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THE PARAMETERIZATION OF ALL DISTURBANCE OBSERVERS FOR PERIODIC OUTPUT DISTURBANCES

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Abstract. *In this paper, we examine the parameterization of all disturbance observers for periodic output disturbances. The disturbance observers have been used to estimate the disturbance in the plant. Several papers on design methods for disturbance observers have been published. Recently, the parameterization of all disturbance observers for plants was clarified. However, no paper clarifies the parameterization of all disturbance observers for periodic output disturbances. In this paper, we propose the parameterization of all disturbance observers for periodic output disturbances.*
Keywords: Estimation, Disturbances, Disturbance observers, Periodic output disturbances, Parameterization

1. **Introduction.** In this paper, we examine the parameterization of all disturbance observers for periodic output disturbances. A disturbance observer is used to estimate the disturbances in the factory plant [1-6]. Several papers on design methods for disturbance observers have been published [4-8]. Currently, the applications of disturbance observers have been used in many control systems such as a motion-control field [3, 5, 9]. A disturbance observer is used in motion control to cancel the disturbance or to make the closed-loop system robustly stable [1, 2, 7-14]. Typically disturbance observers include disturbance signal generators and an observer. Disturbances that are normally considered step disturbances are estimated by the observer. Since the disturbance observer is simple to understand the structure, it is used in many cases [1, 2, 8-13].

Mita et al. pointed out that disturbance observers are not the only alternative design of complete controllers [7]. That is, a control system with a disturbance observer does not guarantee robust stability. Extended H_∞ control in [7] has therefore been proposed as an effective motion control method that cancels disturbances. This implies that using the method in [7], a control system with a disturbance observer could be designed to guarantee robust stability. From another point of view, Kobayashi et al. considered an observer design method for obtaining phase compensation based on disturbance observers [8]. Compared to using a phase compensator, the control system in [8] is simple and easy

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to design. In this way, a robustness analysis of the control system that has observed disturbances has been considered.

Another important control problem is the parameterization problem which is the problem of finding all stable controllers for the plant [15-21]. If the parameterization of all disturbance observers for any disturbances could be obtained, we could express results from previous studies of disturbance observers in a uniform manner. In addition, disturbance observers for any disturbances could be designed systematically. From this point of view, Yamada et al. examined parameterizations of all disturbance observers and all linear functional disturbance observers for plants with any input-output disturbances [22-26]. Ando et al. examined parameterizations of all disturbance observers and all linear functional disturbance observers for plants with any input and output disturbances [27]. However, no paper examines the parameterization of all disturbance observers for periodic output disturbances. Methods in [22-27] can estimate disturbances with finite number of frequency component but cannot estimate disturbances with infinite number of frequency component. In addition, when we control practical systems, many disturbances appear as periodic disturbances, such as robot arms, heat-flow experiments, multi-axis manipulators, noises and vibrations [3, 26, 28, 29]. Therefore, it is important for the control system to attenuate periodic disturbances. In addition, the parameterization motivated by the above, parameterization and disturbance observers are taken into account for periodic output disturbances in this paper. Based on a control system to attenuate periodic disturbances strategy is proposed to improve the performance of the parameterization of all disturbance observers for periodic output disturbances. For these reasons, the purpose of this study is to propose the parameterization of all disturbance observers for periodic output disturbances.

In this paper, we clarify that the periodic output disturbances could be estimated using disturbance observers and propose the parameterization of all disturbance observers for periodic output disturbances. First, the necessary structure and characteristics of disturbance observers for periodic output disturbances are defined. In addition, the problem considered in this paper is explained. Conditions to estimate the periodic output disturbances are clarified. The parameterization of all disturbance observers for periodic output disturbances and that of all linear functional disturbance observers for periodic output disturbances are clarified. In addition, a design method for the linear functional disturbance observer and a procedure for linear functional disturbance observers for periodic output disturbances are clarified. Finally, we offer a numerical example to illustrate the features of the proposed design method. This paper is organized as follows. In Section 2, we formulate the problem considered in this paper. In Section 3, we clarify the conditions to estimate the periodic output disturbances. In Section 4, we propose the parameterization of all disturbance observers for periodic output disturbances. In Section 5, we define the parameterization of all linear functional disturbance observers for periodic output disturbances. In Section 6, we show a design method for the linear functional disturbance observer. In Section 7, we present a procedure for linear functional disturbance observers for periodic output disturbances. In Section 8, we provide a numerical example to illustrate the features of the proposed method. Section 9 gives concluding remarks.

Notations

R	the set of real numbers.
$R(s)$	the set of real rational functions with s.
RH_∞	the set of stable proper real rational functions.
$\mathcal{U}$	the unimodular procession in RH_∞. That is, $P(s) \in \mathcal{U}$ means that $P(s) \in RH_\infty$ and $P^{-1}(s) \in RH_\infty$.

$\bar{\sigma}(\{\cdot\})$	largest singular value of $\{\cdot\}$.
$\text{diag}\,(a_1, \ldots, a_n)$	an $n \times n$ diagonal matrix with a_i as its ith diagonal element.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array}\right]$	the state space description $C(sI - A)^{-1}B + D$.
$\mathcal{L}\{\cdot\}$	the Laplace transformation of $\{\cdot\}$.

