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DEEP GAUSSIAN MULTIVARIATE HOSMER-LEMESHOW FEATURE LEARNING VIA PARTIAL DIFFERENTIAL EQUATION TO FACE RECOGNITION

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Abstract. *Nowadays, numerous well-known image classification methods consist of two steps, namely, feature extraction and classification. Owing to the fact that the classifier performance is largely dependent on the feature characteristics, much of the image classification endeavor advances into the feature acquired. Several research works have been analyzed for face recognition based on the features being learnt, however not found to be computationally efficient. To overcome this limitation, in this work, we propose a new method, called Deep Gaussian Multivariate Hosmer-Lemeshow Feature Learning (DGM-HLFL), via Partial Differential Equation (PDE) for face recognition. The DGM-HLFL via PDE to face recognition is performed in deep neural network. The overall process involves three layers, i.e., one input layer, two hidden layers and one output layer. In the input layer, the face images acquired from two datasets are provided as input. In the first hidden layer, robust feature learning and non-linear mapping is performed using Gaussian multivariate regression function. In the second hidden layer, computationally efficient learnt feature is provided as input where the parameter update and classification using Hosmer-Lemeshow test classifier for robust classification is performed. For the experiments, we used the publicly available Extended Yale B and Pedestrian Intention Estimation (PIE) datasets. A comparative study with other approaches such as conventional feature learning methods via PDE and latest research work reported in the literature are provided in detail to validate our proposed method. The experimental results of face recognition on the two benchmark face datasets show that the proposed DGM-HLFL method via PDE outperforms the state-of-the-art feature learning methods in terms of recognition time, recognition accuracy and false positive rate.*
Keywords: Gaussian multivariate regression, Feature learning, Non-linear internal mapping, Hosmer-Lemeshow test, Face recognition, Partial differential equation

1. Introduction.

In recent years, numerous familiar methods for face recognition involve two steps. They are feature extraction and classification. Owing to the reason that the classifier performance is deliberately dependent on the feature quality, much of the endeavor on image classification for face recognition advances into the arrangement of features and data transformations. As a reason, much awareness in recent years has gone into the linear representation based feature learning methods. This is due to the reason that convex images and Lambertian objects acquired under obscure illumination lie virtually in a roughly linear subspace, referred to as the harmonic plane.

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By employing the subspace characteristic, low rank representation based methods extract feature to apprehend the entire structure globally, hence found to be robust to noise. A novel PDE based method for feature learning called L-PDE was proposed in [1]. The feature learned by employing PDE was found to be highly discriminative in nature. It was also found to be translationally, rotationally invariant and finally robust to illumination variation.

The feature learning process was modeled by means of an evolution process employing PDE via reducing the training error. As a result, it extracted discriminative information from data. Followed by the feature extraction process, a linear classifier for classification was also applied that in turn optimized the entire framework, therefore corroborating the objective of minimizing computational time and maximizing the recognition accuracy.

Despite improvement observed both in terms of computational time and recognition accuracy, the recognition time involved in feature learning for face recognition was not focused. To address this issue, in this work, Gaussian multivariate regression-based feature learning model is introduced in the first hidden layer of deep neural learning. With the employment of the Gaussian multivariate regression PDE pertinent noise-reduced and computationally efficient feature are learnt that in turn paves the way for recognition of face in minimum time.

Owing to the reason that person appearance is largely influenced by several factors like lighting, posture and perspective, person identification rate in recent years has become a challenging area. A video person re-identification algorithm integrating Ordinary Differential Equation and Graph Convolution Network (ODE-GCN) was proposed in [2].

First, a continuous time model was formed by employing ODE with the objective of acquiring hidden information. With this the hidden information between frames that were found to be of significance was obtained. Next, the generated features were provided as input to the graph convolution network to reconstruct them. As a result, recognition accuracy of face was improved in a computationally efficient manner.

Despite improvement observed in the recognition accuracy and time, the false positive rate during the person appearance was not focused. To address this issue, in this work, non-linear internal mapping along with Hosmer-Lemeshow test classifier is used to measure the proportional adjustment between actual output and desired output, therefore reducing the false positive rate in a significant manner.

1.1. **Contributions.** The major contributions of this paper include the following:

1) Construction of DGM-HLFL method via PDE for face recognition;
2) To propose a Gaussian multivariate regression-based feature learning model via nonlinear mapping to learn robust and pertinent features with minimum recognition time;
3) Development of a new Hosmer-Lemeshow test classification algorithm for face recognition;
4) Experimental comparisons with other PDE-based methods demonstrated that the proposed DGM-HLFL method performs more significantly in terms of recognition accuracy, recognition time and false positive rate.

1.2. **Outlines.** The remainder of this paper is organized as follows. Section 2 presents the related work. Section 3 describes our proposed DGM-HLFL method in detail. Section 4 presents the implementation detail, evaluation metrics, and experimental results followed by an in-depth discussion in Section 5. Finally, Section 6 concludes the paper.

2. **Related Works.** Deep neural networks, which consist of numerous multiple nonlinear transformations, have exhibited their dominance over the past few years. The

hierarchical framework involved in transformation is significant in acquiring pertinent information.

An image recognition algorithm was designed in [3] on the basis of PDEs and wavelet transform. The algorithm employed in this work utilized weight coefficients to integrate high-order PDE, also maintained the dominance of the second and fourth-order partial differential recognition. This in turn enhanced the potentiality to preserve image edge information, therefore attaining better recognition results.

In [4], a novel PDE-interpretation of a class of deep Convolutional Neural Networks (CNN) that is specifically utilized in analyzing speech, image, and video data was proposed. A theoretical analysis on deep neural networks and PDE for object recognition was investigated in [5]. For face recognition, deeply learned features require to be not only distinct but also discerning. Owing to the reason that the deeply learned features are found to be inappropriate to accumulate all the probable testing identities for training, label prediction in CNNs is not consistently appropriate. The emerging features are not adequately efficient for face recognition.

A holistic numerical study for neural networks on learning problems for numerical analysis employing approximation theory was proposed in [6]. In most of the prevailing CNNs, the softmax loss function is employed as the supervision signal with the purpose of training the deep model. However, to improve the discerning potentiality of the features that have been deeply learned, a new center loss supervision signal for face recognition was proposed in [7]. In [8], recent advancements in deep learning technique for face recognition were investigated in depth.

In recent years, deep learning algorithms have been applied in various domains, like image classification, image or face recognition, visual tracking, solving PDEs, that in turn stimulate the deep learning technology development. A new generic model based on the adaptive differential equation by means of Fourier periodic expansion function was proposed in [9]. With this type of expansion function it in turn reduced the computational time involved in object recognition. In [10], two distinct functions, called systolic Gaussian elimination and systolic Gauss-Jordan elimination, were integrated for improving incomplete face recognition.

