

## TWO-STEP DESIGN METHOD FOR SLIDING MODE CONTROL TO SUPPRESS SPILLOVER

KEIGO OYAMA<sup>1</sup>, KOTARO HASHIKURA<sup>2</sup>, MD ABDUS SAMAD KAMAL<sup>2</sup>  
AND KOU YAMADA<sup>2</sup>

<sup>1</sup>School of Science and Technology

<sup>2</sup>Division of Mechanical Science and Technology  
Gunma University

1-5-1 Tenjincho, Kiryu 376-8515, Japan

{ t180b028; k-hashikura; maskamal; yamada }@gunma-u.ac.jp

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**ABSTRACT.** *In this paper, we examine a design method for low-sensitive sliding mode control to suppress spillover. The design objective is to improve a system performance while attaining a robust stable sliding mode control under modeling errors, and a dynamic switching surface design using two-step design method is proposed. The basic concept of two-step design method is to reduce sensitivity using the free parameter of the parameterization of all controllers satisfying robust stability condition. The design method enables us to achieve both the robust stability and the low-sensitivity in a more exhaustive way and analytical way than mixed-sensitivity problem. Moreover, the performance evaluation in the frequency domain can ease the design procedure. Hence the proposed method is useful to suppress spillover effectively. In addition, the effectiveness of the proposed method is demonstrated through the numerical example.*

**Keywords:** Sliding mode control, Spillover, Modeling error, Dynamic switching surface design,  $H_\infty$  control, Parameterization

**1. Introduction.** Sliding mode control (SMC), the practical method to guarantee robustness, has been paid great attention for decades [1, 2]. The conventional SMC (i.e., the relative degree of the plant is 1) has two phases: a reaching phase and a sliding phase, which respectively exhibit the different system's behavior. In terms of the sliding phase, the conventional SMC based on an ideal switching control becomes insensitive to some uncertainties if the matching condition is satisfied. This powerful property is achieved by an appropriate control law design. The another design procedure is the so-called switching surface design which determines the sliding mode dynamics. Therefore, it should be designed so that sliding mode is able to have desired dynamics such as fast convergence. Since the switching surface design is of importance, numerous studies have been carried out [1, 2, 3, 4, 5]. The effectiveness of the switching surface design based on the linear control theory was also proved.

Though the conventional SMC is successfully established, there are still some drawbacks. One of them is chattering phenomenon due to switching control, which is regarded as the most important problem to be solved. Many researchers have been proposing the solutions, and the systematical theory is high-order SMC (HOSMC) in [6]. The relative degree of HOSMC is arbitrary unlike the conventional SMC; therefore, the design of HOSMC is more difficult. Though the effectiveness of chattering alleviation was shown

in [6], it was also proved that HOSMC was the worst amplitude of the output chattering in [7]. Judging by this result, HOSMC is not the best method for any systems. For this reason, the conventional SMC is still capturing the attention because of the easier concept.

After the great theoretical developments, the conventional SMC was applied to many applications in [9, 10, 11]. The effectiveness of the conventional SMC was shown, and it reached a practical level. From a practical point of view, model order reduction is necessary to simplify the control problem. The high frequency modes are typically removed from the plant model since the system's behavior is considerably dominated by the low frequency modes. In this case, the high frequency switching control of SMC affects the disregarded resonance modes in the high frequency domain, and possibly results in making SMC system unstable. This is called spillover phenomena, which is regarded as problematic especially in the control of flexible systems [12]. It is known that the matching condition is not suitable for dealing with the modeling errors. Therefore, the switching surface should be designed to guarantee robust stability in order for SMC to work well under the modeling errors.

The conventional SMC, however, cannot have such an additional property because all state variables are used to enforce sliding mode [1]. To deal with this, dynamic switching surface design was proposed in [13]. The concept is to give dynamics to the switching surface by introducing a new state variable and to design the controller regarded as switching surface in frequency domain so as to have low-pass characteristics. The former state space is converted to the extended state space with the additional variable as a consequence. It is successfully performed while retaining the unique nature of SMC. For the purpose of frequency-shaping, classical control theory such as bode diagram or nyquist diagram was utilized. In general, the classical control theory may be imperfect when it is applied to MIMO system; therefore,  $H_\infty$  control theory was adopted in [14] and the frequency-shaped SMC for MIMO system became truly robust. Following that, since the performance is the primary factor to determine the quality of system, mixed sensitivity problem and  $\mu$  synthesis approach were proposed in [15, 16]. Those researches simultaneously achieved a robust stability and a performance criteria and showed the effectiveness in practical applications. Since the mixed sensitivity problem inevitably faces the well-known trade-off between the robust stability and the low sensitivity, it can be an obstacle to achieve the design requirements in more detail. Then,  $\mu$  synthesis, which can give a robust performance, is applied to fulfilling the requirements exhaustively. However,  $\mu$  synthesis cannot offer the globally optimal solution despite involving iterative calculation such as D-K iteration. A more effective way to obtain the switching surface is needed because the same problem is true of it.

In this paper, in order to overcome these problem, we propose a design method for sliding mode control to suppress spillover by expanding the idea of two-step design method proposed in [17] to the switching surface design. The basic concept is to reduce the sensitivity using the free parameter given by the parameterization of all controllers taking account of only robust stability. We preliminarily introduce new state variables to increase a degree of freedom of the design in the same manner as [13], and establish the form of the controller. The additional controller is constructed via two-step design method, and then each of the parameters is embedded in the extended state space and the dynamic switching surface. The design procedure of two-step design method is to

- 1) solve the  $H_\infty$  control problem using Riccati equation, and
- 2) decide the free parameter so as to be the desired system performance.