2. **Problem Formulation.** Consider the plant described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + d(t) \end{cases}, \tag{1}$$

where $x \in R^n$ is the state variable, $u \in R^p$ is the control input, $y \in R^m$ is the output, $d \in R^m$ is the periodic output disturbance with period $T > 0$ satisfying

$$d(t + T) = d(t) \quad (\forall t \geq 0), \tag{2}$$

$A \in R^{n\times n}$, $B \in R^{n\times p}$ and $C \in R^{m\times n}$. It is assumed that (A, B) is stabilizable, (C, A) is detectable, A has no eigenvalue on the imaginary axis and $u(t)$ and $y(t)$ are available, but $d(t)$ is unavailable. The transfer function from $u(s)$ to $y(s)$ in (1) is denoted by

$$y(s) = G(s)u(s) + d(s), \tag{3}$$

where

$$G(s) = C(sI - A)^{-1}B \in R^{m\times p}(s). \tag{4}$$

When the disturbance $d(t)$ is unavailable, a disturbance estimator called the disturbance observer is frequently used. The disturbance observer estimates the disturbance $d(t)$ using the available measurements. Since the available measurements of the plant in (1) are $u(t)$ and $y(t)$ and the disturbance $d(t)$ satisfies (2), the periodic output disturbance $d(t)$ is estimated by the form in

$$\tilde{d}(s) = F_1(s)e^{-sT}y(s) + F_2(s)e^{-sT}u(s), \tag{5}$$

where $F_1(s) \in RH_\infty^{m\times m}$, $F_2(s) \in RH_\infty^{m\times p}$, $\tilde{d}(s) = \mathcal{L}\left\{\tilde{d}(t)\right\}$ and $\tilde{d}(t) \in R^m$. The structure of disturbance observer $\tilde{d}(s)$ in (5) is shown in Figure 1. In the following, we call the system $\tilde{d}(s)$ in (5) a disturbance observer for periodic output disturbances, if the error $e(t)$ between $d(t)$ and $\tilde{d}(t)$ written by

$$e(t) = d(t) - \tilde{d}(t) \tag{6}$$

satisfies

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} \left(d(t) - \tilde{d}(t)\right) = 0 \tag{7}$$

for any initial state $x(0)$, control input $u(t)$ and periodic output disturbance $d(t)$.

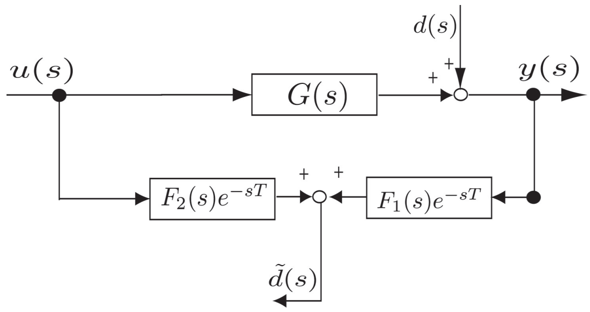

FIGURE 1. Structure of a disturbance observer

If the parameterization of all disturbance observers for periodic output disturbances is obtained, there is a possibility to attenuate periodic output disturbances without using the repetitive control system. In addition, we can design the disturbance observers for periodic output disturbances systematically. However no paper examines the parameterization of all disturbance observers for periodic output disturbances.

The problem considered in this paper is to propose the parameterization of all disturbance observers for periodic output disturbances. That is, we obtain the parameterization of all disturbance observers for $\tilde{d}(s)$ in (5) for periodic output disturbances.

3. **Conditions to Estimate the Periodic Output Disturbances.** In this section, we clarify the conditions of $\tilde{d}(s)$ in (5) to satisfy (7).

The condition of $\tilde{d}(s)$ in (5) satisfying (7) is summarized in the following theorem.

Theorem 3.1. *$\tilde{d}(s)$ in (5) works as a disturbance observer for periodic output disturbances if and only if*

$$F_1(s)N(s) + F_2(s)D(s) = 0, \tag{8}$$

and

$$\left(1 - e^{-s_iT}\right) e(s_i) = 0 \quad \forall s_i \, (i = 0, 1, \ldots), \tag{9}$$

respectively, where

$$s_i = j\omega_i, \tag{10}$$

and

$$\omega_i = \frac{2\pi i}{T} \quad (i = 0, 1, \ldots), \tag{11}$$

and j is the imaginary unit.

Proof: First, the necessity is shown, that is, if $\tilde{d}(s)$ in (5) satisfies (7), then (8) and (9) are satisfied. $\tilde{d}(s)$ in (5) is rewritten as

$$\tilde{d}(s) = \left(F_1(s)e^{-sT}N(s) + F_2(s)e^{-sT}D(s)\right)\xi(s) + F_1(s)e^{-sT}d(s), \tag{12}$$

where $\xi(s)$ is the pseudo-state variable satisfying

$$u(s) = D(s)\xi(s). \tag{13}$$

$N(s) \in RH_\infty^{m\times p}$ and $D(s) \in RH_\infty^{m\times m}$ are coprime factors of $G(s)$ on RH_∞ satisfying

$$G(s) = N(s)D^{-1}(s). \tag{14}$$

$\xi(s)$ in (12) is factorized as

$$\xi(s) = \tilde{\xi}(s) + \bar{\xi}(s) = \frac{1}{1 - e^{-sT}}\tilde{\xi}(s) + \bar{\xi}(s), \tag{15}$$

where $\tilde{\xi}(s)$ is denoted by

$$\tilde{\xi}(s) = \int_0^T e^{-s\tau}\xi(\tau)d\tau, \tag{16}$$

$\tilde{\xi}(s)/\left(1 - e^{-sT}\right)$ means the periodic signal with period T and $\bar{\xi}(s)$ includes all other signals. In addition, $d(s)$ in (12) satisfying (2) is rewritten by

$$d(s) = \frac{1}{1 - e^{-sT}}\hat{d}(s), \tag{17}$$

where

$$\hat{d}(s) = \int_0^T e^{-s\tau} d_i(\tau) d\tau. \tag{18}$$

Then the Laplace transformation of the error $e(t)$ in (6) is written by

$$\begin{aligned} e(s) = {} & \left(I - F_1(s)e^{-sT}\right) \frac{1}{1-e^{-sT}} \hat{d}(s) - (F_1(s)N(s) + F_2(s)D(s)) \frac{e^{-sT}}{1-e^{-sT}} \tilde{\xi}(s) \\ & + (F_1(s)N(s) + F_2(s)D(s)) e^{-sT} \bar{\xi}(s). \end{aligned} \tag{19}$$

