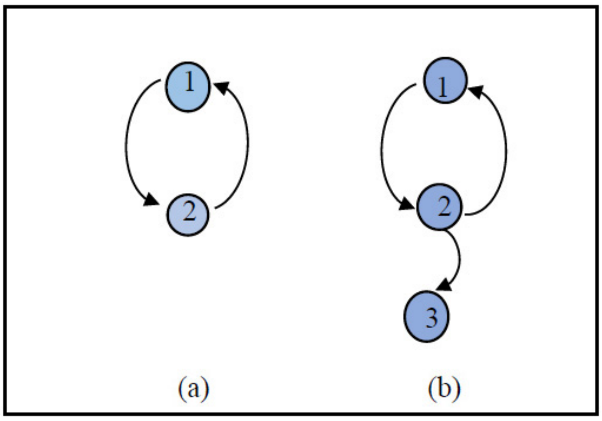


FIGURE 1. (a) The subgraph is irreducible; (b) the subgraph is not irreducible.

is a fuzzy relation on V with $\mu(v_i, v_j) \leq \sigma(v_i) \wedge \sigma(v_j)$, $\forall v_i, v_j \in V$ where $\sigma(v_i) \wedge \sigma(v_j)$ indicates the minimum of $\sigma(v_i)$ and $\sigma(v_j)$.

Definition 2.4. [11] *Let $D_{FG}(\sigma, \mu)$ be a fuzzy graph. Then,*

1) A fuzzy graph $D_{FG_1}(\tau, \omega)$ is called a fuzzy subgraph of $D_{FG}(\sigma, \mu)$ if $\tau(v_i) \leq \sigma(v_i)$, $\forall v_i \in V$ and $\omega(v_i, v_j) \leq \mu(v_i, v_j)$, $\forall v_i, v_j \in V$.

2) A sequence p: $v_i = v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k, \ldots, v_{n-1}, e_n, v_n = v_j$ is a directed path from a vertex v_i to a vertex v_j in $D_{FG}(\sigma, \mu)$ if its sequence p of distinct vertices and edges starting from v_i and ending at v_j such that the membership value $\mu((v_{k-1}, v_k)) > 0$ for $k = 1, \ldots, n$.

3) If v_i and v_j coincide in a directed path p, then p is called a cycle.

It is considered that the underlying crisp graph of the fuzzy graph $D_{FG}(\sigma, \mu)$ is $D(V, E)$. The notion of a path (not cycle) p is highlighted in this work, especially paths that are participated with specific edges, each edge or node plays a crucial functional role in the concept of sum paths as defined in the next section.

3. Sum-ID-Paths Containing a Certain Edge e_k in Irreducible Fuzzy Graphs.

This section contributes essential concepts in this paper. Initially, sum-length of a path in an undirected fuzzy graph was proposed by Vaishnaw and Sharma [20]. Thereby, we modify this definition in a directed fuzzy graph represented by an irreducible graph and defined as follows.

Definition 3.1. *Sum-length of a path in $ID_{FG}(V, E)$. Let $ID_{FG}(V, E)$ be a digraph of an irreducible fuzzy graph. A path $E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n} = (e_1, e_2, \ldots, e_k, \ldots, e_n)$ in $ID_{FG}(V, E)$ is a sequence of distinct vertices and edges in $ID_{FG}(V, E)$ that is initiated by v_i and ended by v_j such that the membership value $\mu((v_{k-1}, v_k)) = \mu(e_k) > 0$, $1 \leq k \leq n$, for each edge $e_k \in E$ and n is the number of edges in this path. Then, a sum-length of a path $E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}$ in $ID_{FG}(V, E)$ is defined as $\left|E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}\right| = \sum_{k=1}^{n} \mu((v_{k-1}, v_k)) = \sum_{k=1}^{n} \mu(e_k) = \mu(e_1) + \mu(e_2) + \cdots + \mu(e_k) + \cdots + \mu(e_n)$ (see Figure 2).*

Then, by using this sum-length $\left|E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}\right|$ of a path in $ID_{FG}(V, E)$, a length between two distinct vertices that goes through a certain edge e_k is introduced in $ID_{FG}(V, E)$ as follows.

Definition 3.2. *Sum-ID-length of a path in $ID_{FG}(V, E)$. Let n be the total number of edges in the path $E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}$ containing an edge e_k in $ID_{FG}(V, E)$ with $|V|$ as the number of all vertices in $ID_{FG}(V, E)$. Then, any sum-length of path $E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}$ that*