For issues pertaining to featuring consistent frameworks that disseminate over time like face recognition, Reduced Basis (RB) method was utilized to provide the result of inefficient Reduced Order Models (ROMs). Then, the high accuracy levels are important, but it was not focused. To overcome this issue, in [11], a novel nonlinear method to set ROMs by utilizing Deep Learning (DL) algorithms was proposed, therefore attaining high degree of accuracy. Yet another deep learning based approach for solving high dimension PDEs was designed in [12], therefore ensuring accuracy and cost. A Squeeze-Extracted Multi-Feature Convolution Neural Network (SE-MCNN) was proposed in [13] with the purpose of enhancing the recognition rate for deep face recognition.

The face is the foremost characteristic for numerous computer vision applications, owing to the distinctiveness of its aspect for each person that makes the face a better feature for face recognition. Due to the observation that humans are the paramount root of insecurity in society, facial recognition can be employed for recognizing anomalistic behavior.

For analyzing the features, utilizing statistical pattern matching concepts, being the integration of Chi Square (CSQ), Hu Moment Invariants (HuMIs), Absolute Difference Probability of White Pixels (AbsDifPWPs) and Geometric Distance Values (GDVs) were proposed for face recognition [14]. In [15], robust face recognition was made by employing L2 normal features, therefore ensuring better face recognition rate. Yet another method for surveillance based on visual attention mechanism towards face detection and recognition for the accuracy of occlusion of face detection was designed in [16].

Emotion plays a pivotal part in interpersonal communication and also enhancing social life. Facial recognition in the current years is tremendously applied in progressing interface between human and computer, and humanoid robots. In [17], a triangulation method was employed with the objective of extracting new types of geometric features to classify six distinct emotional expressions by means of computer-generated markers. Computational intelligent algorithms were designed in [18] for deep facial recognition.

Deep learning framework was developed in [21] for precisely identifying the human actions. Particle swarm optimization detection technique was employed to minimize the features that are reduced in video image sequences with less computational complexity. However, it failed to focus on enhancing the processing speed. Multi Task Cascaded Neural Network (MTCNN) and pre-trained FaceNet was introduced in [22] for discovering the faces and FaceNet with higher accuracy. However, the false positive rate was not considered.

Motivated by the above research papers, in this work, DGM-HLFL via PDE for face recognition is proposed. The elaborate description is provided in the following sections.

3. **DGM-HLFL Method.** Feature learning is contemplated as an analytical pace in pattern recognition like classification of images. Nevertheless, most of the prevailing materials and methods cannot extract features that are discerning and at the same time unvarying under a few changes. This in turn restricts the overall classification performance, specifically subject to small training sets. To address this issue, in this work, DGM-HLFL via PDE for face recognition is proposed. The block diagram of DGM-HLFL method is shown in Figure 1.

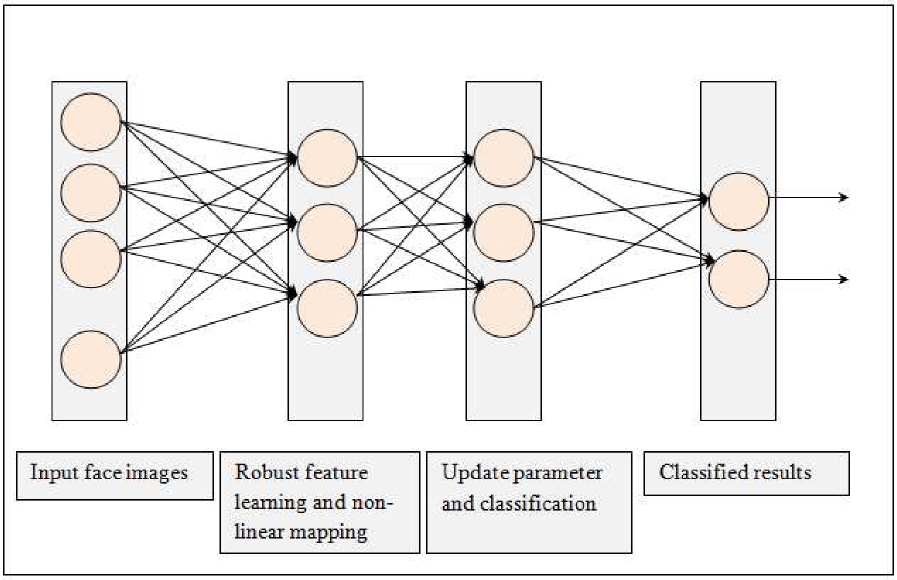


FIGURE 1. Block diagram of DGM-HLFL via Partial Differential Equation (PDE) for face recognition

As shown in Figure 1, the deep feature learning for face recognition consists of three layers. They are one input layer, two hidden layers and one output layer. In the input layer, the face images acquired from Extended Yale B [19] and Pie [20] datasets are provided as input. Next, in the first hidden layer, robust feature learning and non-linear mapping are performed, followed by which parameter update and classification are performed in the second hidden layer. Finally, the classified output results paving way for face recognition is provided in the output layer.

3.1. **Gaussian multivariate regression-based feature learning model.** Learning-based PDE (L-PDE) to face recognition was proposed in [1] with rotationally invariant also reducing the training error by extracting pertinent information required for classification. However, the enhancement of L-PDE through the increase of training samples in terms of recognition time was not remarkable. This was owing to the reason of employing restricted parameters while performing rotational invariants up to the second order in [1] and therefore compromising the recognition time.

With the objective of minimizing the recognition time, in our work, rotational invariants based on Gaussian multivariate regression model are introduced. By applying this model will obviously increase the number of parameters that possess one dependent variable and multiple independent variables. Also, with the advantage of multivariate regression it assists in interpreting the correlations between variables (i.e., dependent variable and independent variables) present in the Extended Yale B [19] and Pie [20] datasets considered for face recognition. This in turn reduces the recognition time involved for face recognition. Figure 2 given below shows the structure of Gaussian multivariate regression-based feature learning model.

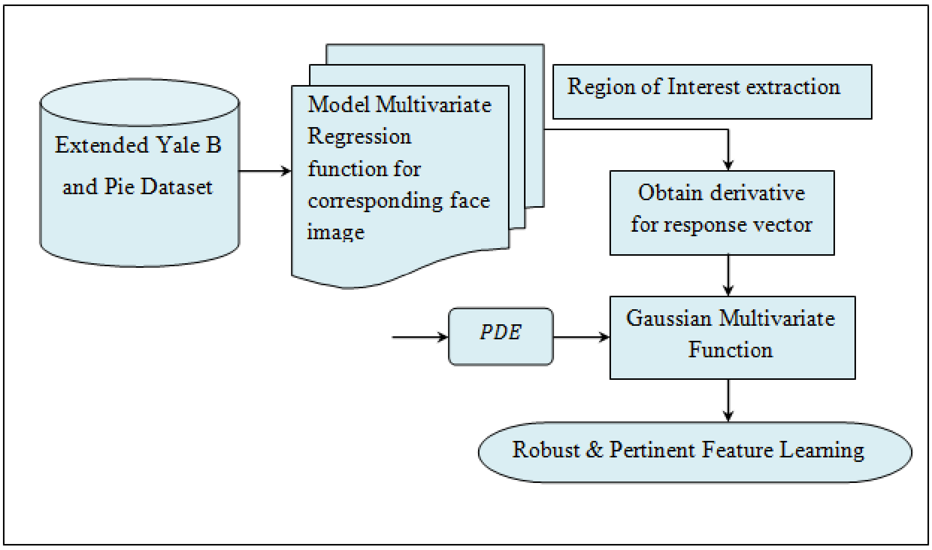


FIGURE 2. Structure of Gaussian multivariate regression-based feature learning model