From the assumption that $e(t)$ satisfies (7) for any $\bar{\xi}(s)$,

$$(F_1(s)N(s) + F_2(s)D(s))\, e^{-sT} \bar{\xi}(s) = 0 \tag{20}$$

is satisfied for any $\bar{\xi}(s)$. That is, we have (8). Substitution of (8) for (19) gives

$$e(s) = \left(I - F_1(s)e^{-sT}\right) \frac{1}{1-e^{-sT}} \hat{d}(s) - (F_1(s)N(s) + F_2(s)D(s)) \frac{e^{-sT}}{1-e^{-sT}} \tilde{\xi}(s). \tag{21}$$

From the assumption that $e(t)$ satisfies (7) and internal model principle [33], (9) is satisfied. We have thus proved the necessity.

Next, the sufficiency is shown. That is, if (8) and (9) are satisfied, then $e(s)$ in (6) satisfies (7). From (8), $e(s)$ in (6) is written by

$$e(s) = \left(I - F_1(s)e^{-sT}\right) \frac{1}{1-e^{-sT}} \hat{d}(s) - (F_1(s)N(s) + F_2(s)D(s)) \frac{e^{-sT}}{1-e^{-sT}} \tilde{\xi}(s). \tag{22}$$

From (9), $e(t)$ in (6) satisfies (7). Thus, the sufficiency is shown.

We have thus proved Theorem 3.1. □

Note that from Theorem 3.1, (8) is a condition of $\tilde{d}(s)$ disturbance observers for any state variable. In addition, (9) is a condition to estimate any periodic signals. Therefore, this is the most important condition to estimate the periodic output disturbances and the mentioned condition could solve the problem.

In this section, we obtained the conditions to estimate the periodic output disturbances. In the next section, using the result of Theorem 3.1, we clarify the parameterization of all disturbance observers for periodic output disturbances.

4. Parameterization of All Disturbance Observers for Periodic Disturbances.

In this section, we propose the parameterization of all disturbance observers $\tilde{d}(s)$ in (5) for periodic output disturbances.

The parameterization is summarized in the following theorem.

Theorem 4.1. *The system $\tilde{d}(s)$ in (5) is the disturbance observer for periodic output disturbances if and only if $F_1(s)$ and $F_2(s)$ are written by*

$$F_1(s) = \tilde{D}(s) + Q(s)\tilde{D}(s), \tag{23}$$

and

$$F_2(s) = -\tilde{N}(s) - Q(s)\tilde{N}(s) \in RH_\infty^{m\times p}, \tag{24}$$

where $\tilde{D}(s) \in RH_\infty^{m\times m}$ and $\tilde{N}(s) \in RH_\infty^{m\times p}$ are coprime factors of $G(s)$ on RH_∞ satisfying

$$G(s) = \tilde{D}(s)^{-1}\tilde{N}(s), \tag{25}$$

respectively. Here $Q(s) \in RH_\infty$ is any function satisfying

$$\tilde{D}(s_i) + Q(s_i)\tilde{D}(s_i) = I \quad \forall s_i\,(i = 0, 1, \ldots). \tag{26}$$

Proof of Theorem 4.1 requires the following lemma.

Lemma 4.1. [18] *Assume that* $A(s) \in RH_\infty^{m\times n}$, $B(s) \in RH_\infty^{q\times p}$, $C(s) \in RH_\infty^{m\times p}$ *and*

$$\text{rank} \left[\begin{array}{cc} A^T(s) & B^T(s) \end{array} \right] = \gamma \tag{27}$$

are satisfied. There exist $X(s) \in RH_\infty^{m\times m}$ *and* $Y(s) \in RH_\infty^{m\times q}$ *satisfying*

$$X(s)A(s) + Y(s)B(s) = C(s), \tag{28}$$

if and only if there exists $U(s) \in \mathcal{U}$ *satisfying*

$$\left[\begin{array}{c} A(s) \\ B(s) \\ C(s) \end{array} \right] = U(s) \left[\begin{array}{c} A(s) \\ B(s) \\ 0 \end{array} \right]. \tag{29}$$

When $X_0(s) \in RH_\infty^{m\times m}$ *and* $Y_0(s) \in RH_\infty^{m\times q}$ *are solutions of (28), then all solutions of (28) are given by*

$$\left[\begin{array}{cc} X(s) & Y(s) \end{array} \right] = \left[\begin{array}{cc} X_0(s) & Y_0(s) \end{array} \right] + Q(s) \left[\begin{array}{cc} W_1(s) & W_2(s) \end{array} \right], \tag{30}$$

where $W_1(s)$ *and* $W_2(s)$ *satisfy*

$$W_1(s)A(s) + W_2(s)B(s) = 0, \tag{31}$$

and

$$\text{rank} \left[\begin{array}{cc} W_1(s) & W_2(s) \end{array} \right] = n + q - \gamma, \tag{32}$$

and $Q(s) \in RH_\infty^{p\times(n+q-\gamma)}$ *is any function.*

Using Theorem 3.1 and Lemma 4.1, Theorem 4.1 is proved.