When the controller is chosen in such a way, the motion in sliding phase becomes robustly stable and low sensitive. In that both the performance criteria are divided and

considered separately, two-step design method is different from mixed sensitivity problem and enables us to deal with more detailed requirement. Moreover, in spite of considering low-sensitivity as well as robust stability, the proposed method provides the analytical solution as opposed to  $\mu$  synthesis. Hence, the main concern is how we can achieve both the design requirements more exhaustively and analytically. In order to show the effectiveness, the numerical example is shown. This paper is organized as follows. In Section 2, the problem considered in this paper is described. Section 3 explains the switching surface design on the basis of two-step design method. Section 4 explains the design of sliding mode controller. In Section 5, a numerical example is illustrated to demonstrate the effectiveness of the proposed method. Section 6 gives concluding remarks.

**Notations**

$R$	the set of real numbers.
$R_+$	the set of non-negative real numbers.
$C$	the set of complex numbers.
$R(s)$	the set of real rational functions with $s$ .
$RH_\infty$	the set of stable proper real rational functions.
$A^T$	transpose of $A$ .
$A^\dagger$	pseudo inverse of $A$ .
$A^\perp$	orthogonal complement of $A$ .
$\left[ \begin{array}{c c} A & B \\ \hline C & D \end{array} \right]$	the shorthand for state space realization of $C(sI - A)^{-1}B + D$ .
$\rho(\{\cdot\})$	the spectral radius of $\{\cdot\}$ .
$\bar{\sigma}(\{\cdot\})$	the maximum singular value of $\{\cdot\}$ .
$ \{\cdot\} $	the absolute value of $\{\cdot\} \in C$ .
$\ \{\cdot\}\ $	the Euclidean norm of $\{\cdot\}$ .
$\ \{\cdot\}\ _\infty$	the $H_\infty$ norm of $\{\cdot\}$ .
$\text{diag}(a_1, \dots, a_n)$	an $n \times n$ diagonal matrix with $a_i$ as its $i$ -th diagonal element.
$S(\{\cdot\})$	the dynamic switching surface of $\{\cdot\}$ as a linear operator.

**2. Problem Formulation.** Consider a linear time invariant system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where  $x(t) \in R^n$  is the state variable,  $u(t) \in R^m$  is the input,  $A \in R^{n \times n}$  and  $B \in R^{n \times m}$ . Assume that  $(A, B)$  is controllable. The regular form given by a similarity transformation is as follows:

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) \tag{2}$$

and

$$\dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t), \tag{3}$$

where  $x_1(t) \in R^{n-m}$ ,  $x_2(t) \in R^m$ , and  $B_2$  is assumed to be of full rank. Note that  $(A_{11}, A_{12})$  is controllable, since  $(A, B)$  is assumed to be controllable.

In many cases, the actual plant includes uncertainty. It is well-known that in some cases, the uncertainty makes the control system unstable. The stability problem of control system including uncertainty is known as the robust stability problem, which is important to solve. Then, we assume that the actual plant  $P(s)$  is written by

$$P(s) = (I + \Delta(s))P_0(s), \tag{4}$$

where  $P_0(s)$  is the nominal plant written by

$$P_0(s) = \left[ \begin{array}{c|c} A_{11} & A_{12} \\ \hline I & 0 \end{array} \right] \tag{5}$$

and  $\Delta(s) \in R^{(n-m) \times (n-m)}$  is the uncertainty. Both  $P(s)$  and  $P_0(s)$  are assumed to have no zero and pole on the imaginary axis and the same number of unstable poles. In addition, the set of  $\Delta$  is all rational functions satisfying

$$\bar{\sigma}\{\Delta(j\omega)\} < |W_T(j\omega)| \quad \forall \omega \in R_+, \tag{6}$$

where  $W_T(s)$ , which is called by the weighting function, is an asymptotically stable rational function.

(2) shows the sliding mode dynamics which depends on the switching surface. The new state variable is introduced to design a dynamic switching surface [13] and we have

$$\begin{cases} \dot{z}_e(t) = Fz_e(t) + Gx_1(t) \\ S(x_1(t)) = Hz_e(t) + Lx_1(t) \end{cases} \tag{7}$$

where  $z_e(t) \in R^l$  is the auxiliary state variable, and the matrices are of appropriate dimensions. A switching function is defined as

$$\psi(t) = S(x_1(t)) + x_2(t) = Hz_e(t) + Lx_1(t) + x_2(t). \tag{8}$$

Consequently, the extended state space representation is written by

$$\begin{bmatrix} \dot{z}_e(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} F & G & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_e(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ B_2 \end{bmatrix} u(t) \tag{9}$$

$$\psi(t) = [ H \quad L \quad I ] \begin{bmatrix} z_e(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}. \tag{10}$$

We now obtain a realization of transfer function to formulate the design problem:

$$C(s) = \left[ \begin{array}{c|c} F & G \\ \hline H & L \end{array} \right], \tag{11}$$

which is obviously considered to be a feedback controller which acts on (2) regarded as the nominal plant  $P_0(s)$  in (5).

The problem considered in this paper is to obtain the feedback controller  $C(s)$  in (11) which makes the actual plant  $P(s)$  in (4) robustly stable, and then the switching surface is derived from the controller  $C(s)$  in (11).

**3. Design of Switching Surface Using Two-Step Design Method.** In this section, we explain the detail of a two-step design method for dynamic switching surface. At first, we will deal mainly with the basic idea of two-step design method as a powerful tool to satisfy the requirement while avoiding the controversial points in previous studies. Then we will explain the design procedure in more detail. Finally, we will summarize it.

In order to design the feedback controller  $C(s)$  in (11) which makes the actual plant  $P(s)$  in (4) robustly stable and low sensitive, we expand the idea of two-step design method proposed in [19]. The basic idea is summarized as follows.

1) Robust stability condition of the control system is given by

$$\|W_T(s)T(s)\|_\infty < 1, \tag{12}$$

where  $T(s)$  is the complementary sensitivity function of the form

$$T(s) = P_0(s)C(s) (I + P_0(s)C(s))^{-1}. \tag{13}$$

In order to formulate  $H_\infty$  control problem, the generalized plant in Figure 1 is constructed so that the transfer function from  $w(s)$  to  $z(s)$  is equal to  $W_T(s)T(s)$ . By using  $H_\infty$  control problem based on Riccati equation, the parameterization of all controllers  $C(s)$  satisfying (12) is obtained.