As shown in the above figure, with two datasets, Extended Yale B and Pie provided as input, with this input most pertinent features are extracted with minimum time. With this objective, Gaussian multivariate regression function is applied. Let us consider a multivariate regression function as given below.

$$q_i = FIp_i + T \quad (1)$$

$$r_i = q_i + \epsilon_i \quad (2)$$

From the above Equations (1) and (2), '$p_i \in R^j$' refers to the observation vectors, '$FI \in R^{i*j}$' represents the '$i * j$' dimensional face image matrices, 'r_i' refers to the response vector, '$T \in R^i$' refers to the intersection point vector, '$q_i \in R^i$' denotes the resultant value vector and '$\epsilon_i \in R^i$' refers to the Gaussian noise. Then, the regression model is considered as evaluating the best '*j-dimensional*' sub-space due to the reason interpolation of '*j-dimensional*' sub-space (i.e., for derivatives of a function) that is mathematically formulated as given below.

$$q_1 = FI_{11}p_1 + FI_{12}p_2 + \cdots + FI_{1j}p_j + T_1 \tag{3}$$

$$q_2 = FI_{21}p_1 + FI_{22}p_2 + \cdots + FI_{2j}p_j + T_2 \tag{4}$$

$$q_3 = FI_{31}p_1 + FI_{32}p_2 + \cdots + FI_{3j}p_j + T_3 \tag{5}$$

$$\vdots \tag{6}$$

$$q_i = FI_{i1}p_1 + FI_{i2}p_2 + \cdots + FI_{ij}p_j + T_i \tag{7}$$

From the above Equations (3)-(7), the Region of Interest (ROI) is interpolated by the prior probability density function mathematically expressed as given below.

$$F(FI) = F\left(FI_{11}, \ldots, FI_{ij}, T_1, \ldots, T_i | I\right) \tag{8}$$

From the above Equation (8), for the correlation coefficients '$(FI_{11}, \ldots, FI_{ij}, T_1, \ldots, T_i)$', the intersection point vector remains invariant under rotations of the coordinate system. With the regression model, invariant rotation in '*n-dimension*' is indicated as a progression of rotations throughout rotation axes that are said to be orthogonally perpendicular to planes traversed by congruously chosen pairs of vector coordinate in the face images. This is designed on the basis of the fact that any orthogonal matrix or rotation matrices possess one dependent variable and multiple independent variables are said to be written in a unique manner as a by-product of '$2 * 2$' rotations. Then, the derivative for the response vector 'r_i' to interpolate is mathematically expressed as given below.

$$\frac{\partial}{\partial r_i}\left[f\left(r_1, r_2, r_3, \ldots, r_n\right) h_i\left(r_1, r_2, r_3, \ldots, r_n\right)\right] = 0 \tag{9}$$

Then, to increase the number of parameters, the proposed model is represented around rotation axis that is perpendicular to plane traversed by the linear combination of vectors 's_i' and 's_j' as rotation in the '$p_i q_h$' plane, respectively. This is mathematically formulated as given below.

$$p_i{}' = p_i - \epsilon q_j \tag{10}$$

$$q_j{}' = \epsilon p_i + q_j \tag{11}$$

Substituting the above coordinate resultant values from (10) and (11) to the derivative in (8) yields the rotation transformation in the face images as given below.

$$FI_{ji}{}' = FI_{ji} + \left(1 + FI_{ji}^2\right)\epsilon \tag{12}$$

$$FI_{kl}{}' = FI_{kl} + \left(FI_{jl} + FI_{ki}\right)\epsilon \tag{13}$$

$$T_k{}' = T_k + \left(FI_{ki}T_j\right)\epsilon \tag{14}$$

Finally, the Gaussian multivariate regression PDE for pertinent feature is yielded as given below.

$$\sum_{k=1}^{i}\sum_{l=1}^{j} \frac{\partial}{\partial FI_{kl}}\left[F\left(FI_{jl}FI_{ki}\right)\right] + \frac{\partial}{\partial FI_{ji}}F \tag{15}$$

From the above Equation (15), pertinent feature learnt for further classification of face recognition is obtained in a computationally efficient manner.

3.2. **Non-linear internal mapping.** In numerous image classification tasks, like face recognition, illumination variation is contemplated as a major issue. To make nearly invariant under gray-level scaling, a linear combination of transformed fundamental differential invariants was employed in [1], therefore improving recognition accuracy. However, the error involved during the mapping process was not concerned. To address this issue, in our work, non-linear internal mapping is performed that makes a detailed comparison between the actual response and the target response with the objective of minimizing the sum of square of error, therefore reducing the false positive rate significantly. The non-linear internal mapping function for each response vector is mathematically formulated as given below.

$$r_i\left(t_i + P\right) = f\left[r_i(t_i), r_i(t_{i-1}), \ldots\right] \tag{16}$$

From the above Equation (16), the response vector 'r_i' at time interval 't_i' with the predicted output vector 'P' is estimated with the purpose of modeling an internal mapping. With this it reproduces the expected output, followed by which with the object of reducing false positive rate it is formulated as given below.

$$E = \sum_{i=1}^{n} \left(P_i - y_i\right)^2 \tag{17}$$

From the above Equation (17), 'P_i' and 'y_i' represent the predicted output vector and the actual output vector of the 'i' output image. The pseudo code representation of Gaussian multivariate regression-based feature learning is given below.

Algorithm 1: Gaussian multivariate regression-based feature learning

Input: Dataset 'DS' [Extended Yale B and Pie], Face Image '$FI = FI_1, FI_2, FI_3, \ldots, FI_n$'
Output: Noise-reduced and computationally efficient feature learning '$FL = fl_1, fl_2, \ldots, fl_n$'
Step 1: **Begin**
Step 2: **For** each Face Image 'FI' with Dataset 'DS'
Step 3: Formulate multivariate regression function as in Equations (1) and (2)
Step 4: Formulate regression model as evaluating the best '*j-dimensional*' sub-space in an '*n-dimensional*' as in Equations (3)-(7)
Step 5: Model Region of Interest (ROI) with prior probability density function as in Equation (8)
Step 6: Estimate derivative for the response vector as in Equation (9)
Step 7: Formulate invariant rotation by linear vector combination as in Equations (10) and (11)
Step 8: Substitute resultant value in ROI with prior probability density function as in Equations (12)-(14)
Step 9: Estimate Gaussian multivariate regression PDE as in Equation (15)
Step 10: Estimate non-linear internal mapping as in Equation (16)
Step 11: Estimate sum of the square of the error as in Equation (17)
Step 12: **Return (feature learnt)**
Step 13: **End for**
Step 14: **End**

As given in the above algorithm, with Extended Yale B and Pie datasets provided as input, the objective in our work remains in learning the pertinent and robust features and classifies the image in an accurate manner, therefore forming base of face recognition. With this objective, to start with a multivariate regression function for the respective face image is modeled, followed by which, the region of interest is obtained by means of rotation progression that is found to be orthogonally perpendicular to each other. Next, with the extracted region of interest, the rotation transformation is made by means of Gaussian multivariate regression PDE, therefore learning pertinent feature. With this pertinent features formed as input to the classifier contribute to the objective of reducing the recognition time. Also by performing non-linear internal mapping, proportional adjustment between actual output and desired output is obtained, therefore reducing the false positive rate in a significant manner.