Proof: From Theorem 3.1, $\tilde{d}(s)$ works as a disturbance observer for periodic output disturbances if and only if $F_1(s) \in RH_\infty^{m\times m}$ and $F_2(s) \in RH_\infty^{m\times p}$ satisfy (8). From Lemma 4.1, all solutions $F_1(s)$ and $F_2(s)$ of (8) are given by (23) and (24), respectively, since

$$\tilde{D}(s)N(s) - \tilde{N}(s)D(s) = 0, \tag{33}$$

and Lemma 4.1, where $\tilde{D}(s) \in RH_\infty^{p\times p}$ and $\tilde{N}(s) \in RH_\infty^{p\times m}$ are coprime factors of $G(s)$ on RH_∞ satisfying

$$G(s) = \tilde{D}^{-1}(s)\tilde{N}(s). \tag{34}$$

The rest is to prove $\tilde{d}(s)$ in (5) works as a disturbance observer for periodic output disturbances if and only if $Q(s)$ in (23) and (24) satisfies (26). From Theorem 3.1, $\tilde{d}(s)$ in (5) works as a disturbance observer for periodic output disturbances if and only if $e(s)$ in (6) satisfies (9).

The necessity is shown. That is, if $\tilde{d}(s)$ in (5) works as a disturbance observer for periodic output disturbances, then $Q(s)$ in (23) and (24) satisfies (26). From (23) and (24), $e(s)$ in (6) is written by

$$e(s) = \left\{ I - F_1(s)e^{-sT} \right\} \frac{1}{1 - e^{-sT}} \hat{d}(s). \tag{35}$$

This equation yields

$$\left(1 - e^{-s_iT}\right) e(s_i) = \left\{ I - \left(\tilde{D}(s_i) + Q(s_i)\tilde{D}(s_i) \right) \right\} \tilde{d}(s_i) = 0. \tag{36}$$

We have (26). Thus, we proved the necessity.

Next, the sufficiency is shown. That is, we show that if $Q(s)$ in (23) and (24) satisfies (26), then (9) is satisfied. $e(s)$ in (6) is written by (35). Substituting (26) to (35), it is obvious that (9) is satisfied. In this way, the sufficiency has been proved.

From the above discussion, we have thus proved Theorem 4.1. □

Note that from Theorem 4.1, when $G(s)$ is stable, if $Q(s)$ is settled by

$$Q(s) = \tilde{D}^{-1}(s) - I, \tag{37}$$

then $Q(s)$ in (37) satisfies (26). However, when $G(s)$ is unstable, it is difficult to set $Q(s)$ satisfying (26). For the unstable plant $G(s)$, a disturbance observer for periodic output disturbances is often used to attenuate disturbances effectively in [30], even if the system $\tilde{d}(s)$ in (5) satisfying (9) could not be designed. This means that in order to attenuate periodic disturbances, it is enough to estimate $(I - F(s))\tilde{d}(s)$, where $F(s) \in RH_\infty$ is any function. From this point of view, in the next section, when $G(s)$ is unstable, we define a linear functional disturbance observer for periodic output disturbances and clarify the parameterization of all linear functional disturbance observers for periodic output observers.

5. **Parameterization of All Linear Functional Disturbance Observers for Periodic Output Disturbances.** In this section, we define a linear functional disturbance observer and present the parameterization of all linear functional disturbance observers for periodic output disturbances.

We call $\tilde{d}(s)$ in (5) the linear functional disturbance observer for periodic output disturbances if $\tilde{d}(s)$ is written by

$$\left(1 - e^{-s_i T}\right) e(s_i) = F(s_i)\hat{d}(s_i) \tag{38}$$

is satisfied, where $F(s) \in RH_\infty^{m\times m}$ is any function satisfying

$$\bar{\sigma}\left\{F(s_i)\right\} \simeq 0 \quad \forall s_i\, (i = 0, 1, \dots, n_{\max}), \tag{39}$$

and $n_{\max}$ is the maximum frequency satisfying (39). Since the available measurements of the plant $G(s)$ in (1) are $u(t)$ and $y(t)$ and the disturbance $d(t)$ satisfies (2), the periodic disturbance $d(t)$ is estimated by the form in (5), where $F_1(s) \in RH_\infty^{m\times m}$ and $F_2(s) \in RH_\infty^{m\times p}$.

The parameterization of all linear functional disturbance observers for periodic output disturbances is summarized in the following theorem.

Theorem 5.1. *The system $\tilde{d}(s)$ in (5) is the linear functional disturbance observer for periodic output disturbances if and only if $F_1(s)$, $F_2(s)$ and $F(s)$ are described by*

$$F_1(s) = \tilde{D}(s) + Q(s)\tilde{D}(s), \tag{40}$$

$$F_2(s) = -\tilde{N}(s) - Q(s)\tilde{N}(s), \tag{41}$$

and

$$F(s) = I - F_1(s) = I - \left(\tilde{D}(s) + Q(s)\tilde{D}(s)\right), \tag{42}$$

respectively, where $Q(s) \in RH_\infty^{m\times m}$ is any function satisfying

$$\bar{\sigma}\left\{I - F_1(s_i)\right\} = \bar{\sigma}\left\{I - \left(\tilde{D}(s_i) + Q(s_i)\tilde{D}(s_i)\right)\right\} \simeq 0 \quad \forall s_i\, (i = 0, 1, \dots, n_{\max}). \tag{43}$$

Proof: First, the necessity is shown. That is, we show that if the system $\tilde{d}(s)$ in (5) is a linear functional disturbance observer for periodic output disturbances, then (40), (41), (42) and (43) are satisfied. From (5), (12), (13), (14), (15) and (16), for the system $\tilde{d}(s)$ in (5), $e(s)$ is written as (19). From the assumption that $e(s)$ satisfies (7) for any $\bar{\xi}(s)$, (21) holds for any $\bar{\xi}(s)$. That is, we have (9). From (33) and Lemma 4.1, all solutions of $F_1(s)$ and $F_2(s)$ to satisfy (9) are given by (40) and (41), respectively. Substitution of (9) to (21) gives (22). From (22) the assumption that $e(s)$ satisfies (38), we have (42) and (43). In this way, the necessity has been proved.