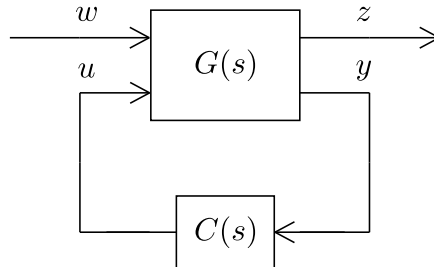


FIGURE 1. Block diagram of  $H_\infty$  control problem

- 2) The parameterization of all controllers  $C(s)$  satisfying (12) includes the free parameter. Using the free parameter, the control system can be low sensitive. By the state space description of obtained controller  $C(s)$ , the dynamic switching surface in (7) is designed.

In comparison with the existing approaches, two-step design method is easy to make a design plan because we only consider the condition for robustness at first. The next thing we do is just to tune the time constants as described later. Clearly the design is analytical, so we do not have to suffer from the global optimality unlike  $\mu$  synthesis approach.

According to the basic idea shown above, we will introduce the design procedure. When the state space realization of  $W_T(s)P_0(s)$  is written as

$$W_T(s)P_0(s) = \left[ \begin{array}{c|c} A_{11} & A_{12} \\ \hline C_T & D_T \end{array} \right], \tag{14}$$

the state space representation for the generalized plant is of the form

$$\begin{bmatrix} \dot{x}_1(t) \\ z(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{12} \\ C_T & 0 & D_T \\ -I & I & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ w(t) \\ x_2(t) \end{bmatrix}, \tag{15}$$

where  $z(t) \in R^p$ ,  $y(t) \in R^{n-m}$  and  $w(t) \in R^r$ . In addition, the following assumptions are valid:

- 1)  $(A_{11}, -I)$  is detectable
- 2)  $\text{rank} \begin{bmatrix} A_{11} - j\omega I & A_{12} \\ C_T & D_T \end{bmatrix} = n$
- 3)  $\text{rank} \begin{bmatrix} A_{11} - j\omega I & 0 \\ -I & I \end{bmatrix} = 2(n - m)$ .

The solvability of the  $H_\infty$  control problem is to satisfy the following conditions:

- 1) The Riccati equation

$$\begin{aligned} X \left( A_{11} - A_{12}D_T^\dagger C_T \right) + \left( A_{11} - A_{12}D_T^\dagger C_T \right)^T - X A_{12} (D_T^T D_T)^{-1} A_{12}^T X \\ + (D_T^\dagger C_T)^T D_T^\dagger C_T^T = 0 \end{aligned} \tag{16}$$

has the stabilizing solution  $X \geq 0$  to make  $A_{11} - A_{12}D_T^\dagger C_T + A_{12}(D_T^T D_T)^{-1}A_{12}^T X$  have no eigenvalue in the closed right half plane.

2) The Riccati equation

$$YA_{11} + A_{11}Y + Y(C_T^T C_T - I)Y = 0 \tag{17}$$

has the stabilizing solution  $Y \geq 0$  to make  $A_{11} + Y(C_T^T C_T - I)$  have no eigenvalue in the closed right half plane.

3) The spectral radius condition

$$\rho(XY) < 1$$

is satisfied.

Following these preliminary conditions, the parameterization of all controllers  $C(s)$  satisfying (12) is given by

$$C(s) = C_{11}(s) + C_{12}(s)Q(s)(I - C_{22}(s)Q(s))^{-1}C_{21}(s), \tag{18}$$

where

$$\begin{bmatrix} C_{11}(s) & C_{12}(s) \\ C_{21}(s) & C_{22}(s) \end{bmatrix} = \left[ \begin{array}{c|cc} A_c & B_{c1} & B_{c2} \\ \hline C_{c1} & D_{c11} & D_{c12} \\ C_{c2} & D_{c21} & D_{c22} \end{array} \right], \tag{19}$$

$$\begin{aligned} A_c &= A_{11} - A_{12} \left( D_T^\dagger C_T + E_{12}^{-1} A_{12}^T X \right) - (I - XY)^{-1} Y E_{21}^{-1}, \\ B_{c1} &= (I - XY)^{-1} Y E_{21}^{-1}, \\ B_{c2} &= (I - XY)^{-1} (A_{12} + Y C_T^T D_T) E_{12}^{-1/2}, \\ C_{c1} &= -D_T^\dagger C_T - E_{12}^{-1} A_{12}^T X, \\ C_{c2} &= -E_{21}^{-1/2}, \\ D_{c11} &= 0, \\ D_{c12} &= E_{12}^{-1/2}, \\ D_{c21} &= I, \\ D_{c22} &= 0, \\ E_{12} &= D_{12}^T D_{12}, \\ E_{21} &= I. \end{aligned}$$

The free parameter  $Q(s)$  is an arbitrary rational function satisfying  $\|Q(s)\|_\infty < 1$ .

Next step is to design  $Q(s)$ .  $Q(s)$  is designed as

$$C_{22}(s)Q(s) = \bar{Q}(s)C_{22i}(s), \tag{20}$$

where  $C_{22i}(s)$  is the inner function of  $C_{22}(s)$  satisfying  $C_{22i}(0) = I$  and  $\bar{Q}(s)$  is

$$\bar{Q}(s) = \text{diag} \left\{ \frac{1}{(1 + sT_1)^{\alpha_1}}, \dots, \frac{1}{(1 + sT_{n-m})^{\alpha_{n-m}}} \right\}, \tag{21}$$