3.3. **Hosmer-Lemeshow test classifier for face recognition.** When obtaining the noise-reduced and computationally efficient learnt feature '$FL = fl_1, fl_2, \ldots, fl_n$' from the input face image '$FI = FI_1, FI_2, FI_3, \ldots, FI_n$', a classifier is required for classification. In the training phase, we minimize an error function to acquire both the learnt feature and the parameters in the classifier. To start with, training face image samples are first acquired and the total number of samples involved in simulation is obtained. Then, for each input face image, a non-linear internal mapping is performed for final classification process.

In our work with the objective of increasing the recognition accuracy, a statistical testing classifier employing Hosmer-Lemeshow test for classification called Hosmer-Lemeshow test classifier for face recognition is proposed. Figure 3 shows the structure of Hosmer-Lemeshow classification model.

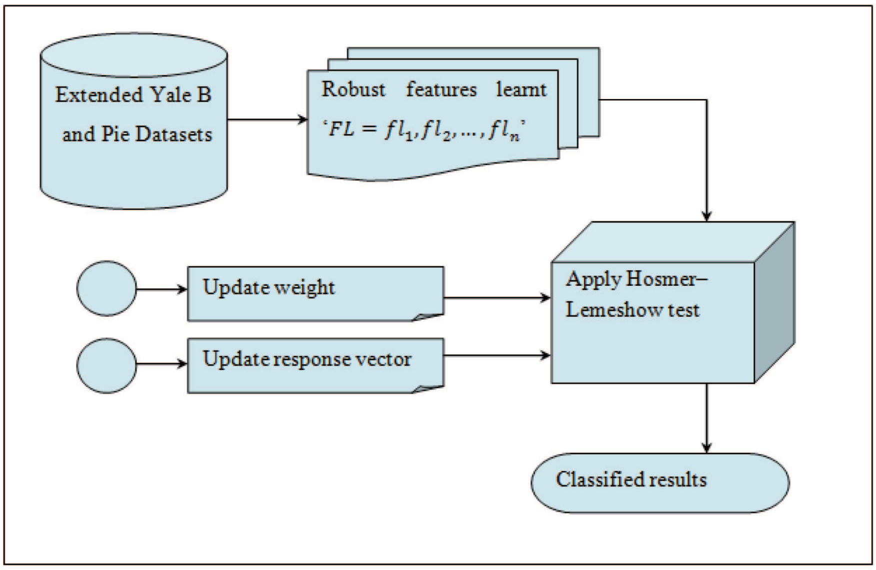


Figure 3. Structure of Hosmer-Lemeshow classification

As shown in the above figure, the learning model for classification of face images is formulated as identifying a function '$q_i = FIp_i + T$' and parameters 'W' as a function of a classifier to minimize the error involved during face recognition. This is mathematically formulated as given below.

$$W \rightarrow Min(CE) = E\left(J^T J + \alpha FL\right)^{-1} J^T EV \tag{18}$$

From the above Equation (18), the objective remains in minimizing the classification error '$Min(CE)$' for estimating weight 'W' by means of a Jacobian matrix 'J' of derivatives corresponding to error with respect to each weight for learnt feature 'FL'. The objective behind the utilization of Jacobian matrix for estimating weight 'W' remains in obtaining nonlinear relationships without any prior assumptions about nature of relationship between learnt feature images, therefore contributing to recognition accuracy. Next, a statistical classifier using Hosmer-Lemeshow test is employed in our work for performing classification. This test is mathematically formulated as given below.

$$H = \left[\frac{(Obs_S - Exp_S)^2}{Exp_S} + \frac{(Obs_F - Exp_F)^2}{Exp_F}\right] \tag{19}$$

From the above Equation (19), Hosmer-Lemeshow test result 'H' is estimated based on the observed success rate 'Obs_S', observed failure rate 'Obs_F', expected success rate

'Exp_S' and expected failure rate 'Exp_F', respectively. Followed by the weight update performed using Hosmer-Lemeshow test, with the computational overhead incurred during the sample update ('A' as in [1]), the sample update in our work for the corresponding response vector 'r_i' is performed by means of non-linear internal mapping for the respective response function. This is mathematically expressed as given below.

$$(r_i^n)^{j+1} = (r_i^n)^j - \beta \frac{\partial E^j}{\partial (r_i^n)^j} \tag{20}$$

$$\frac{\partial E^j}{\partial (r_i^n)^j} = \sum_{i=1}^{n} \frac{\partial E}{\partial r_i \left(t_i + P\right)} \tag{21}$$

From the above Equations (20) and (21), non-linear internal map-based back propagation is measured and updating the corresponding response vector is denoted as 'r_i'. With this multiple processes are performed in a single iteration, optimizing resource utilization and therefore corroborating the objective, i.e., minimizing the recognition time and maximizing the recognition accuracy via parameter optimization. The pseudo code representation of Hosmer-Lemeshow test classifier for face recognition is given below.

Algorithm 2: Hosmer-Lemeshow test classifier for face recognition

Input: Dataset 'DS' [Extended Yale B and Pie], Face Image '$FI = FI_1, FI_2, FI_3, \ldots, FI_n$'
Output: Robust classification
Step 1: **Initialize** learnt feature '$FL = fl_1, fl_2, \ldots, fl_n$' Step 2: **Begin** Step 3: **For** each Dataset 'DS' with Face Image 'FI' **//Update parameters** Step 4: **Repeat** Step 5: Update function of a classifier to minimize the error as in Equation (18) Step 6: Perform classification employing Hosmer-Lemeshow test as in Equation (19) Step 7: **If** '$H \geq 0.5$ and $H < 1$' Step 8: Match in face recognition Step 9: **End if** Step 10: **If** '$H > 0$ and $H < 0.5$' Step 11: Not match in face recognition Step 12: **End if** Step 13: Update response vector 'r_i' as in Equations (20) and (21) Step 14: **Until** $(\boldsymbol{Min}\,(\boldsymbol{CE}))$ Step 15: **Return** (classified results) Step 16: **End for** Step 17: **End**

As given in the above Hosmer-Lemeshow test classifier for face recognition algorithm, the objective remains in robust classification of face images with minimum falsification. With this objective, a statistical testing classifier called Hosmer-Lemeshow test for classification is performed. Also, to ensure robust classification with minimum false positive rate, the weight and the response vector are updated based on the learnt features. Finally, templates are matched with the classified results for ensuring accurate face recognition.