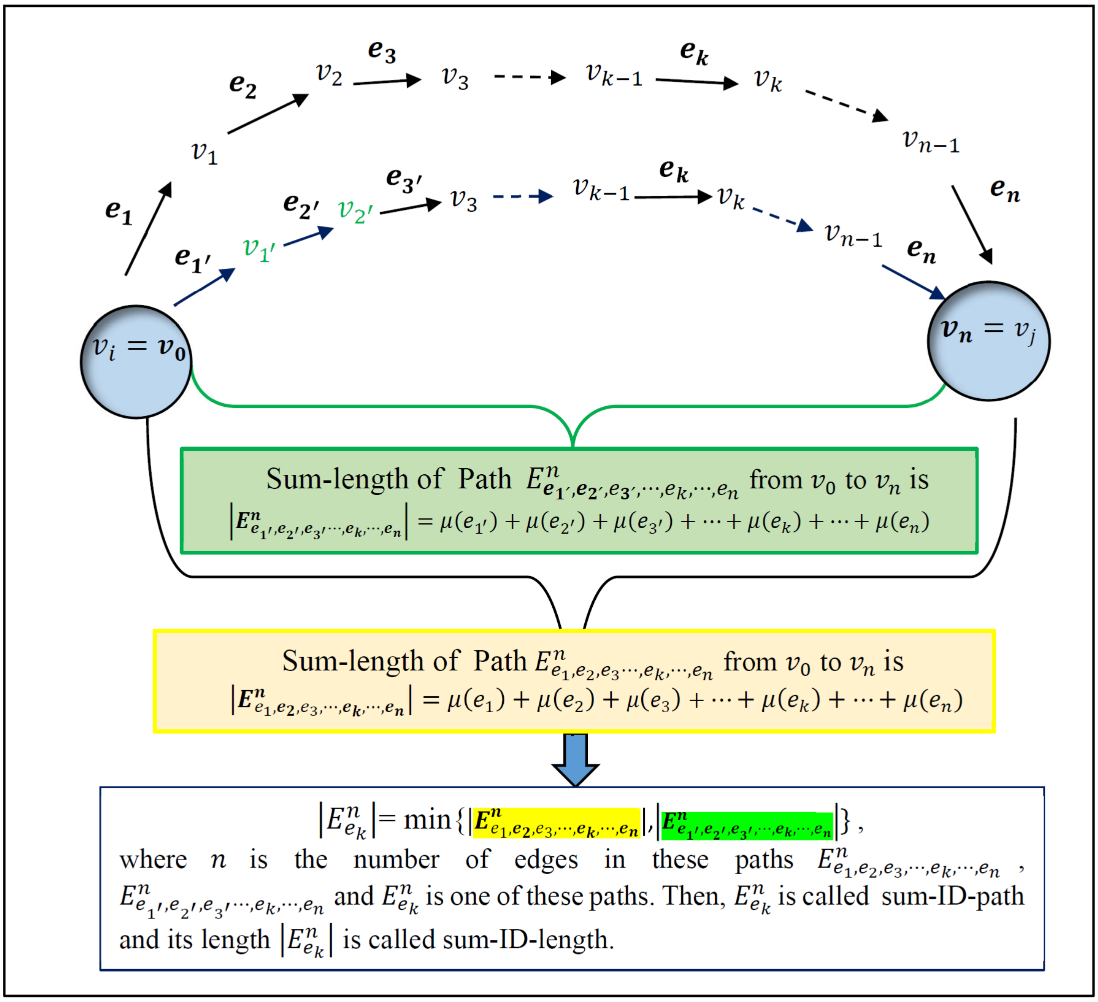


FIGURE 2. Illustration of sum-length $\left|E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}\right|$ and sum-ID-length $\left|E^n_{e_k}\right|$ of a path between v_0 and v_n that contains an edge e_k in $ID_{FG}(V, E)$

goes through the edge e_k and given by $\left|E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}\right| = \sum_{k=1}^{n} \mu(e_k)$ with the total number of edges in these paths the same and equal to n, denoted by $\left|E^n_{e_k}\right|$, is defined by

$$\left|E^n_{e_k}\right| = \min_{n=1,2,\ldots,|V|-1} \left\{\left|E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}\right|\right\} = \min_{n=1,2,\ldots,|V|-1} \sum_{k=1}^{n} \mu(e_k) \tag{1}$$

A path $E^n_{e_1,e_2,\ldots,e_k,\ldots,e_n}$ containing an edge e_k of length $\left|E^n_{e_k}\right|$ is called a sum-ID-path $E^n_{e_k}$ and its length is called a sum-ID-length (see Figure 2).

By participating in specific edges, each edge or node plays a crucial functional role in Definition 3.2. Notice that all vertices in a path should be distinct, hence the minimum sum-length of a path in Equation (1) is taken over all possible pathways connecting v_i to v_j (i.e., from $n = 1$ to $n = |V| - 1$).

Remark 3.1. *If $\mu((v_{k-1}, v_k)) = \mu(e_k) = 1$ for every edge $e_k \in E$ in Definition 3.2, then the sum-ID-length $\left|E^n_{e_k}\right|$ of a path $E^n_{e_k}$ between v_i and v_j gives the length of the shortest path between v_i and v_j that contains an edge e_k in $ID_{FG}(V, E)$. Therefore, this definition gives the classical concept of length in crisp graph.*

4. Vector Space Related to an Irreducible Fuzzy Graph.

Let $|E|$ be the number of all edges in an irreducible fuzzy graph $ID_{FG}(V, E)$. The vector space of all sum-ID-paths and edge-disjoint unions of these paths in $ID_{FG}(V, E)$ is defined and called the path-edge space of $ID_{FG}(V, E)$ as explained in this section.

This vector space is constructed by the following way. For each sum-ID-path $E^n_{e_k} = (e_1, e_2, \ldots, e_k, \ldots, e_n)$ one can relate a vector representation $\vec{E}^n_{e_k} \in \{0, 1\}^{|E|}$, $1 \leq k \leq |E|$, with the ith component equal to 1 if the edge e_k is in this path $E^n_{e_k}$. Obviously, each edge agrees with the direction of the sum-ID-path $E^n_{e_k}$ due to the fact that each edge in this path is traversed in the same direction (see Definition 3.2). Also, the ith component of $\vec{E}^n_{e_k}$ is equal to 0 if the edge e_k is not part of the sum-ID-path $E^n_{e_k}$. In other words, each sum-ID-path $E^n_{e_k}$ has a vector representation in the form of $\vec{E}^n_{e_k} = \left(x_{e_1}, x_{e_2}, \ldots, x_{e_k}, \ldots, x_{e_{|E|}}\right)$ with the entries $x_{e_1}, x_{e_2}, \ldots, x_{e_k}, \ldots, x_{e_{|E|}}$ equal to 1 or 0 and $x_{e_k} = 1$.

The above explanation can be illustrated by Figure 3 in Section 6 to clarify the vector representation $\vec{E}^n_{e_k}$ of the sum-ID-path $E^n_{e_k}$ and also the vector representation of the unions of edge-disjoint of any two sum-ID-paths in $ID_{FG}(V, E)$.

Thus, by all these vector representations $\vec{E}^n_{e_k} \in \{0, 1\}^{|E|}$, the set of all sum-ID-paths and edge-disjoint unions of these paths in $ID_{FG}(V, E)$ can form a vector space over $\{0, 1\}^{|E|}$ associated with every irreducible fuzzy graph $ID_{FG}(V, E)$. Therefore, the path-edge space of $ID_{FG}(V, E)$ is introduced as follows.