$T_i > 0$  ( $i = 1, \dots, n - m$ ), and  $\alpha_i$  ( $i = 1, \dots, n - m$ ) is an arbitrary positive integer to make  $Q(s)$  be proper and satisfy  $\|Q(s)\|_\infty < 1$ . The state space representation of  $Q(s)$ ,  $\bar{Q}(s)$  and  $C_{22i}(s)$  is obtained by the method in [19]. Using the result in [19], the state space description of  $C(s)$  can be obtained by (11). Then, the extended state space and the dynamic switching surface in (9) is settled by

$$\dot{x}_e(t) = A_e x_e(t) + B_e u(t) \tag{22}$$

$$\psi(t) = S_e x_e(t), \tag{23}$$

where

$$A_e = \begin{bmatrix} F & G & 0 \\ 0 & A_{11} & A_{12} \\ 0 & A_{21} & A_{22} \end{bmatrix}, \tag{24}$$

$$B_e = \begin{bmatrix} 0 \\ 0 \\ B_2 \end{bmatrix}, \tag{25}$$

$$S_e = [ H \quad L \quad I ] \tag{26}$$

and

$$x_e(t) = \begin{bmatrix} z_e(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}. \tag{27}$$

Finally, the design procedure of switching surface using two-step design method is summarized as follows.

- 1) Establish the extended state-space representation in (9).
- 2) Construct the generalized plant for  $H_\infty$  control problem.
- 3) Solve the  $H_\infty$  control problem and get the parameterization of all  $H_\infty$  controllers with free parameter  $Q(s)$ .
- 4) Determine the free parameter  $Q(s)$  by using the filtered right inverse system described in [19].
- 5) Obtain the switching surface corresponding to (11).

**4. Design of Sliding Mode Controller.** In this section, we explain a design of controller enforcing sliding mode in the prescribed switching surface.

The controller consists of linear component and discontinuous component. The linear part of the controller is given by equivalent control method in [1], and the discontinuous part is the switching control obtained by unit vector control method in [2]. Using equivalent control method in [1] and unit vector control method in [2], the control input  $u(t)$  is given by

$$u(t) = u_{eq}(t) + u_{sw}(t), \tag{28}$$

where  $u_{eq}(t)$  is the linear component and called the equivalent control and  $u_{sw}(t)$  is the discontinuous component and called the switching control. According to [1], on the equivalent control, the following lemma holds true.

**Lemma 4.1.** [1] *Assume that  $\dot{\psi}(t) = 0$ . The linear component, which guarantees the asymptotic stability of the system on the switching surface, is given by*

$$u_{eq}(t) = -(S_e B_e)^{-1} S_e A_e x_e(t). \tag{29}$$

From Lemma 4.1, the stability of SMC is summarized as the following theorem.

**Theorem 4.1.** *If the control input  $u(t)$  in (28) is used, where the equivalent control  $u_{eq}(t)$  is given by (29) and the switching control is given by*

$$u_{sw}(t) = -\alpha (S_e B_e)^{-1} \frac{\psi(t)}{\|\psi(t)\|}, \tag{30}$$

*then the system with the actual plant  $P(s)$  in (4) and the control input  $u(t)$  in (28) is asymptotically stable, where  $\alpha$  is an arbitrary positive number.*

**Proof:** Lyapunov function  $V(t)$  is set as

$$V(t) = \frac{1}{2}\psi(t)^T\psi(t). \quad (31)$$

From (28), (29) and (30), the time derivative of the Lyapunov function  $V(t)$  in (31) is calculated as

$$\begin{aligned} \dot{V}(t) &= \psi(t)^T\dot{\psi}(t) \\ &= \psi(t)^T(S_eA_ex_e(t) + S_eB_ey(t)) \\ &= \psi(t)^T\left\{S_eA_ex_e(t) - S_eB_e\left((S_eB_e)^{-1}S_eA_ex_e(t) + \alpha(S_eB_e)^{-1}\frac{\psi(t)}{\|\psi(t)\|}\right)\right\} \\ &= -\psi(t)^T\left(\alpha\frac{\psi(t)}{\|\psi(t)\|}\right) \end{aligned} \quad (32)$$

Since  $\alpha > 0$ , we have

$$\dot{V}(t) \leq -\alpha\|\psi(t)\|. \quad (33)$$

Thus, the system with the actual plant  $P(s)$  in (4) and the control input  $u(t)$  in (28) is asymptotically stable.  $\square$

Note that the state variables rapidly converge to the origin while involving the large chattering if we select  $\alpha$  to be large.

**5. Numerical Example.** In this section, numerical examples are illustrated to show the effectiveness of the proposed method.

**5.1. Example 1.** Consider a following second order system:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (34)$$

At first, we select

$$W_T(s) = \frac{2s + 5}{15} \quad (35)$$

as the weighting function. Solving the robust stability problem using  $H_\infty$  control based on the Riccati equation provides the parameterization of all robust stabilizing controllers in (18). The free parameter  $Q(s)$  is given by (20) where

$$\bar{Q}(s) = \frac{1}{1 + 0.14s} \quad (36)$$

Then the controller  $C(s)$  obtained by (11) offers the extended state space and the dynamic switching surface in (22) and (23), respectively.

The control input  $u(t)$  is given by (28), where  $u_{eq}(t)$  and  $u_{sw}(t)$  are written by (29) and (30), respectively. The parameter  $\alpha$  in (30) is selected as

$$\alpha = 2. \quad (37)$$

When  $\Delta(s)$  in (4) is given by

$$\Delta(s) = \frac{1.0 \times 10^4 s + 1.5 \times 10^4}{9s^3 + 68s^2 + 90115s + 2.3 \times 10^5}, \quad (38)$$

$\Delta(s)$  in (48) satisfies (6) since the number of unstable poles of  $P(s)$  is equal to that of  $P_0(s)$  and the gain plot of  $W_T(s)$  and the maximum singular value plots of  $\Delta(s)$  are shown in Figure 2, where the solid line shows the gain plot of  $W_T(s)$  and the dashed line shows the maximum singular value plot of  $\Delta(s)$ . Clearly,  $\Delta(s)$  has a resonance peak in the high frequency domain, and it can devastate the sliding mode control system.