Next, the sufficiency is shown. That is, we show that if (40), (41), (42) and (43) are satisfied, then the $\tilde{d}(s)$ is a linear functional disturbance observer. Since $e(s)$ in (6) is written by (19), substituting (40), (41), (42) and (43) to (19), it is obvious that (38) is satisfied. In this way, the sufficiency has been proved.

From the above, we have thus proved Theorem 5.1. □

Note that from Theorem 5.1, $\tilde{d}(s)$ satisfing (38) and (39), then (40), (41), (42) and (43) are satisfied to be solved by Theorem 5.1.

6. **Design Method for Linear Functional Disturbance Observers.** In this section, we show a design method for a linear functional disturbance observer $\tilde{d}(s)$.

In order to design the linear functional disturbance observer $\tilde{d}(s)$ for periodic output disturbances, $Q(s)$ in (40) and (41) needs to satisfy (43).

When $G(s)$ is unstable, $Q(s)$ is set as

$$Q(s) = \hat{Q}(s)\left(I - \tilde{D}(s)\right)\tilde{D}_o^{-1}(s), \tag{44}$$

where $\tilde{D}_o(s) \in RH_\infty^{m\times m}$ is an outer function of $\tilde{D}(s)$ satisfying

$$\tilde{D}(s) = \tilde{D}_o(s)\tilde{D}_i(s), \tag{45}$$

$\tilde{D}_i(s) \in RH_\infty^{m\times m}$ is a co-inner function of $\tilde{D}(s)$ satisfying $\tilde{D}_i(0) = I$ and $\tilde{D}_i(s)\tilde{D}_i(-s)^T = I$, $\hat{Q}(s) \in RH_\infty^{m\times m}$ is any function satisfying

$$\bar{\sigma}\left\{I - \hat{Q}(s_i)\tilde{D}_i(s_i)\right\} \simeq 0 \quad \forall s_i\,(i = 0, 1, \ldots, n_{\max}). \tag{46}$$

From the above, we showed a design of the linear functional disturbance observer $\tilde{d}(s)$ for periodic output disturbances, $Q(s)$ in (44) satisfied (45) and (46) based on Theorem 5.1. $Q(s)$ in (44) is designed using the method described in Section 7.

7. **Procedure for Linear Functional Disturbance Observers for Periodic Output Disturbances.** In this section, we show a design procedure for linear functional disturbance observer for periodic output disturbances satisfying Theorem 5.1.

A design procedure is summarized as follows.

Procedure

1) Obtain coprime factors $\tilde{N}(s) \in RH_\infty^{m\times p}$ and $\tilde{D}(s) \in RH_\infty^{m\times m}$ of $G(s) \in R(s)^{m\times p}$ satisfying (25). The parameterization of all linear functional disturbance observers is given by (5), where $F_1(s)$, $F_2(s)$ and $F(s)$ are written by (40), (41) and (42), respectively.
2) The maximum frequency range $n_{\max}$ in (43) to estimate the periodic disturbance $d(s)$ is settled.
3) Factorize $\tilde{D}(s)$ as (45) satisfying $\tilde{D}_i(0) = I$.
4) Settle $Q(s) \in RH_\infty^{m\times m}$ satisfying (43). In order to satisfy (43), $Q(s) \in RH_\infty^{m\times m}$ is set according to (44), where $\hat{Q}(s)$ is a low-pass filter satisfying $\hat{Q}(0) = I$, as

$$\hat{Q}(s) = \mathrm{diag}\left\{\frac{k_1}{(1+s\tau_1)^{\alpha_1}}, \ldots, \frac{k_m}{(1+s\tau_m)^{\alpha_m}}\right\}, \tag{47}$$

α_i $(i = 1, 2, \ldots, m)$ is an arbitrary positive integer and k_i $(i = 1, 2, \ldots, m)$ satisfying τ_i $(i = 1, \ldots, m)$

$$\sigma\left\{I - \hat{Q}(s_i)\tilde{D}_i(s_i)\right\} \simeq 0, \quad \forall s_i\,(i = 1, \ldots, m) \tag{48}$$

are real numbers.

5) Substituting $Q(s)$ for (40), (41) and (42), $F_1(s)$, $F_2(s)$ and $F(s)$ are obtained. Then we could design disturbance observer $\tilde{d}(s)$ for periodic output disturbances as (5).

In this section, we showed a procedure for linear functional disturbance observers for periodic output disturbances.

8. **Numerical Example.** In this section, we show numerical examples to illustrate the effectiveness of the proposed parameterizations.

Firstly, we show that the proposed design method of the disturbance observer for the stable plant in this paper could estimate the periodic disturbance more effectively than the other design method of disturbance observers. To compare the effectiveness of the proposed design method in this paper, we show a result that the disturbance observer designed by using a design method of [4] and a proposed method in this paper estimates the periodic disturbance for a Single-Input/Single-Output stable plant. Next, we show that the linear functional disturbance observer for the periodic disturbances designed by using the proposed design method in this paper could estimate the periodic disturbances for Single-Input/Single-Output unstable plant.