Definition 4.1. *For a graph $ID_{FG}(V, E)$, let $S_{PE}(ID_{FG})$ denote the set of all sum-ID-paths and unions of edge-disjoint paths in $ID_{FG}(V, E)$, and an empty graph $\varnothing$ which means all vertices with no edges. Then, $S_{PE}(ID_{FG})$ is called the path-edge space of $ID_{FG}(V, E)$.*

The following step is to consider $S_{PE}(ID_{FG})$ as a vector space over $\{0, 1\}^{|E|}$. The vector addition of two sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$ in $S_{PE}(ID_{FG})$ is given by symmetric difference:

$$E^n_{e_k} \boxplus E^m_{e_l} = \left(E^n_{e_k} \cup E^m_{e_l}\right) \backslash \left(E^n_{e_k} \cap E^m_{e_l}\right) \text{ for each } E^n_{e_k},\ E^m_{e_l} \in S_{PE}(ID_{FG}) \tag{2}$$

Note that the union or intersection of two sum-ID-paths may give up to be sum-ID-paths. However, this operator given by Equation (2) preserves the character of being sum-ID-paths or edge-disjoint unions of these paths in $ID_{FG}(V, E)$. Then, $S_{PE}(ID_{FG})$ is closed under symmetric difference so that a collection of sets closed under this sum operation can be described algebraically as a vector space over $\mathbb{Z}_2$ (with addition and multiplication modulo 2) as seen in Theorem 4.1, where a scalar multiplication $\boxdot$ between a scalar $\lambda \in \{0, 1\}$ and an element in $S_{PE}(ID_{FG})$ is given by

$$1 \boxdot E^n_{e_k} = E^n_{e_k} \text{ and } 0 \boxdot E^n_{e_k} = \varnothing \text{ for each } E^n_{e_k} \in S_{PE}(ID_{FG}),\ \varnothing \in S_{PE}(ID_{FG}) \tag{3}$$

Theorem 4.1. *The path-edge space of $ID_{FG}(V, E)$, $S_{PE}(ID_{FG})$, is a vector space over $\{0, 1\}^{|E|}$ (with addition and multiplication modulo 2).*

Proof: Let $E^n_{e_k}$, $E^m_{e_l}$ be two sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$ in $S_{PE}(ID_{FG})$. Then, clearly, the vector addition $E^n_{e_k} \boxplus E^m_{e_l} = \left(E^n_{e_k} \cup E^m_{e_l}\right) \backslash \left(E^n_{e_k} \cap E^m_{e_l}\right)$ is an element in $S_{PE}(ID_{FG})$. This element may be not sum-ID-path but is certainly unions of edge-disjoint of sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$. Then, $S_{PE}(ID_{FG})$ is closed under the vector addition.

Moreover, all the requirements of the vector addition are satisfied for all $E^n_{e_k}$, $E^m_{e_l}$, $E^t_{e_j}$ in $S_{PE}(ID_{FG})$ as follows.

a) $E^n_{e_k} \boxplus E^m_{e_l} = E^m_{e_l} \boxplus E^n_{e_k}$, meaning the vector addition is commutative. This can be readily observed by definition of the vector addition as given in Equation (2).

b) $\left(E^n_{e_k} \boxplus E^m_{e_l}\right) \boxplus E^t_{e_j} = E^n_{e_k} \boxplus \left(E^m_{e_l} \boxplus E^t_{e_j}\right)$, meaning the vector addition is associative. This condition is proven as follows. Assume that $E^n_{e_k} \boxplus E^m_{e_l}$ is edge-disjoint unions of sum-ID-paths $E^n_{e_k}$ and $E^m_{e_l}$ which is denoted by $E_{E^n_{e_k} \boxplus E^m_{e_l}}$. Then, take again a vector addition of $\left(E^n_{e_k} \boxplus E^m_{e_l}\right)$ and $E^t_{e_j}$ which, easily, are equal to edge-disjoint unions of these paths $\left(E^n_{e_k} \boxplus E^m_{e_l}\right)$ and $E^t_{e_j}$, and denoted by $E_{\left(E^n_{e_k} \boxplus E^m_{e_l}\right) \boxplus E^t_{e_j}}$ which means unions of edge-disjoint of sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$, $E^t_{e_j}$.

In the right side of (b), assume that $E^m_{e_l} \boxplus E^t_{e_j}$ is edge-disjoint unions of sum-ID-paths $E^m_{e_l}$, $E^t_{e_j}$ in $S_{PE}(ID_{FG})$ and denoted by $E_{E^m_{e_l} \boxplus E^t_{e_j}}$. Then, take again a vector addition of $E^n_{e_k}$ and $\left(E^m_{e_l} \boxplus E^t_{e_j}\right)$ which, clearly, are equal to edge-disjoint unions of these paths $E^n_{e_k}$ and $\left(E^m_{e_l} \boxplus E^t_{e_j}\right)$ and denoted by $E_{E^n_{e_k} \boxplus \left(E^m_{e_l} \boxplus E^t_{e_j}\right)}$ which also means unions of edge-disjoint of sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$, $E^t_{e_j}$; i.e., $E_{\left(E^n_{e_k} \boxplus E^m_{e_l}\right) \boxplus E^t_{e_j}} = E_{E^n_{e_k} \boxplus \left(E^m_{e_l} \boxplus E^t_{e_j}\right)}$.

c) The empty graph $\varnothing$ is the additive identity element in $S_{PE}(ID_{FG})$, in view of the fact that $E^n_{e_k} \boxplus \varnothing = \left(E^n_{e_k} \cup \varnothing\right) \setminus \left(E^n_{e_k} \cap \varnothing\right) = E^n_{e_k} \setminus \varnothing = E^n_{e_k} = \varnothing \boxplus E^n_{e_k}$.

d) For every element in $S_{PE}(ID_{FG})$, there is the additive inverse in $S_{PE}(ID_{FG})$, for the reason that $E^n_{e_k} \boxplus E^n_{e_k} = \varnothing$ if and only if $E^n_{e_k} = E^n_{e_k}$. Therefore, each element $E^n_{e_k}$ is its own negative.

By definition of the scalar multiplication in Equation (3), the multiplication of vector by the scalar 1 is the identity operation, $1 \boxdot E^n_{e_k} = E^n_{e_k}$ and multiplication of vector by the scalar 0 takes every element in $S_{PE}(ID_{FG})$ to the empty graph, $0 \boxdot E^n_{e_k} = \varnothing$. Therefore, $S_{PE}(ID_{FG})$ is closed under scalar multiplication. Furthermore, the requirements of the scalar multiplication are satisfied for all $E^n_{e_k}$ and $E^m_{e_l}$ in $S_{PE}(ID_{FG})$ as follows.

e) $\lambda \boxdot \left(E^n_{e_k} \boxplus E^m_{e_l}\right) = \left(\lambda \boxdot E^n_{e_k}\right) \boxplus \left(\lambda \boxdot E^m_{e_l}\right)$ for any scalar λ. Note that $1 \boxdot \left(E^n_{e_k} \boxplus E^m_{e_l}\right) = E^n_{e_k} \boxplus E^m_{e_l} = \left(1 \boxdot E^n_{e_k}\right) \boxplus \left(1 \boxdot E^m_{e_l}\right)$ and $0 \boxdot \left(E^n_{e_k} \boxplus E^m_{e_l}\right) = \varnothing = \varnothing \boxplus \varnothing = \left(0 \boxdot E^n_{e_k}\right) \boxplus \left(0 \boxdot E^m_{e_l}\right)$.