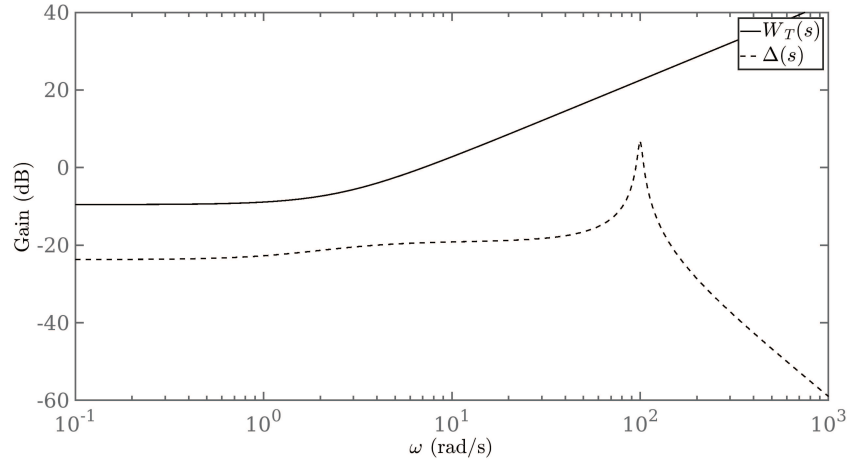


FIGURE 2. The gain plot of  $W_T(s)$  and the maximum singular value plot of  $\Delta(s)$

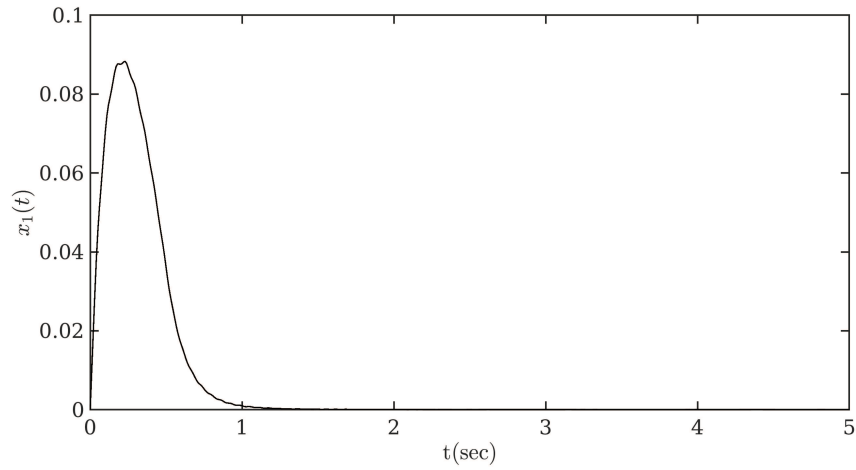


FIGURE 3. Trajectory of the state variables  $x_1(t)$

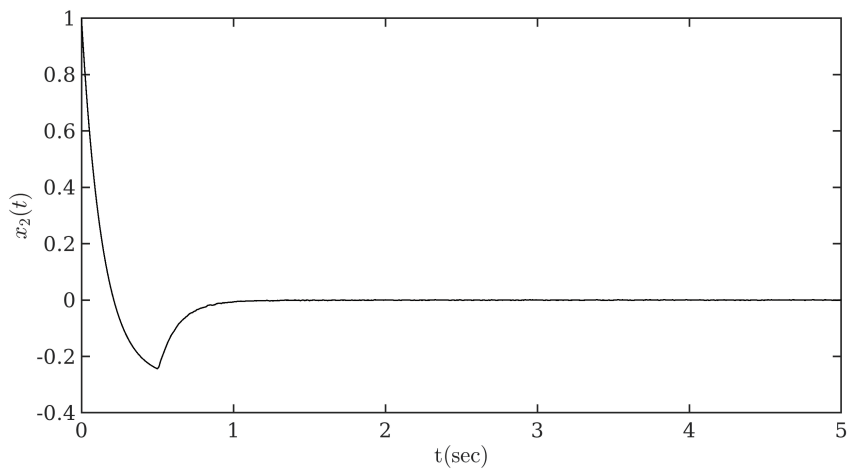


FIGURE 4. Trajectory of the state variables  $x_2(t)$

Figure 3 and Figure 4 show the trajectory of the state variables which have

$$x_1(0) = 0 \tag{39}$$

and

$$x_2(0) = 1 \tag{40}$$

as the initial condition. From Figure 3 and Figure 4, it can be seen that the state variables rapidly reach the origin even if the actual plant  $P(s)$  has the uncertainty  $\Delta(s)$ .

5.2. **Example 2.** Consider a linear time invariant system in (1), where

$$A = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{41}$$

Since eigenvalues of  $A$  are  $-1, -5, 2$  and  $7$ , the system is unstable. The similarity transformation provides

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}, \quad x_2(t) = \begin{bmatrix} x_{21}(t) \\ x_{22}(t) \end{bmatrix}, \tag{42} \\ A_{11} &= \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -3 & 0 \\ 0 & -12 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}, \\ B_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 3 & 0 \\ 0 & 12 \end{bmatrix} \end{aligned}$$

The actual plant  $P(s)$  is given by (4), where the nominal plant  $P_0(s)$  is given by (5) and the weighting function  $W_T(s)$  satisfying (6) is set as

$$W_T(s) = \text{diag} \left\{ \frac{65s + 1}{2100}, \frac{65s + 1}{2100} \right\}. \tag{43}$$

The free parameter  $Q(s)$  in (18) is given by (20) where

$$\bar{Q}(s) = \text{diag} \left\{ \frac{1.0 \times 10^{-6}s + 1}{(0.07s + 1)^2}, \frac{1.0 \times 10^{-6}s + 1}{(0.08s + 1)^2} \right\}. \tag{44}$$

After obtaining the extended state space and the dynamic switching surface, the control input is obtained by selecting  $\alpha$  as

$$\alpha = 50. \tag{45}$$

When  $\Delta(s) = 0$ , that is, the actual plant  $P(s)$  is equal to  $P_0(s)$ , the trajectory of the state variable  $x(t)$  which have

$$x_1(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{46}$$

and

$$x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{47}$$

as the initial condition is shown in Figure 5 and Figure 6. Here Figures 5(a) and 5(b) show the trajectory of  $x_{11}(t)$  and  $x_{12}(t)$ , Figures 6(a) and 6(b) show that of  $x_{21}(t)$  and  $x_{22}(t)$ , respectively. From Figure 5 and Figure 6, it can be seen that the state variables rapidly reach the origin.