8.1. **Numerical Example 1. A numerical example of disturbance observers for step disturbance for the stable plant.** Consider the problem to estimate the periodic disturbance by designing a disturbance observer using a design method in [4] for stable plant $G(s)$ given as

$$G(s) = \frac{s+1}{s^2+5s+6}. \tag{49}$$

The period T of the periodic disturbance $d(t)$ is

$$T = \pi.$$

The disturbance observer is denoted as

$$\tilde{d}(s) = Q(s)G(s)^{-1}y(s) + Q(s)u(s), \tag{50}$$

where $Q(s)$ in (50) is the filter satisfying $\lim_{s\to 0} Q(s) = 1$. $Q(s)$ in (50) is settled by

$$Q(s) = \frac{1}{(s+1)^2}.$$

When the control input $u(t)$ and the periodic output disturbance $d(t)$ are given by

$$u(t) = 0,$$

and

$$d(t) = \sum_{i=1}^{5} \sin(it),$$

respectively, the response curve of disturbance are estimated by using a design method [4] for the step disturbance. The response curves of disturbance estimations are shown in Figure 2. Here, the dotted line shows the periodic output disturbances of $d(t)$ and the solid line shows the disturbance observer of $\tilde{d}(t)$. Figure 2 shows that the disturbance observer $\tilde{d}(s)$ in (50) for step disturbance could not estimate $\tilde{d}(t)$ effectively.

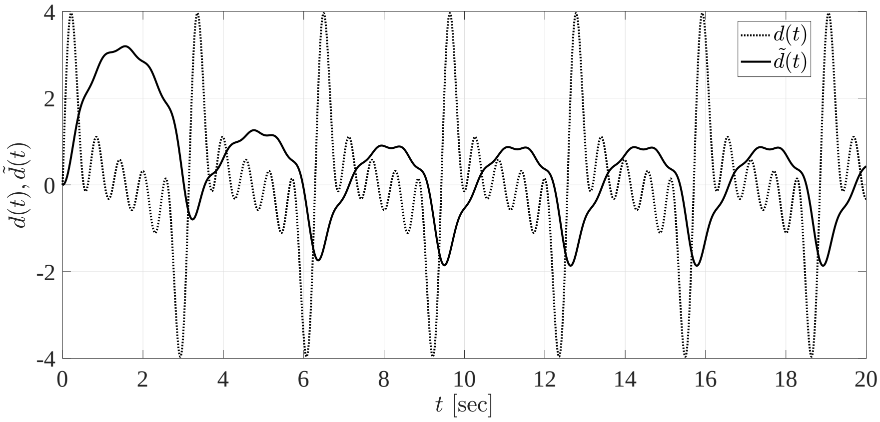

FIGURE 2. Response curves of the disturbance estimation by using a design method of [4]

8.2. **Numerical Example 2. A numerical example of disturbance observers for step disturbance for the stable plant.** Consider the problem to obtain the parameterization of all disturbance observers for stable plant $G(s)$ written by

$$G(s) = \frac{s+1}{s^2+5s+6}. \tag{51}$$

The period T of the periodic disturbance $d(t)$ is

$$T = \pi.$$

Coprime factorization of $G(s)$ in (51) satisfying (25) is given by

$$\tilde{N}(s) = G(s) = \frac{s+1}{s^2+5s+6},$$

and

$$\tilde{D}(s) = \frac{s^2+5s+6}{s^2+13s+42}.$$

From Theorem 4.1, the parameterization of all disturbance observers $\tilde{d}(s)$ for stable plant $G(s)$ in (51) is given by (5), where

$$F_1(s) = \frac{s^2+5s+6}{s^2+13s+42} + Q(s)\frac{s^2+5s+6}{s^2+13s+42},$$
$$F_2(s) = -\frac{s+1}{s^2+5s+6} - Q(s)\frac{s+1}{s^2+5s+6},$$

and $Q(s) \in RH_\infty$ is any function.

Next using obtained parameterization, we design a disturbance observer $\tilde{d}(s)$ for the periodic output disturbances, that is, $Q(s)$ is settled satisfying (26). In order to satisfy (26), $Q(s)$ is settled by (37).

When the control input $u(t)$ and the periodic output disturbance $d(t)$ are given by

$$u(t) = 0,$$

and

$$d(t) = \sum_{i=1}^{5} \sin(it),$$

respectively, the response curves of disturbance are estimated by using a proposed method. The response curves of disturbance estimations are shown in Figure 3. Here, the dotted line shows the periodic output disturbances of $d(t)$ and the solid line shows the disturbance observer of $\tilde{d}(t)$. Figure 3 shows that disturbance observer $\tilde{d}(s)$ in (5) for step disturbance could estimate $\tilde{d}(t)$ effectively.

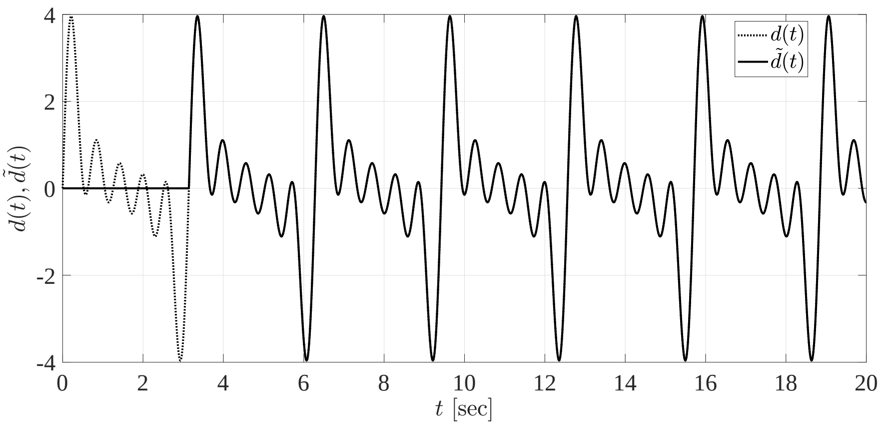

FIGURE 3. Response curves of the disturbance estimation

In this way, it is shown that using the obtained parameterization of all disturbance observers for periodic output disturbances, we could easily design a disturbance observer for step disturbance.