f) $(\lambda +_2 \mu) \boxdot E^n_{e_k} = \left(\lambda \boxdot E^n_{e_k}\right) \boxplus \left(\mu \boxdot E^n_{e_k}\right)$ for any scalars λ and μ. Note that $(1 +_2 1) \boxdot E^n_{e_k} = 0 \boxdot E^n_{e_k} = \varnothing = E^n_{e_k} \boxplus E^n_{e_k} = \left(1 \boxdot E^n_{e_k}\right) \boxplus \left(1 \boxdot E^n_{e_k}\right)$; $(1 +_2 0) \boxdot E^n_{e_k} = 1 \boxdot E^n_{e_k} = E^n_{e_k} = E^n_{e_k} \boxplus \varnothing = \left(1 \boxdot E^n_{e_k}\right) \boxplus \left(0 \boxdot E^n_{e_k}\right)$; $(0 +_2 1) \boxdot E^n_{e_k} = 1 \boxdot E^n_{e_k} = E^n_{e_k} = \varnothing \boxplus E^n_{e_k} = \left(0 \boxdot E^n_{e_k}\right) \boxplus \left(1 \boxdot E^n_{e_k}\right)$; $(0 +_2 0) \boxdot E^n_{e_k} = 0 \boxdot E^n_{e_k} = \varnothing = \varnothing \boxplus \varnothing = \left(0 \boxdot E^n_{e_k}\right) \boxplus \left(0 \boxdot E^n_{e_k}\right)$.

g) $(\lambda \cdot_2 \mu) \boxdot E^n_{e_k} = \lambda \boxdot \left(\mu \boxdot E^n_{e_k}\right)$. Observe that $(1 \cdot_2 \mu) \boxdot E^n_{e_k} = \mu \boxdot E^n_{e_k} = 1 \boxdot \left(\mu \boxdot E^n_{e_k}\right)$; $(0 \cdot_2 \mu) \boxdot E^n_{e_k} = 0 \boxdot E^n_{e_k} = \varnothing = 0 \boxdot \left(\mu \boxdot E^n_{e_k}\right)$.

h) Finally, by definition of the scalar multiplication (3), the multiplication of vector by the scalar 1 is the identity operation, $1 \boxdot E^n_{e_k} = E^n_{e_k}$ for each $E^n_{e_k} \in S_{PE}(ID_{FG})$.

Then, $S_{PE}(ID_{FG})$ is a vector space over $\{0,1\}^{|E|}$; hence every element in $S_{PE}(ID_{FG})$ has a vector representation in $\{0,1\}^{|E|}$. □

By Theorem 4.1, it is shown that every element $E^n_{e_k} \boxplus E^m_{e_l} = E_{E^n_{e_k} \boxplus E^m_{e_l}}$ is appointed by an element $\vec{E}^n_{e_k} +_2 \vec{E}^m_{e_l} = \left(x_{e_1}, x_{e_2}, \ldots, x_{e_k}, \ldots, x_{e_l}, \ldots, x_{e_{|E|}}\right)$ in $\{0,1\}^{|E|}$ with addition modulo 2, for all vector representations $\vec{E}^n_{e_k}$, $\vec{E}^m_{e_l}$ of the sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$ respectively, and the element $E^n_{e_k} \boxplus E^n_{e_k} = \varnothing$ has also a vector representation $\left(x_{e_1}, x_{e_2}, \ldots, x_{e_k}, \ldots, x_{e_l}, \ldots, x_{e_{|E|}}\right) = (0, 0, \ldots, 0, \ldots, 0, \ldots, 0)$.

Moreover, the length of edge-disjoint unions $E^n_{e_k} \boxplus E^m_{e_l} = E_{E^n_{e_k} \boxplus E^m_{e_l}}$ of sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$ in the path-edge space of $ID_{FG}(V,E)$, $S_{PE}(ID_{FG})$, is successively introduced as follows.

Definition 4.2. *Let $S_{PE}(ID_{FG})$ be a vector space over $\{0,1\}^{|E|}$. Then, the length of the element $E^n_{e_k} \boxplus E^m_{e_l} = E_{E^n_{e_k} \boxplus E^m_{e_l}}$ in $S_{PE}(ID_{FG})$ is defined as* $\left|E^n_{e_k} \boxplus E^m_{e_l}\right| = \left|E_{E^n_{e_k} \boxplus E^m_{e_l}}\right| =$

$\sum_{\left(e_i \in E_{E_{e_k}^n \boxplus E_{e_l}^m}\right)} \mu(e_i)$. *The summation indicates that the sum of the length of the element* $E_{e_k}^n \boxplus E_{e_l}^m$ *is the sum of length of unions of edge-disjoint of sum-ID-paths* $E_{e_k}^n$, $E_{e_l}^m$.

To explicate the length of the element $E_{e_k}^n \boxplus E_{e_l}^m = E_{E_{e_k}^n \boxplus E_{e_l}^m}$ in $S_{PE}(ID_{FG})$, let $E_{e_k}^n = (e_1, e_2, e_3, \ldots, e_{k-1}, e_k, \ldots, e_{n-1}, e_n)$ and $E_{e_l}^m = (e_1, e_2, e_{3'}, \ldots, e_{l-1}, e_l, \ldots, e_{m-1}, e_m)$ be two elements in $S_{PE}(ID_{FG})$ with $E_{e_k}^n \cap E_{e_l}^m = \{e_1, e_2\}$ then $E_{e_k}^n \boxplus E_{e_l}^m = E_{E_{e_k}^n \boxplus E_{e_l}^m} = (e_3, \ldots, e_{k-1}, e_k, \ldots, e_{n-1}, e_n, e_{3'}, \ldots, e_{l-1}, e_l, \ldots, e_{m-1}, e_m)$. Subsequently, by Definition 4.2, we get $\left|E_{e_k}^n \boxplus E_{e_l}^m\right| = \left|E_{E_{e_k}^n \boxplus E_{e_l}^m}\right| = \mu(e_3) + \cdots + \mu(e_{k-1}) + \mu(e_k) + \cdots + \mu(e_{n-1}) + \mu(e_n) + \mu(e_{3'}) + \cdots + \mu(e_{l-1}) + \mu(e_l) + \cdots + \mu(e_{m-1}) + \mu(e_m) = \sum_{\left(e_i \in E_{E_{e_k}^n \boxplus E_{e_l}^m}\right)} \mu(e_i)$.