When  $\Delta(s)$  in (4) is given by

$$\begin{aligned} &\Delta(s) \\ &= \text{diag} \left\{ \frac{1.6 \times 10^5 s - 8 \times 10^3}{3s^3 + 630s^2 + 486000s + 9.6 \times 10^7}, \frac{1.6 \times 10^5 s - 8 \times 10^3}{3s^3 + 630s^2 + 486000s + 9.6 \times 10^7} \right\}, \tag{48} \end{aligned}$$

$\Delta(s)$  in (48) satisfies (6) since the number of unstable poles of  $P(s)$  is equal to that of  $P_0(s)$  and the gain plot of  $W_T(s)$  and the maximum singular value plots of  $\Delta(s)$  are shown in Figure 7, where the solid line shows the gain plot of  $W_T(s)$  and the dashed line shows the maximum singular value plot of  $\Delta(s)$ . Clearly,  $\Delta(s)$  has a resonance peak in the high frequency domain.

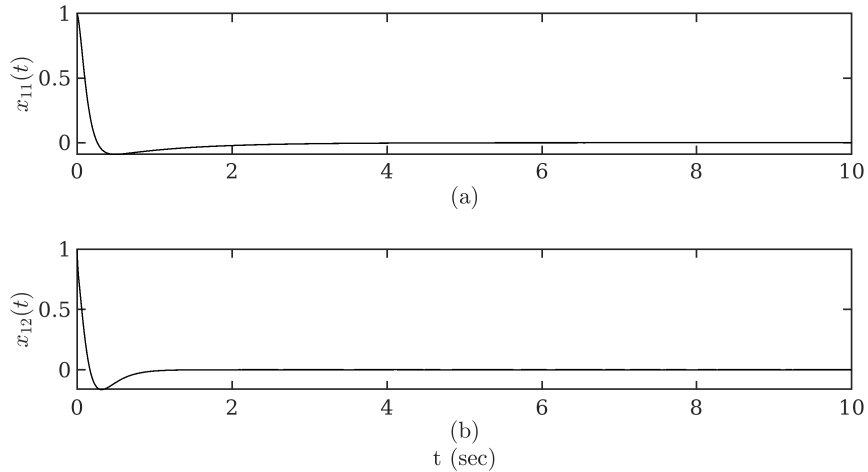


FIGURE 5. Trajectory of the state variables  $x_1(t)$  (a)  $x_{11}(t)$  and (b)  $x_{12}(t)$  when  $\Delta(s) = 0$

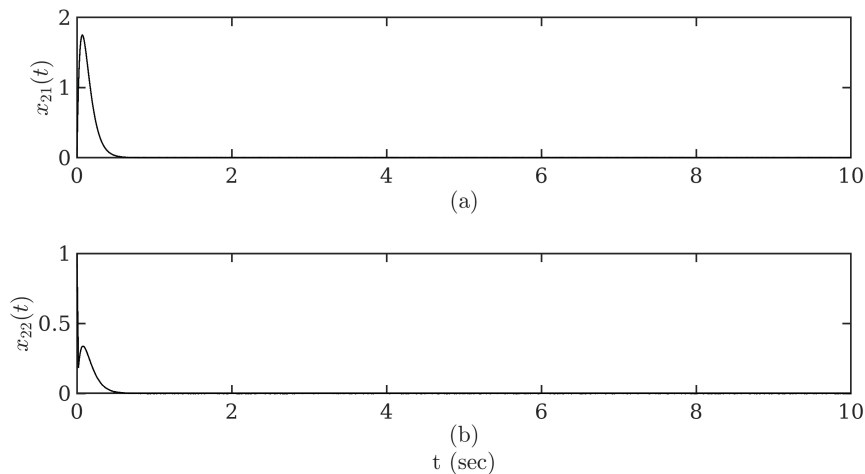


FIGURE 6. Trajectory of the state variables  $x_2(t)$  (a)  $x_{21}(t)$  and (b)  $x_{22}(t)$  when  $\Delta(s) = 0$

Figure 8 and Figure 9 show the trajectory of the state variables which have  $x_1(0)$  in (46) and  $x_2(0)$  in (47) as the initial condition. Here Figure 8(a) and Figure 8(b) show the trajectory of  $x_{11}(t)$  and  $x_{12}(t)$ , and Figure 9(a) and Figure 9(b) show that of  $x_{21}(t)$  and  $x_{22}(t)$ . From Figure 8 and Figure 9, it can be seen that the state variables rapidly reach the origin even if the actual plant  $P(s)$  has the uncertainty  $\Delta(s)$ . According to [12], in order to avoid spillover phenomenon, spillover can be suppressed by considering either control spillover or observation spillover. Thus, we hope there is no chattering in the trajectory of  $x_2$  which is added as an input for sliding mode equation in (2).