8.3. **Numerical Example 3. A numerical example of disturbance observers for periodic output disturbances.** Consider the problem to obtain the parameterization of all disturbance observers for stable plant $G(s)$ written by

$$G(s) = \frac{s+1}{s^2+6s+7}. \tag{52}$$

The period T of the periodic disturbance $d(t)$ is

$$T = \pi.$$

Coprime factorization of $G(s)$ in (52) satisfying (25) is given by

$$\tilde{N}(s) = G(s) = \frac{s+1}{s^2+6s+7},$$

and

$$\tilde{D}(s) = \frac{s^2+6s+7}{s^2+13s+42}.$$

From Theorem 4.1, the parameterization of all disturbance observers $\tilde{d}(s)$ for stable plant $G(s)$ in (52) is given by (5), where

$$F_1(s) = \frac{s^2+6s+7}{s^2+13s+42} + Q(s)\frac{s^2+6s+7}{s^2+13s+42},$$
$$F_2(s) = -\frac{s+1}{s^2+6s+7} - Q(s)\frac{s+1}{s^2+6s+7},$$

and $Q(s) \in RH_\infty$ is any function.

Next using obtained parameterization, we design a disturbance observer $\tilde{d}(s)$ for the periodic output disturbances, that is, $Q(s)$ is settled satisfying (26). In order to satisfy (26), $Q(s)$ is settled by (37).

When the control input $u(t)$ and the periodic output disturbance $d(t)$ are given by

$$u(t) = 0,$$

and

$$d(t) = \begin{cases} 1 + \dfrac{1}{\pi}t, & 2\pi i \leq t < \pi + 2\pi i \quad (\forall i = 0, 1, \ldots) \\ \dfrac{1}{\pi}t - 1, & \pi + 2\pi i \leq t < 2\pi + 2\pi i \quad (\forall i = 0, 1, \ldots) \end{cases},$$

respectively, the response curves of disturbance are estimated by using a proposed method. The response curves of disturbance estimations are shown in Figure 4. Here, the dashed line shows the periodic output disturbances of $d(t)$ and the solid line shows the disturbance observer of $\tilde{d}(t)$. Figure 4 shows that disturbance observer $\tilde{d}(s)$ in (5) for step disturbance could estimate $\tilde{d}(t)$ effectively. The response of the error $e(t)$ in (6) is shown in Figure 5. Here, the solid line shows the response of $e(t)$. Figure 5 shows that disturbance observer $\tilde{d}(s)$ in (5) for periodic output disturbances could estimate $d(t) - \tilde{d}(t)$ effectively.

In this way, it is shown that using the obtained parameterization of all disturbance observers for periodic output disturbances, we could easily design the disturbance observer for periodic output disturbances.

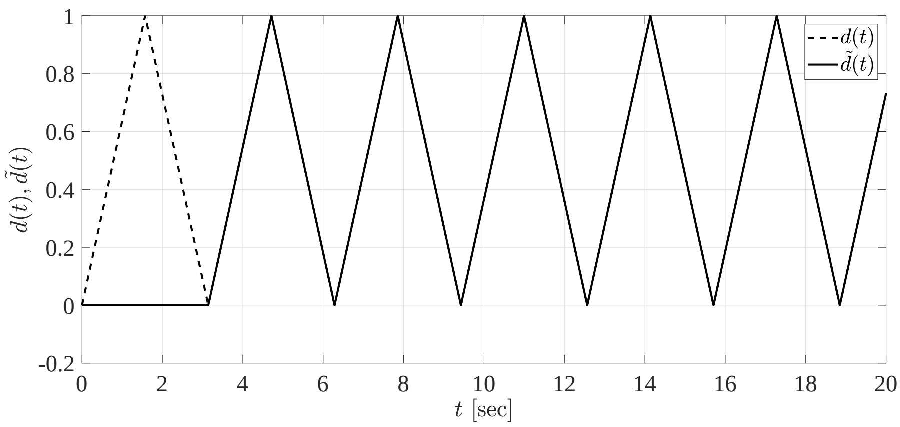

FIGURE 4. Response curves of the disturbance estimation

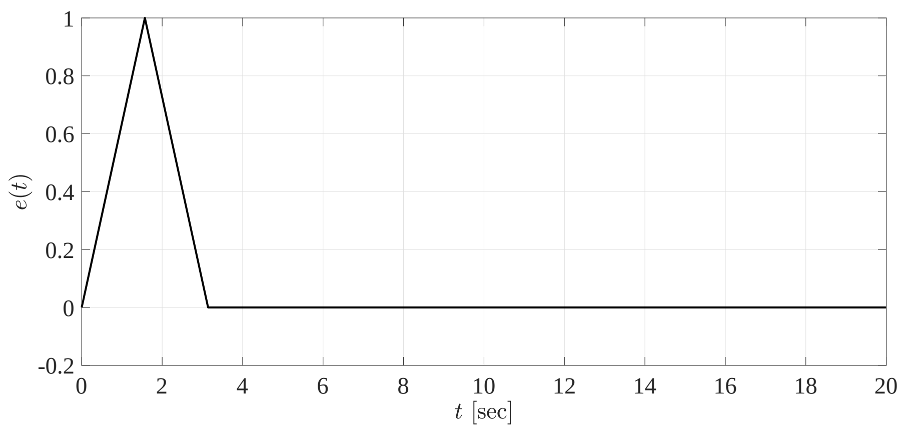

FIGURE 5. The response of the error $e(t)$ in (6)