5. Normed Space Related to an Irreducible Fuzzy Graph.

In this section, one can extend the investigation into edges or links go through sum-ID-paths in an irreducible digraph of fuzzy graph $ID_{FG}(V, E)$ and construct a new kind of normed space of $ID_{FG}(V, E)$ which will be called normed path-edge space. Firstly, for given a sum-ID-path $E_{e_k}^n$ in a path-edge space $S_{PE}(ID_{FG})$, from Definition 3.2, $E_{e_k}^n$ has a length $\left|E_{e_k}^n\right|$ where $E_{e_k}^n$ containing the edge e_k with n is the number of all edges in this path. Now, take another sum-ID-path $E_{e_k}^m$ containing the edge e_k with m and a length $\left|E_{e_k}^m\right|$. By going on searching for sum-ID-paths containing the edge e_k but with different lengths, one can supply a methodical way for creating a new normed space from a given vector space in Theorem 4.1 with depending on the following norm:

$$\|E_{e_k}\| = \max_{n=1,2,3,\ldots,|V|-1} \left\{\left|E_{e_k}^n\right|\right\} = \max\left\{\left|E_{e_k}^1\right|, \left|E_{e_k}^2\right|, \left|E_{e_k}^3\right|, \ldots, \left|E_{e_k}^{|V|-1}\right|\right\} \tag{4}$$

From Definition 3.2, the right side of Equation (4) is well defined as every sum-ID-path $E_{e_k}^n$ has a length and a norm of E_{e_k} is one of the maximum length of paths $E_{e_k}^1, E_{e_k}^2, E_{e_k}^3, \ldots, E_{e_k}^{|V|-1}$. Thus, E_{e_k} is one of these paths. This definition is formulated as a maximization over all paths of the minimum sum-ID-length of these paths $E_{e_k}^1, E_{e_k}^2, E_{e_k}^3, \ldots, E_{e_k}^{|V|-1}$. Therefore, the norm of sum-ID-path E_{e_k} in $S_{PE}(ID_{FG})$ can be also written by

$$\|E_{e_k}\| = \sum_{\left(e_i \in E_{e_k}\right)} \mu(e_i), \text{ for each } e_k \in E \text{ and } 1 \le k \le |E| \tag{5}$$

This reveals that the notion of the norm of a vector in the path-edge space of $ID_{FG}(V, E)$, $S_{PE}(ID_{FG})$, is a generalization of the notion of length in $S_{PE}(ID_{FG})$. Thereafter, from Definition 4.2, the norm of the element $E_{e_k} \boxplus E_{e_l} = E_{E_{e_k} \boxplus E_{e_l}}$ in $S_{PE}(ID_{FG})$ is given in the following:

$$\|E_{e_k} \boxplus E_{e_l}\| = \left\|E_{E_{e_k} \boxplus E_{e_l}}\right\| = \sum_{\left(e_i \in E_{E_{e_k} \boxplus E_{e_l}}\right)} \mu(e_i) \tag{6}$$

The path-edge space of $ID_{FG}(V, E)$ can form a normed space with the function as presented by Equation (5). In other words, the relationship between a normed space and a graph $ID_{FG}(V, E)$ can now be detected by the following theorem.

Theorem 5.1. *Let* $S_{PE}(ID_{FG})$ *be a path-edge space of* $ID_{FG}(V, E)$ *over* $\{0, 1\}^{|E|}$ *and* $\|E_{e_k}\| = \max_{n=1,2,3,\ldots,|V|-1} \left\{\left|E_{e_k}^n\right|\right\} = \max\left\{\left|E_{e_k}^1\right|, \left|E_{e_k}^2\right|, \left|E_{e_k}^3\right|, \ldots, \left|E_{e_k}^{|V|-1}\right|\right\}$ *be a real-valued function on* $S_{PE}(ID_{FG})$. *Then,* $S_{PE}(ID_{FG})$ *is a normed space.*

Proof: By Theorem 4.1, $S_{PE}(ID_{FG})$ is a vector space over $\{0, 1\}^{|E|}$. It remains to fulfill the following four axioms of a norm.

a) $\|E_{e_k}\| \geq 0$ for all sum-ID-path E_{e_k} in $S_{PE}(ID_{FG})$ because $\|E_{e_k}\|$ is the maximum of nonnegative numbers.
b) $\|E_{e_k}\| = 0$ if and only if $E_{e_k} = \varnothing$ because $\varnothing$ is the empty graph which is the zero vector in $S_{PE}(ID_{FG})$ under the vector addition as presented in Equation (2).
c) $\|\alpha \boxdot E_{e_k}\| = |\alpha| \, \|E_{e_k}\|$ for all sum-ID-path E_{e_k} in $S_{PE}(ID_{FG})$ and all scalars $\{0, 1\}$ (with addition and multiplication modulo 2).

By definition of the scalar multiplication in Equation (3), $\|1 \boxdot E_{e_k}\| = \|E_{e_k}\| = |1| \, \|E_{e_k}\|$ and $\|0 \boxdot E_{e_k}\| = \|\varnothing\| = |0| \, \|E_{e_k}\|$. Observe that in congruence modulo 2, the two congruence classes of 0 and 1 are, respectively, $[0]_2 = \{0, \pm 2, \pm 4, \pm 6, \dots\}$ and $[1]_2 = \{\pm 1, \pm 3, \pm 5, \pm 7, \dots\}$.

d) $\|E_{e_k} \boxplus E_{e_l}\| \leq \|E_{e_k}\| + \|E_{e_l}\|$ for all sum-ID-paths E_{e_k}, E_{e_l} in $S_{PE}(ID_{FG})$.