In these results, spillover can be suppressed by  $H_\infty$  control scheme while achieving a fast convergence by the design of the free parameter  $Q(s)$ . The conventional SMC is unable

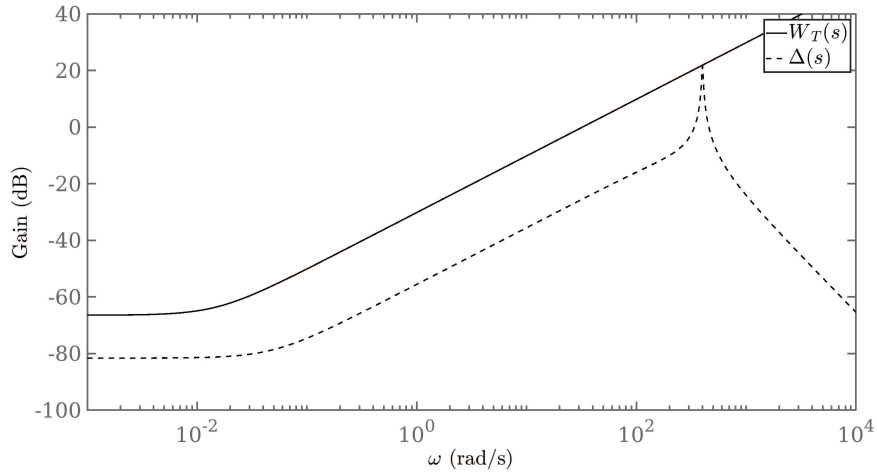


FIGURE 7. The gain plot of  $W_T(s)$  and the maximum singular value plot of  $\Delta(s)$

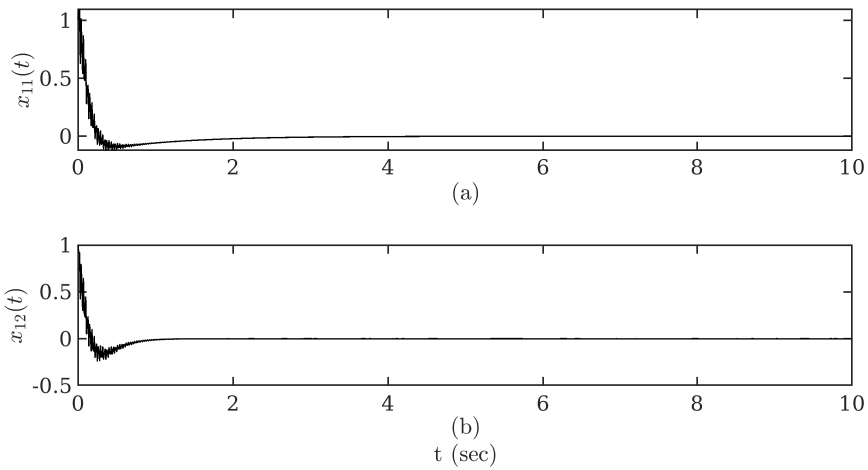


FIGURE 8. Result of the proposed method:  $x_1(t)$  (a)  $x_{11}(t)$  and (b)  $x_{12}(t)$

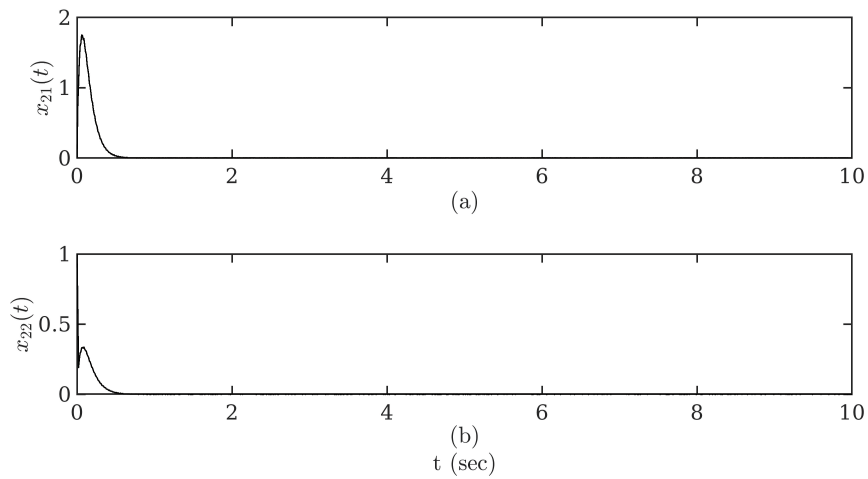


FIGURE 9. Result of the proposed method:  $x_2(t)$  (a)  $x_{21}(t)$  and (b)  $x_{22}(t)$

to eliminate the effect of the high-frequency resonance; however, the dynamic controller design enables us to suppress spillover.

Moreover, we conducted our numerical simulation for mixed-sensitivity approach. The system, the weighting function for complementary sensitivity function and the uncertainty are the same. The weighting function for sensitivity function is chosen as

$$W_S(s) = \text{diag} \left\{ \frac{0.1}{s + 2.0 \times 10^{13}}, \frac{0.1}{s + 2.0 \times 10^{13}} \right\}. \quad (49)$$

in consequence of tuning.

Figure 10 and Figure 11 show the trajectory of state variables which have  $x_1(0)$  in (46) and  $x_2(0)$  in (47). It can be clearly seen that the state variables in the proposed method reach the origin faster than mixed-sensitivity approach. Moreover, the design procedure of the proposed method is easier than mixed-sensitivity approach. We select the weighting function for complementary sensitivity function, which the selection is considerably clear because it depends on the desired control requirement. Regarding the low-pass filter in (21) which is another parameter, what needs to be done is just to tune so that system becomes low-sensitive as much as possible. Without suffering the selection of the weighting functions due to trade-off, we are able to suppress spillover.