4. **Experiments.** In this section, an experimental evaluation of the DGM-HLFL via PDE to face recognition is made by performing comparisons with L-PDE [1] and ODE-GCN [2]. The experimental measure is conducted using two datasets namely, Extended Yale B dataset [19] and Pie dataset [20] and simulations are carried out on MATLAB.

The experiment is implemented using MATLAB version 12.0 with system specifications core i5 processor with 233 MHz, 8GB RAM, 1 TB hard disk. The face images in the range from 150-1500 are utilized to evaluate the simulation. First, the description of two datasets used in our work is provided.

4.1. **Extended Yale B dataset.** The Extended Yale Face Database B is taken from http://vision.ucsd.edu/∼leekc/ExtYaleDatabase/ExtYaleB.html. The Extended Yale Face Database B [19] comprises 16128 images acquired from 28 distinct human subjects under 9 poses and 64 illumination conditions. All test image data utilized in this experiment are aligned and cropped manually, and then finally re-sized to 168×192 images. The data format is the same as the Yale Face Database B.

4.2. **Pie dataset.** The experimental evaluation is performed by using the Pie dataset [http://robotics.csie.ncku.edu.tw/Databases/FaceDetect_PoseEstimate.htm#Our_Database]. The Pie dataset [20] consists of 6660 images acquired from 90 distinct subjects. Each subject in turn possesses 74 images, with 37 images of them acquired every 5 degree from right profile, referred to as $+90°$ and the remaining 37 images of them acquired every 5 degree from left profile, referred to as $-90°$ respectively in the pan rotation. These are 74 images in a Zip file. The first part of the image is 'A' and 'B' that denote the real-shot or synthesized image, respectively. The two digit numbers are represented as subject number. The rest of the filename indicates the corresponding facial pose.

5. **Discussion.** In this section, the performances of the DGM-HLFL method and the existing related approaches namely L-PDE [1] and ODE-GCN [2] are discussed with three metrics, i.e., recognition accuracy, recognition time, and false positive rate.

5.1. **Qualitative analysis.** In this section, the qualitative analysis of DGM-HLFL via PDE for face recognition is provided. To start with the face images are acquired from two different datasets, Extended Yale Face Database B dataset (i.e., Figure 4(a)) and Pie dataset (i.e., Figure 4(b)) as given below.

(a) (b)

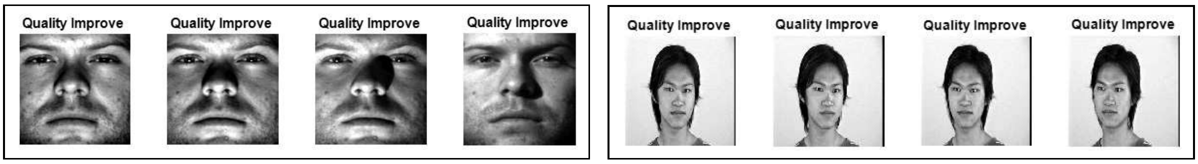
FIGURE 4. (a) Input face images from Extended Yale Face Database B dataset; (b) input face images from Pie dataset

Next, with the above set of input face images, a Gaussian multivariate regression function is applied to the input images with the purpose of learning pertinent features and their resultant output is obtained as given in Figures 5(a) and 5(b) for two different datasets.

Then, non-linear internal mapping is applied to the regressive features of the corresponding face images to learn the noise-reduced and computationally efficient feature as given in Figures 6(a) and 6(b) for two datasets.

Finally, the non linear weight update is performed to the learnt feature that forms the basis for final classification. The result obtained is as given in Figures 7(a) and 7(b).

With the above non linear weight updated performed in our proposed method, sum of square error minimized output vector is obtained, therefore improving the recognition accuracy with minimum time and false positive rate.



(a) (b)

FIGURE 5. (a) Gaussian multivariate regression images from Extended Yale Face Database B dataset; (b) Gaussian multivariate regression images from Pie dataset

5.2. **Performance analysis of recognition accuracy.** The first and foremost parameter of significance for face recognition is the recognition accuracy. The accuracy rate with which the face is recognized is referred to as the recognition accuracy. This is mathematically formulated as given below.

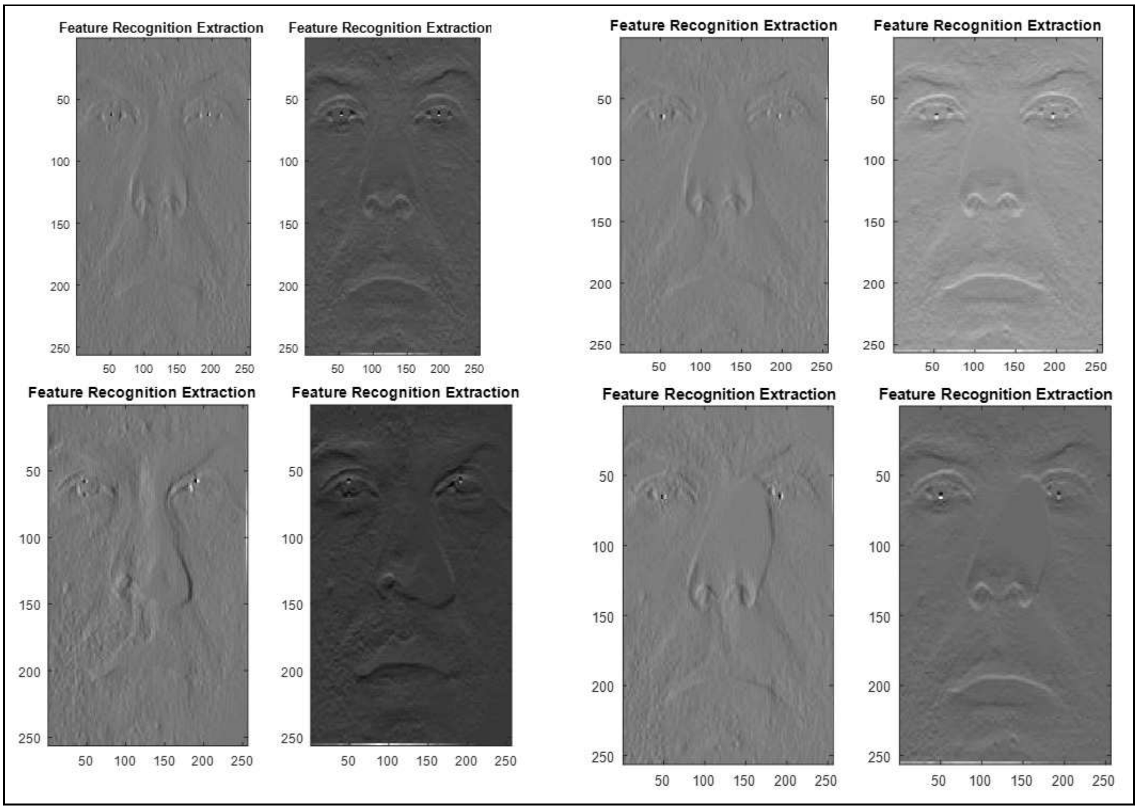
$$R_{acc} = \sum_{i=1}^{n} \frac{FI_{AR}}{FI_i} \tag{22}$$

From the above Equation (22), the recognition accuracy 'R_{acc}' is measured based on the face images involved in the simulation process 'FI_i' and the number of face images accurately recognized 'FI_{AR}'. It is measured in terms of percentage (%).