This axiom is proven as follows. Assume that $E_{e_k} \boxplus E_{e_l}$ is edge-disjoint unions of these paths E_{e_k}, E_{e_l} in $S_{PE}(ID_{FG})$ and denoted by $E_{E_{e_k} \boxplus E_{e_l}}$. Then, from Equation (6), the norm of this element $E_{E_{e_k} \boxplus E_{e_l}}$ in $S_{PE}(ID_{FG})$ is given as $\|E_{e_k} \boxplus E_{e_l}\| = \left\|E_{E_{e_k} \boxplus E_{e_l}}\right\| = \sum_{\left(e_i \in E_{E_{e_k} \boxplus E_{e_l}}\right)} \mu(e_i) \leq \sum_{\left(e_i \in E_{e_k}\right)} \mu(e_i) + \sum_{\left(e_i \in E_{e_l}\right)} \mu(e_i)$ (since $\left[\sum_{\left(e_i \in E_{E_{e_k} \boxplus E_{e_l}}\right)} \mu(e_i)\right]$ is the sum of the norm of just unions of edge-disjoint of two sum-ID-paths E_{e_k}, E_{e_l}) $= \max\left\{\left|E^1_{e_k}\right|, \left|E^2_{e_k}\right|, \dots, \left|E^{|V|-1}_{e_k}\right|\right\} + \max\left\{\left|E^1_{e_l}\right|, \left|E^2_{e_l}\right|, \dots, \left|E^{|V|-1}_{e_l}\right|\right\}$ (by Equation (5)) $= \|E_{e_k}\| + \|E_{e_l}\|$. This concludes the proof. □

Theorem 5.1 establishes that a normed space with an irreducible fuzzy graph can be constructed by the path-edge space $S_{PE}(ID_{FG})$ and called the normed path-edge space of $ID_{FG}(V, E)$. In addition, the basic sum-ID-paths for each edge $e_k \in E$ in $ID_{FG}(V, E)$ can be determined with its norm $\|E_{e_k}\|$ as seen in the following section.

6. Application.

Figure 3 serves as an example of a digraph of an irreducible fuzzy graph $ID_{FG}(V, E)$ having the set of vertices $V = \{v_0, v_1, v_2, v_3\}$ with $|V| = 4$ and the membership values $\mu(e_i) = \mu((v_{i-1}, v_i))$ for each edge $e_k \in E$ with $|E| = 6$ given as follows:

$$\mu(e_1) = \mu((v_1, v_0)) = 0.7, \quad \mu(e_2) = \mu((v_1, v_2)) = 0.1, \quad \mu(e_3) = \mu((v_2, v_3)) = 0.5$$
$$\mu(e_4) = \mu((v_0, v_3)) = 0.6, \quad \mu(e_5) = \mu((v_3, v_1)) = 0.4, \quad \mu(e_6) = \mu((v_3, v_0)) = 0.3$$

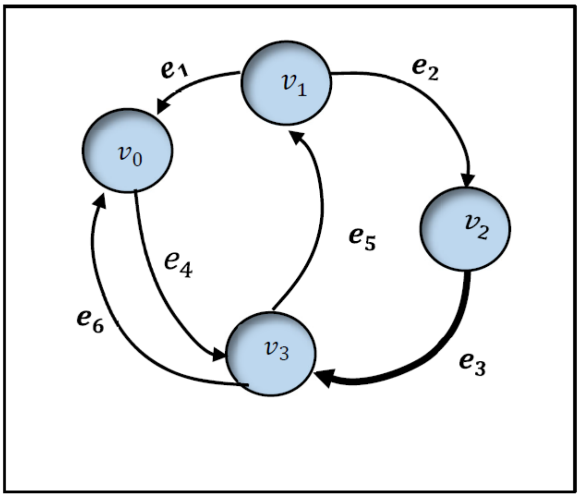


FIGURE 3. Sum-ID-paths in $ID_{FG}(V, E)$ with $|V| = 4$ and $|E| = 6$

One sees easily that all possible paths containing an edge, for instance e_3, are as follows:

1) $\{\mu((v_2, v_3))\}$, i.e., the path $E^1_{e_3} = (e_3)$ of length one $v_2 \to v_3$, namely e_3, with sum-length $\left|E^1_{e_3}\right| = \mu(e_3) = 0.5$
2) $\{\mu((v_1, v_2)), \mu((v_2, v_3))\}$, i.e., the path $E^2_{e_2,e_3} = (e_2, e_3)$ of length two with sum-length $\left|E^2_{e_2,e_3}\right| = \mu(e_2) + \mu(e_3) = 0.1 + 0.5 = 0.6$
3) $\{\mu((v_2, v_3)), \mu((v_3, v_1))\}$, i.e., the path $E^2_{e_3,e_5} = (e_3, e_5)$ of length two with sum-length $\left|E^2_{e_3,e_5}\right| = \mu(e_3) + \mu(e_5) = 0.5 + 0.4 = 0.9$
4) $\{\mu((v_2, v_3)), \mu((v_3, v_0))\}$, i.e., the path $E^2_{e_3,e_6} = (e_3, e_6)$ of length two with sum-length $\left|E^2_{e_3,e_6}\right| = \mu(e_3) + \mu(e_6) = 0.5 + 0.3 = 0.8$
5) $\{\mu((v_1, v_2)), \mu((v_2, v_3)), \mu((v_3, v_0))\}$, i.e., the path $E^3_{e_2,e_3,e_6} = (e_2, e_3, e_6)$ of length three with sum-length $\left|E^3_{e_2,e_3,e_6}\right| = \mu(e_2) + \mu(e_3) + \mu(e_6) = 0.1 + 0.5 + 0.3 = 0.9$
6) $\{\mu((v_2, v_3)), \mu((v_3, v_1)), \mu((v_1, v_0))\}$, i.e., the path $E^3_{e_3,e_5,e_1} = (e_3, e_5, e_1)$ of length three with sum-length $\left|E^3_{e_3,e_5,e_1}\right| = \mu(e_3) + \mu(e_5) + \mu(e_1) = 0.5 + 0.4 + 0.7 = 1.6$.

However, there are six various paths containing an edge e_3 and then, by using the sum-ID-length $\left|E^n_{e_k}\right|$ defined in Equation (1), we get the following:

- The path $E^1_{e_3} = (e_3)$ is a sum-ID-path $E^1_{e_3}$ with its sum-ID-length as $\left|E^1_{e_3}\right| = 0.5$
- The path $E^2_{e_2,e_3} = (e_2, e_3)$ is a sum-ID-path $E^2_{e_3}$ with its sum-ID-length as $\left|E^2_{e_3}\right| = 0.6$, since $\left|E^2_{e_3}\right| = \min\left\{\left|E^2_{e_2,e_3}\right|, \left|E^2_{e_3,e_5}\right|, \left|E^2_{e_3,e_6}\right|\right\} = \min\{0.6, 0.9, 0.8\} = 0.6$
- The path $E^3_{e_2,e_3,e_6} = (e_2, e_3, e_6)$ is a sum-ID-path $E^3_{e_3}$ with its sum-ID-length as $\left|E^3_{e_3}\right| = 0.9$, since $\left|E^3_{e_3}\right| = \min\left\{\left|E^3_{e_2,e_3,e_6}\right|, \left|E^3_{e_3,e_5,e_1}\right|\right\} = \min\{0.9, 1.6\} = 0.9$.