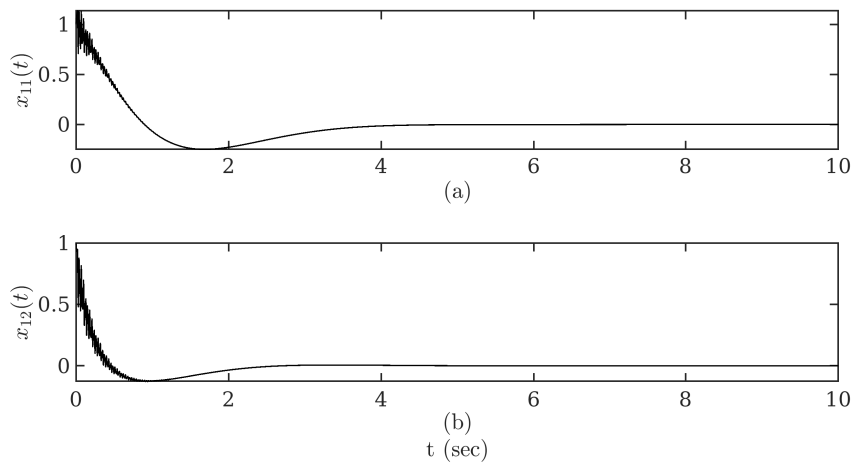


FIGURE 10. Result of mixed-sensitivity approach:  $x_1(t)$  (a)  $x_{11}(t)$  and (b)  $x_{12}(t)$

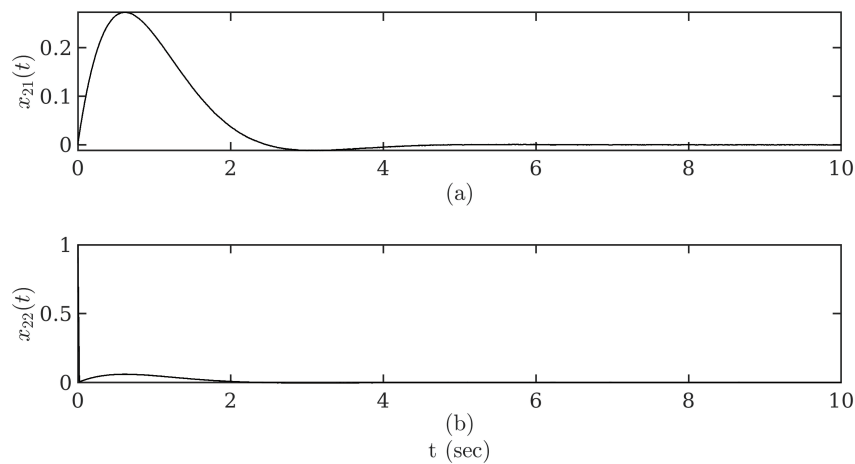


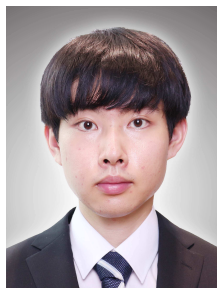
FIGURE 11. Result of mixed-sensitivity approach:  $x_2(t)$  (a)  $x_{21}(t)$  and (b)  $x_{22}(t)$

**6. Conclusions.** In this paper, we have proposed a design method for sliding mode control to suppress spillover by expanding the idea of two-step design method. The dynamic switching surface is effectively given so that the extended system became robust stable and low-sensitive. A numerical example demonstrated that the proposed method asymptotically stabilized the system with uncertainty to the origin. If we use mixed-sensitivity problem approach, it is difficult to design the weighting functions to surpass the maximum singular value of uncertainty because there is the trade-off. In that sense, two-step design method allows us to design dynamic switching surface efficiently. Thus, we conclude that the proposed method is useful to suppress spillover in a more effective way and an analytical way.

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## Author Biography



**Keigo Oyama** received the B.S. degree in Mechanical Science and Technology from Gunma University, Gunma, Japan, in 2022. He is currently a Master student from Gunma University and majors in Mechanical Science and Technology. His research interest lies in robust control theory including sliding mode control theory,  $H_\infty$  control theory and disturbance observer.



**Kotaro Hashikura** received the B.S. degree of Mechanical Engineering, the M.S. degree of Informatics, and the Dr. degree of Engineering from Kyushu Institute of Technology, Fukuoka, Japan in 2006, from Kyoto University, Kyoto, Japan in 2010, and from Tokyo Metropolitan University, Tokyo, Japan in 2014, respectively. From 2014 until 2018, he had been a Project Research Associate at the Faculty of System Design, Tokyo Metropolitan University. He is currently an Assistant Professor at the Department of Mechanical Science and Technology, Gunma University, Japan. His research interests include time-delay-related control techniques, such as deadbeat, preview-prediction and repetitive controls. He is a member of IEEE, ISCIE and SICE.



**Md Abdus Samad Kamal** received the B.Sc. degree in Electrical and Electronic Engineering from Khulna University of Engineering and Technology (KUET), Khulna, Bangladesh in 1997, Master and Doctor degrees from Kyushu University from Graduate School of Information Science and Electrical Engineering, Japan in 2003 and 2006, respectively. He was a post-doctoral fellow in Kyushu University till November 2006. He is currently an Associate Professor at the Department of Mechanical Science and Technology, Gunma University, Japan. His current research interests are reinforcement learning, intelligent transportation systems, and multi-agent systems. He is a member of IEEE and SICE.



**Kou Yamada** received B.S. and M.S. degrees from Yamagata University, Yamagata, Japan in 1987 and 1989, respectively, and a Dr. Eng. degree from Osaka University, Osaka, Japan in 1997. From 1991 to 2000, he was with the Department of Electrical and Information Engineering, Yamagata University, Yamagata, Japan as a research associate. From 2000 to 2008, he was an associate professor in the Department of Mechanical System Engineering, Gunma University, Gunma, Japan. Since 2008, he has been a professor in the Department of Mechanical System Engineering, Gunma University, Gunma, Japan. His research interests include robust control, repetitive control, process control, and control theory for inverse systems and infinite-dimensional systems. Dr. Yamada received the 2005 Yokoyama Award in Science and Technology, the 2005 Electrical Engineering/Electronics, Computer, Telecommunication, and Information Technology International Conference (ECTI-CON2005) Best Paper Award, the Japanese Ergonomics Society Encouragement Award for an Academic Paper in 2007, the 2008 Electrical Engineering/Electronics, Computer, Telecommunication, Information Technology International Conference (ECTI-CON2008) Best Paper Award, and the 4th International Conference on Innovative Computing, Information and Control Best Paper Award in 2009, the 14th International Conference on Innovative Computing, Information and Control Best Paper Award in 2019, and Outstanding Achievement Award from Kanto Branch of Japanese Society for Engineering Education in 2022. He is a member of IEEE and SICE.