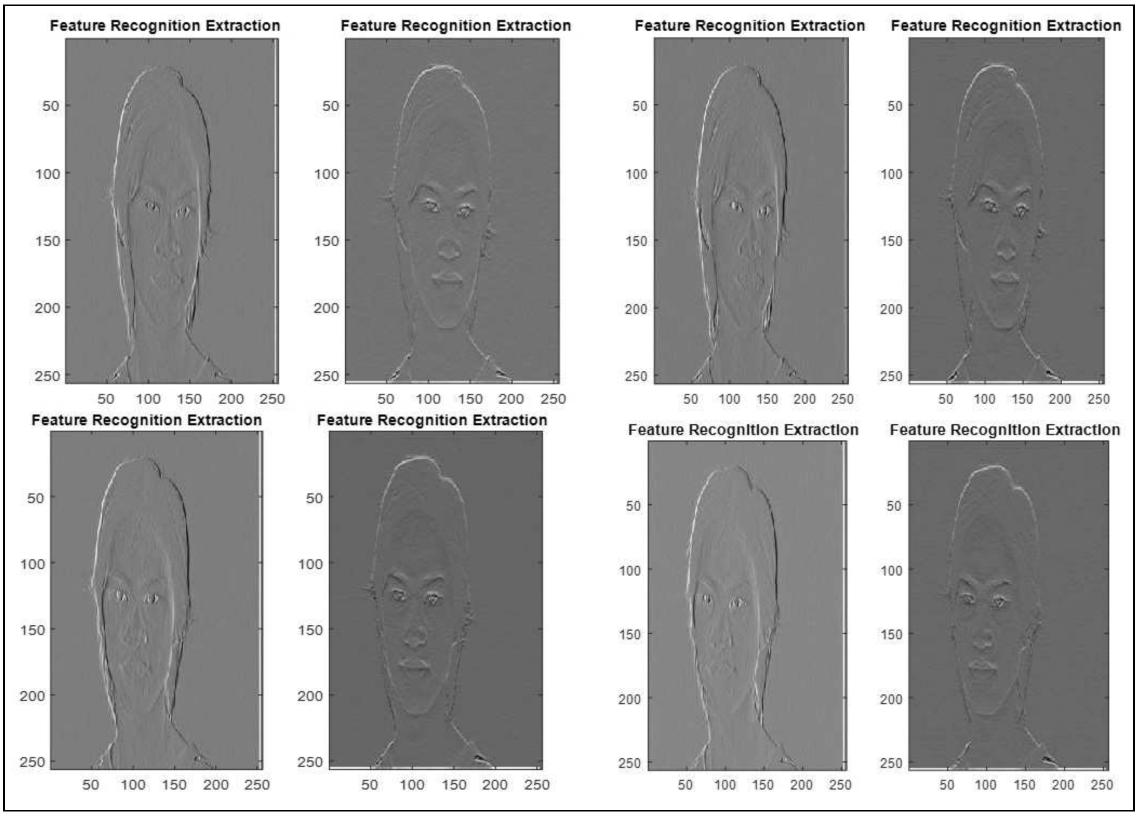
Figures 8(a) and 8(b) illustrate the performance measure of recognition accuracy for 1500 different images using two datasets, Extended Yale Face Database B dataset and Pie dataset, respectively. In Figures 8(a) and 8(b), x axis refers to the different face images acquired from two different datasets and y axis refers to the recognition accuracy using the three methods, DGM-HLFL, L-PDE [1] and ODE-GCN [2], respectively. From Figure 8 it is inferred that increasing the number of face images acquired for simulation obviously increases the derivative for response vector. This in turn reduces the recognition accuracy.

However, with simulations conducted using 150 face images, 145 face images were accurately recognized using DGM-HLFL, 140 face images using [1] and 135 face images were accurately recognized using [2], respectively (Extended Yale Face Database B dataset). As a result, the recognition accuracies using the three methods DGM-HLFL, [1] and [2] (Extended Yale Face Database B dataset) were observed to be 96.66%, 93.33% and 90% respectively, and (Pie dataset) were observed to be 95.33%, 91.33% and 87.33% respectively. From the simulation results, the recognition accuracy using DGM-HLFL was found to be better than [1] and [2]. The reason behind the improvement was due to the application of Hosmer-Lemeshow test classifier for face recognition algorithm. By applying this classifier, the weight and the response vector for each face image were updated on the basis of the learnt features. Also by introducing non-linear internal map-based back propagation, multiple processes were performed in a single iteration, optimizing resource utilization and therefore corroborating the objective, i.e., maximizing the recognition accuracy via parameter optimization. This in turn improved the recognition accuracy for face using DGM-HLFL by 3% compared to [1], [2] for Extended Yale Face Database B dataset. In a similar manner, recognition accuracy involved in the face recognition process using DGM-HLFL was improved by 4% compared to [1] and 8% compared to [2] for Pie dataset.

5.3. **Performance analysis of recognition time.** The second parameter of significance via partial differential equation for face recognition is the recognition time. In other words, recognition time refers to the time consumed in recognizing the face. It is mathematically

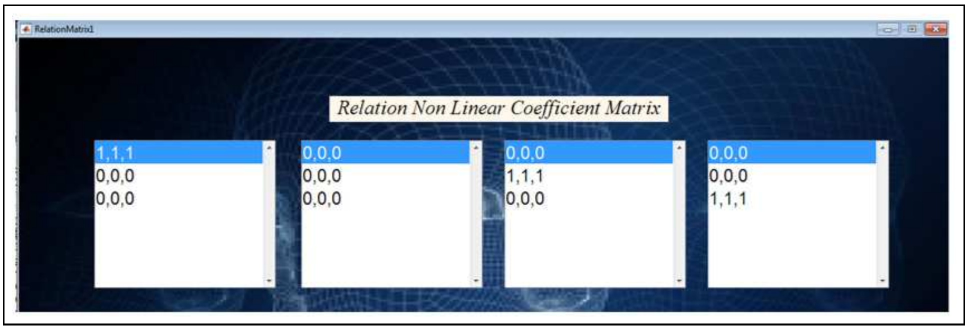


(a)