Thus, other lengths which contain any edge e_k in this example of $ID_{FG}(V, E)$ can be easily computed by using the sum-ID-length $\left|E^n_{e_k}\right|$ given in Equation (1). Now, the preceding concepts in foregoing Sections 3 and 4 are used to identify the sum-ID-length $\left|E^n_{e_k}\right|$ for all edges e_k, $k = 1, 2, 3, 4, 5, 6$, with its vector representation $\vec{E}^n_{e_k} = (x_{e_1}, x_{e_2}, x_{e_3}, x_{e_4}, x_{e_5}, x_{e_6})$ in the following:

1) The sum-ID-paths that contain the edge e_1 are
$E^1_{e_1} = (e_1)$ with length $\left|E^1_{e_1}\right| = 0.7 \to \vec{E}^1_{e_1} = (1, 0, 0, 0, 0, 0)$
$E^2_{e_1} = (e_5, e_1)$ with length $\left|E^2_{e_1}\right| = 1.1 \to \vec{E}^2_{e_1} = (1, 0, 0, 0, 1, 0)$
$E^3_{e_1} = (e_3, e_5, e_1)$ with length $\left|E^3_{e_1}\right| = 1.6 \to \vec{E}^3_{e_1} = (1, 0, 1, 0, 1, 0)$
2) The sum-ID-paths that contain the edge e_2 are
$E^1_{e_2} = (e_2)$ with length $\left|E^1_{e_2}\right| = 0.1 \to \vec{E}^1_{e_2} = (0, 1, 0, 0, 0, 0)$
$E^2_{e_2} = (e_5, e_2)$ with length $\left|E^2_{e_2}\right| = 0.5 \to \vec{E}^2_{e_2} = (0, 1, 0, 0, 1, 0)$
$E^3_{e_2} = (e_2, e_3, e_6)$ with length $\left|E^3_{e_2}\right| = 0.9 \to \vec{E}^3_{e_2} = (0, 1, 1, 0, 0, 1)$
3) The sum-ID-paths that contain the edge e_3 are
$E^1_{e_3} = (e_3)$ with length $\left|E^1_{e_3}\right| = 0.5 \to \vec{E}^1_{e_3} = (0, 0, 1, 0, 0, 0)$
$E^2_{e_3} = (e_2, e_3)$ with length $\left|E^2_{e_3}\right| = 0.6 \to \vec{E}^2_{e_3} = (0, 1, 1, 0, 0, 0)$
$E^3_{e_3} = (e_2, e_3, e_6)$ with length $\left|E^3_{e_3}\right| = 0.9 \to \vec{E}^3_{e_3} = (0, 1, 1, 0, 0, 1)$
4) The sum-ID-paths that contain the edge e_4 are
$E^1_{e_4} = (e_4)$ with length $\left|E^1_{e_4}\right| = 0.6 \to \vec{E}^1_{e_4} = (0, 0, 0, 1, 0, 0)$
$E^2_{e_4} = (e_4, e_5)$ with length $\left|E^2_{e_4}\right| = 1 \to \vec{E}^2_{e_4} = (0, 0, 0, 1, 1, 0)$
$E^3_{e_4} = (e_4, e_5, e_2)$ with length $\left|E^3_{e_4}\right| = 1.1 \to \vec{E}^3_{e_4} = (0, 1, 0, 1, 1, 0)$
5) The sum-ID-paths that contain the edge e_5 are
$E^1_{e_5} = (e_5)$ with length $\left|E^1_{e_5}\right| = 0.4 \to \vec{E}^1_{e_5} = (0, 0, 0, 0, 1, 0)$
$E^2_{e_5} = (e_5, e_2)$ with length $\left|E^2_{e_5}\right| = 0.5 \to \vec{E}^2_{e_5} = (0, 1, 0, 0, 1, 0)$

$E^3_{e_5} = (e_4, e_5, e_2)$ with length $\left|E^3_{e_5}\right| = 1.1 \to \vec{E}^3_{e_5} = (0,1,0,1,1,0)$
6) The sum-ID-paths that contain the edge e_6 are
$E^1_{e_6} = (e_6)$ with length $\left|E^1_{e_6}\right| = 0.3 \to \vec{E}^1_{e_6} = (0,0,0,0,0,1)$
$E^2_{e_6} = (e_3, e_6)$ with length $\left|E^2_{e_6}\right| = 0.8 \to \vec{E}^2_{e_6} = (0,0,1,0,0,1)$
$E^3_{e_6} = (e_2, e_3, e_6)$ with length $\left|E^3_{e_6}\right| = 0.9 \to \vec{E}^3_{e_6} = (0,1,1,0,0,1)$.

Then, it is interesting to show that the unions of edge-disjoint $E^n_{e_k} \boxplus E^m_{e_l}$ of sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$ in this example are determined by definition of the vector addition as given in Equation (2). For instance, the union of edge-disjoint of $E^1_{e_3}$ and $E^3_{e_3}$ is obviously determined by the set of edges $G = \{e_2, e_6\}$ which means the set of edges G in its vector representation is equal to $(0,1,0,0,1)$. Consequently, by Definition 4.2, we get the length of this element $\left|E^1_{e_3} \boxplus E^3_{e_3}\right| = \left|E_{E^1_{e_3} \boxplus E^3_{e_3}}\right| = \mu(e_2) + \mu(e_6) = 0.1 + 0.3 = 0.4$. Thereby, for all edge-disjoint unions of sum-ID-paths $E^n_{e_k}$, $E^m_{e_l}$ in Figure 3, we obtain the length of these edge-disjoint unions $\left|E^n_{e_k} \boxplus E^m_{e_l}\right|$.