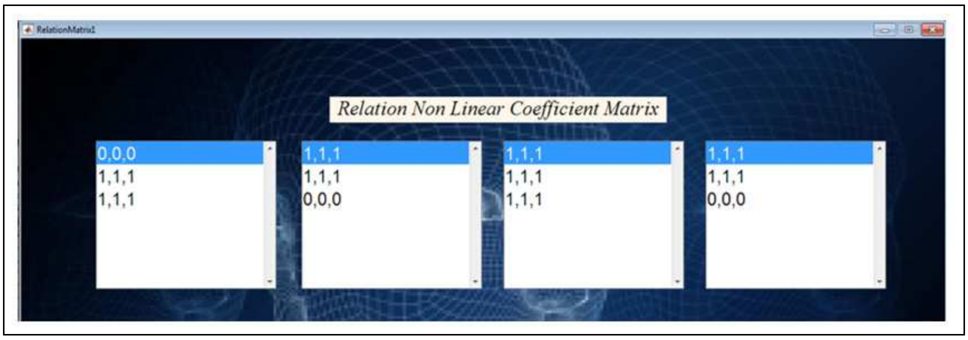


(b)

FIGURE 6. (a) Feature learnt from Extended Yale Face Database B dataset; (b) feature learnt from Pie dataset

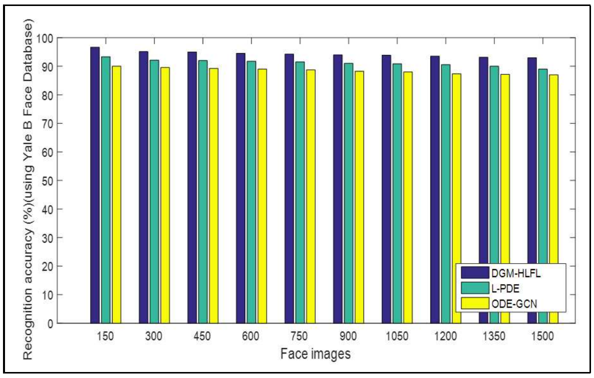


(a)

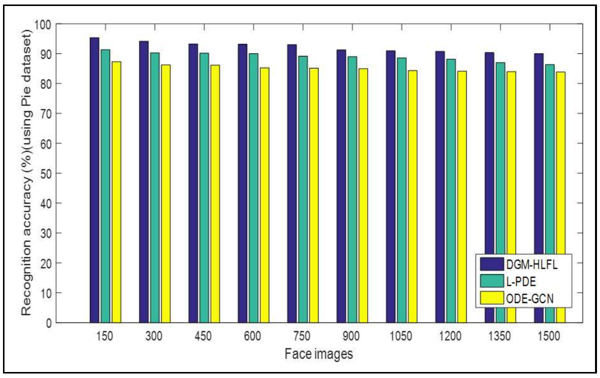


(b)

FIGURE 7. (a) Non linear weight update process using Extended Yale Face Database B dataset; (b) non linear weight update process using Pie dataset



(a)



(b)

FIGURE 8. (a) Recognition accuracy using Extended Yale Face Database B dataset; (b) recognition accuracy using Pie dataset

formulated as given below.

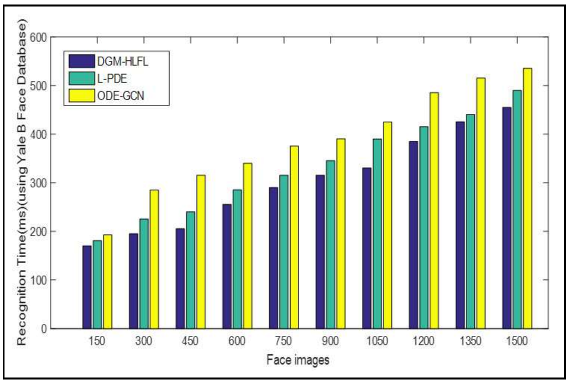
$$R_{time} = \sum_{i=1}^{n} FI_i * Time[FR] \tag{23}$$

From Equation (23), the recognition time 'R_{time}' is measured based on the same face images involved in simulation 'FI_i' and the time consumed in the face recognition process '$Time[FR]$'. It is measured in terms of milliseconds (ms).

Figures 9(a) and 9(b) illustrate the performance measure of recognition time with respect to 1500 distinct face images collected at different time instances using two different datasets, Extended Yale Face Database B dataset and Pie dataset, respectively. With the

Table 1. Quantitative analysis of recognition accuracy using DGM-HLFL, L-PDE [1] and ODE-GCN [2]

Face images	**Recognition accuracy (%) (Using Extended Yale Face Database B dataset)**			**Recognition accuracy (%) (Using Pie dataset)**		
	DGM-HLFL	L-PDE	ODE-GCN	DGM-HLFL	L-PDE	ODE-GCN
150	96.66	93.33	90	95.33	91.33	87.33
300	95.15	92.15	89.55	94.15	90.25	86.25
450	95	92	89.25	93.25	90.15	86.15
600	94.55	91.75	89	93.15	90	85.25
750	94.25	91.55	88.75	93	89.15	85.15
900	94	91	88.25	91.25	89	85
1050	93.85	90.85	88	90.95	88.55	84.35
1200	93.55	90.55	87.35	90.75	88.15	84.15
1350	93.15	90	87.15	90.35	87	84
1500	93	89	87	90	86.35	83.85


(a)

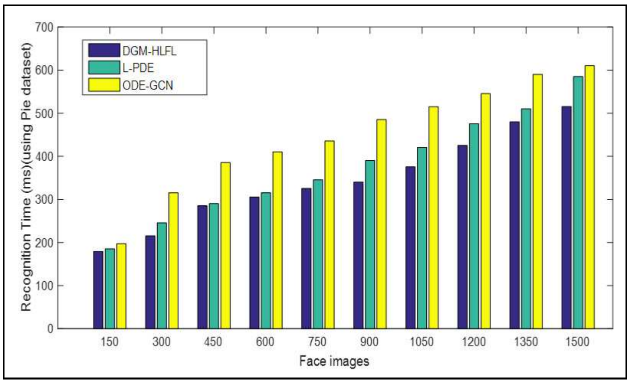

(b)

Figure 9. (a) Recognition time using Extended Yale Face Database B dataset; (b) recognition accuracy using Pie dataset

Table 2. Quantitative analysis of recognition time using DGM-HLFL, L-PDE [1] and ODE-GCN [2]

Face images	**Recognition time (ms) (Using Extended Yale Face Database B dataset)**			**Recognition time (ms) (Using Pie dataset)**		
	DGM-HLFL	L-PDE	ODE-GCN	DGM-HLFL	L-PDE	ODE-GCN
150	170.25	180.75	192.75	179.25	185.25	197.25
300	195.15	225.35	285.15	215.55	245.55	315.55
450	205.35	240.15	315.55	285.35	290.35	385.45
600	255.35	285.35	340.25	305.55	315.55	410.55
750	290.15	315.55	375.55	325.55	345.55	435.55
900	315.55	345.55	390.55	340.15	390.15	485.55
1050	330.45	390.15	425.15	375.55	420.55	515.25
1200	385.15	415.55	485.55	425.55	475.55	545.55
1350	425.55	440.25	515.55	480.15	510.35	590.15
1500	455.15	490.15	535.55	515.55	585.15	610.35