Therefore, it is clearly seen that the above sum-ID-paths and edge-disjoint unions of these paths in this example of $ID_{FG}(V,E)$ is easily checked for all requirements of vector addition and scalar multiplication as in Theorem 4.1. Therefore, the path-edge space in this example, $S_{PE}(ID_{FG})$, is a vector space over $\{0,1\}^{|E|}$. At this juncture, we apply the new type of this vector space in Section 5 and by Theorem 5.1, the path-edge space $S_{PE}(ID_{FG})$ clearly satisfies the four axioms of the norm given in Equation (4). Therefore, the basic sum-ID-paths for each edge $e_k \in E$ in this example can be determined with its norm $\|E_{e_k}\|$ as follows:

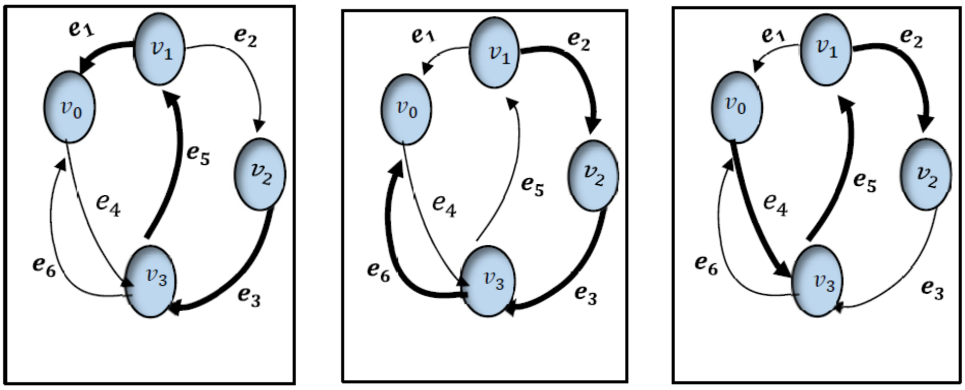
1) The sum-ID-path that contains the edge e_1 is $E_{e_1} = (e_3, e_5, e_1)$ with norm $\|E_{e_1}\| = 1.6$, since $\|E_{e_1}\| = \max\left\{\left|E^1_{e_1}\right|, \left|E^2_{e_1}\right|, \left|E^3_{e_1}\right|\right\} = \max\{0.7, 1.1, 1.6\} = 1.6$
2) The sum-ID-path that contains the edge e_2 is $E_{e_2} = (e_2, e_3, e_6)$ with norm $\|E_{e_2}\| = 0.9$, since $\|E_{e_2}\| = \max\left\{\left|E^1_{e_2}\right|, \left|E^2_{e_2}\right|, \left|E^3_{e_2}\right|\right\} = \max\{0.1, 0.5, 0.9\} = 0.9$
3) The sum-ID-path that contains the edge e_3 is $E_{e_3} = (e_2, e_3, e_6)$ with norm $\|E_{e_3}\| = 0.9$, since $\|E_{e_3}\| = \max\left\{\left|E^1_{e_3}\right|, \left|E^2_{e_3}\right|, \left|E^3_{e_3}\right|\right\} = \max\{0.5, 0.6, 0.9\} = 0.9$
4) The sum-ID-path that contains the edge e_4 is $E_{e_4} = (e_4, e_5, e_2)$ with norm $\|E_{e_4}\| = 1.1$, since $\|E_{e_4}\| = \max\left\{\left|E^1_{e_4}\right|, \left|E^2_{e_4}\right|, \left|E^3_{e_4}\right|\right\} = \max\{0.6, 1.0, 1.1\} = 1.1$
5) The sum-ID-path that contains the edge e_5 is $E_{e_5} = (e_4, e_5, e_2)$ with norm $\|E_{e_5}\| = 1.1$, since $\|E_{e_5}\| = \max\left\{\left|E^1_{e_5}\right|, \left|E^2_{e_5}\right|, \left|E^3_{e_5}\right|\right\} = \max\{0.4, 0.5, 1.1\} = 1.1$
6) The sum-ID-path that contains the edge e_6 is $E_{e_6} = (e_2, e_3, e_6)$ with norm $\|E_{e_6}\| = 0.9$, since $\|E_{e_6}\| = \max\left\{\left|E^1_{e_6}\right|, \left|E^2_{e_6}\right|, \left|E^3_{e_6}\right|\right\} = \max\{0.3, 0.8, 0.9\} = 0.9$

It is worth noticing that $E_{e_2} = E_{e_3} = E_{e_6}$ and $E_{e_4} = E_{e_5}$. Hence, this method is able to identify the main sum-ID-paths which are exactly three paths as seen in Figure 4.

Finally, the example given on this work establishes the simple and efficiency of the functional technique to classify $ID_{FG}(V,E)$ as a normed structure and at the same time, the successful technique is able to identify the main sum-ID-paths E_{e_k} which are the maximizations over all paths of the minimum sum-ID-length of these paths $E^1_{e_k}$, $E^2_{e_k}$, $E^3_{e_k}$ that travel through certain edges e_k, $k = 1,2,3,4,5,6$. In comparison with the existing methods that used the fuzzy max-min approach, our approach has a higher performance using the normed structure of a graph without employing the algebraic structure represented by the adjacency matrix of a finite graph which was used by conventional methods.

7. **Conclusions.**

In this paper, we establish functional method to find all sum-ID-paths, edge-disjoint unions of these paths and path-edge space of $ID_{FG}(V,E)$ which is proved as a vector space. These modern concepts led to constructing a new kind of normed space which is a normed path-edge space of $ID_{FG}(V,E)$. Eventually, a practical example was



(a) (b) (c)

Figure 4. (a) sum-ID-path $E_{e_1} = (e_3, e_5, e_1)$; (b) sum-ID-paths $E_{e_2} = E_{e_3} = E_{e_6} = (e_2, e_3, e_6)$; (c) sum-ID-paths $E_{e_4} = E_{e_5} = (e_4, e_5, e_2)$

presented which demonstrates that the method in this paper can be used successfully for classification of any irreducible fuzzy graph to normed structure. Furthermore, the main sum-ID-paths can be determined with respect to this norm with irreducible fuzzy graph.

The classification of an irreducible fuzzy graph as normed structure contributes to the growth of the structural examples in the theory of functional analysis. As future work our results can be extended and unified with many properties which describe attitude some vertices go through paths in the normed path-edge space such as convergence, Cauchyness and bicompleteness.

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Author Biography

Umilkeram Qasim Obaid received her Ph.D. degree in Mathematics from Faculty of Science, University of Technology Malaysia (UTM), Malaysia, 2016. She is currently working as a lecturer at Department of Mathematics, College of Science, Mustansiriyah University in Baghdad, Iraq. Her research interests include functional analysis theory, graph theory, fuzzy graph theory and mathematical analysis.