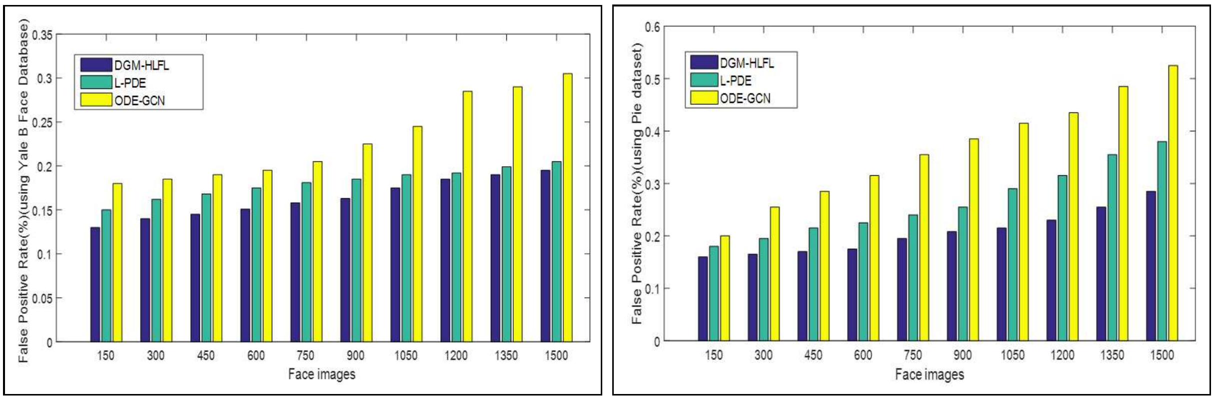
increase in the number of face images, acquired from two datasets [19] and [20], the mean time, i.e., amount of delay in executing certain operations for recognition, also increases. This is because certain face images are rotational invariant to some characteristics like light, and illumination whereas certain other face images are not said to be so. Therefore, the increase in recognition time is also found to be proportionally increasing. However, with the simulations conducted with '150' face images and time consumed in recognition single face image using Extended Yale Face Database B dataset as '1.135 ms', the recognition time was found to be '1.205 ms' using [1] and '1.285 ms' using [2]. With this, the recognition time was observed to be '170.25 ms', '180.75 ms' and '192.75 ms' using DGM-HLFL, L-PDE [1] and ODE-GCN [2], respectively. From this performance evaluation, it is inferred that the overall recognition time using DGM-HLFL is found to be minimum when compared to [1] and [2]. This is because of the incorporation of Gaussian multivariate regression-based feature learning model. By applying this model, in our work rotational invariants are performed on the basis of Gaussian multivariate regression function, therefore increasing the number of parameters that possess one dependent variable and multiple independent variables. Also, by means of multivariate regression it assists in interpreting the correlations between dependent variable and independent variables. This in turn reduces the recognition time involved for face recognition using DGM-HLFL that was minimized by 9% compared to [1] and 22% compared to [2] for Extended Yale Face Database B dataset. In a similar manner, recognition time involved for face recognition using DGM-HLFL was reduced by 8% compared to [1] and 23% compared to [2] for Pie dataset.

5.4. **Performance analysis of false positive rate.** Finally, the parameter of significance for face recognition is the false positive rate. In statistical analysis, when performing multiple comparisons for face recognition, a false positive ratio is referred to as the probability of falsely rejecting the true face with the stored templates for face recognition. In other words, the false positive rate is measured as the ratio between the number of negative events (i.e., falsification of images with correctly recognized) wrongly categorized as positive and the total number of actual negative events (i.e., total number of actually falsified images). The false positive rate is mathematically expressed as given below.

$$FPR = \frac{FP}{FP + TN} \tag{24}$$

From the above Equation (24), the false positive rate 'FPR' is measured based on the number of false positive 'FP', and the number of true negative 'TN', respectively.

Figures 10(a) and 10(b) illustrate the performance measure of false positive rate for 1500 different face images using two datasets, Extended Yale Face Database B dataset and Pie dataset, respectively. From the figure, it is inferred that the false positive rate is found to be inversely proportional to the number of face images provided as input. In other words, increasing the number of face images results in the increase in the false positive rate. However, for simulations performed with 150 images, the false positive rates using the three methods were found to be 20, 23 [1] and 28 [2] using Extended Yale Face Database B dataset and 24, 28 [1] and 30 [2] using Pie dataset. As a result, the false positive rate was observed to be 0.13, 0.15 [1] and 0.18 [2] when applied with Extended Yale Face Database B dataset, 0.16, 0.18 [1] and 0.2 [2] when applied with Pie dataset, respectively. With this the false positive rate was found to be less using DGM-HLFL compared to [1] and [2]. The reason behind the improvement was due to the application of non-linear internal mapping function. The function was designed in such a manner as to reduce the sum of square error for each response vector. With this the false positive rate using DGM-HLFL was said to be reduced by 10% compared to [1] and 28% compared



(a) (b)

FIGURE 10. (a) False positive rate using Extended Yale Face Database B dataset; (b) false positive rate using Pie dataset

TABLE 3. Quantitative analysis of false positive rate using DGM-HLFL, L-PDE [1] and ODE-GCN [2]

Face images	**False positive rate (%) (Using Extended Yale Face Database B dataset)**			**False positive rate (%) (Using Pie dataset)**		
	DGM-HLFL	L-PDE	ODE-GCN	DGM-HLFL	L-PDE	ODE-GCN
150	0.13	0.15	0.18	0.16	0.18	0.2
300	0.14	0.162	0.185	0.165	0.195	0.255
450	0.145	0.168	0.19	0.17	0.215	0.285
600	0.151	0.175	0.195	0.175	0.225	0.315
750	0.158	0.181	0.205	0.195	0.24	0.355
900	0.163	0.185	0.225	0.208	0.255	0.385
1050	0.175	0.19	0.245	0.215	0.29	0.415
1200	0.185	0.192	0.285	0.23	0.315	0.435
1350	0.19	0.199	0.29	0.255	0.355	0.485
1500	0.195	0.205	0.305	0.285	0.38	0.525

to [2] for Extended Yale Face Database B dataset. In a similar manner, false positive rate using DGM-HLFL was minimized by 21% compared to [1] and 42% compared to [2] for Pie dataset.

6. **Conclusion.** Feature learning via PDE for face recognition is the foremost research topic in the domain of image processing. In order to enhance the performance of face recognition, a PDE algorithm based on Gaussian multivariate regression-based feature learning via non-linear mapping Hosmer-Lemeshow test classification algorithm is proposed. Firstly, the Gaussian multivariate regression-based feature learning is used to learn robust and pertinent features with minimum recognition time. Then, illumination variation based on non-linear internal mapping is formed to concentrate on the false positive rate issue. Finally, Hosmer-Lemeshow test classifier for face recognition is designed to get the classification results. Experimental results show that this method can significantly improve the performance of face recognition, which is of immense importance to the research of face recognition. In future, adaptive boosting classifier is used for future extensions and advancements to accurately classify the facial images. In addition, increasing the number of parameters is used in the proposed method. It is the most effective way to develop the performance of face recognition.

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