8.4. **Numerical Example 4. A numerical example for linear functional disturbance observer.** Consider the problem to obtain the parameterization of all linear functional disturbance observers for periodic output disturbances for unstable plant $G(s)$ described by

$$G(s) = \frac{s+1}{s^2+43s-350}. \tag{53}$$

The period T of the periodic disturbances is

$$T = \pi.$$

A pair of coprime factors $\tilde{N}(s) \in RH_\infty$ and $\tilde{D}(s) \in RH_\infty$ of $G(s)$ in (53) satisfying (25) is given by

$$\tilde{N}(s) = \frac{-2s-2}{s^2+1007s+7000},$$

and

$$\tilde{D}(s) = \frac{-2s+100}{s+1000}. \tag{54}$$

From Theorem 5.1, the parameterization of all linear functional disturbance observers $\tilde{d}(s)$ is given by (5), where

$$F_1(s) = \frac{-2s+100}{s+1000} + Q(s)\frac{-2s+100}{s+1000},$$
$$F_2(s) = \frac{2s+2}{s^2+1007s+7000} + Q(s)\frac{2s+2}{s^2+1007s+7000},$$
$$F(s) = 1 - \frac{-2s+100}{s+1000} - Q(s)\frac{-2s+100}{s+1000},$$

and $Q(s) \in RH_\infty$ is any function.

Next using obtained parameterization, we design a linear functional disturbance observer $\tilde{d}(s)$ for the periodic output disturbances by using the procedure described in Section 7, that is, $Q(s)$ is settled satisfying (26). The maximum frequency range $n_{\max}$ in (43) to estimate the periodic disturbance $d(s)$ is settled by

$$n_{\max} = 3.$$

$\tilde{D}(s)$ in (54) is factorized as (45), where

$$\tilde{D}_o(s) = \frac{2s+100}{s+1000},$$

and

$$\tilde{D}_i(s) = \frac{-s+50}{s+50}.$$

In order to satisfy (43), $\hat{Q}(s)$ is settled by

$$\hat{Q}(s) = 1. \tag{55}$$

In order to confirm that $\hat{Q}(s)$ in (55) satisfies (43), we show the gain plot of $1 - \hat{Q}(s)\tilde{D}_i(s)$ in Figure 6. Figure 6 shows $\hat{Q}(s)$ in (55) satisfies (43). $Q(s)$ is set by (44) and written by

$$Q(s) = \frac{1.5s+450}{s+50}.$$

From (40), (41) and (42), we have $F_1(s)$, $F_2(s)$ and $F(s)$ are designed as

$$F_1(s) = \frac{-5s^2-750s+50000}{s^2+1050s+50000},$$

$$F_2(s) = \frac{5s^2 + 1005s + 1000}{s^3 + 1057s^2 + 57350s + 350000},$$

and

$$F(s) = \frac{6s^2 + 1800s}{s^2 + 1050s + 50000}.$$

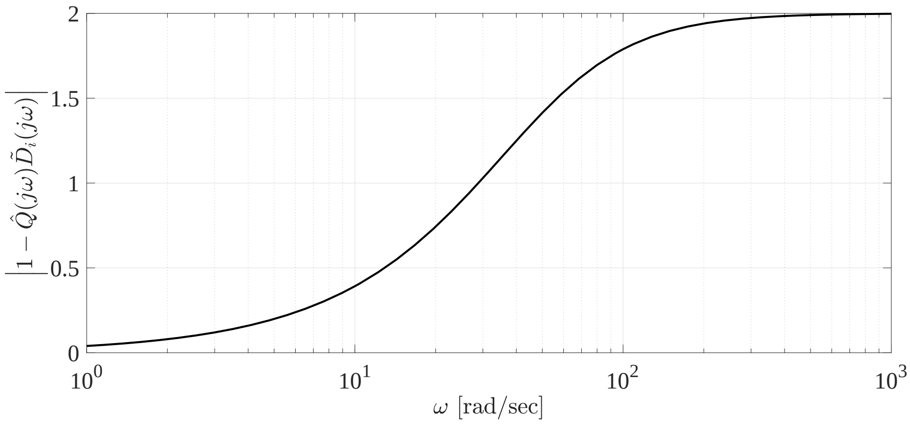

FIGURE 6. The gain plot of $1 - \hat{Q}(s)\tilde{D}_i(s)$

When the control input $u(t)$ and the periodic output disturbance $d(t)$ are given by

$$u(t) = 0,$$

and

$$d(t) = \begin{cases} 1 + \dfrac{1}{\pi}t, & 2\pi i \le t < \pi + 2\pi i \quad (\forall i = 0, 1, \ldots) \\ \dfrac{1}{\pi}t - 1, & \pi + 2\pi i \le t < 2\pi + 2\pi i \quad (\forall i = 0, 1, \ldots) \end{cases},$$

respectively, the response curves of disturbance are estimated by using a proposed method. The response curves of disturbance estimations are shown in Figure 7. Here, the dashed line shows the periodic output disturbances of $d(t)$ and the solid line shows the disturbance observer of $\tilde{d}(t)$. Figure 7 shows that disturbance observer $\tilde{d}(s)$ in (5) for step disturbance could estimate $\tilde{d}(t)$ effectively. The response to the error $e(t)$ in (6) is shown in Figure 8. Here, the solid line shows the response of $e(t)$. Figure 8 shows that linear functional disturbance observer $\tilde{d}(s)$ in (5) for periodic output disturbances could estimate $d(t)-\tilde{d}(t)$ effectively.

We have thus shown that using the parameterization of all linear functional disturbance observers for periodic output disturbances, we could easily design a linear functional disturbance observer for periodic output disturbances.

9. **Conclusions.** In this paper, we have proposed parameterizations of all disturbance observers and of all linear functional disturbance observers for periodic output disturbances. We have shown that the proposed method could attenuate periodic disturbances effectively without using repetitive controllers. A design method and a design procedure of linear functional disturbance observers are presented. Finally, we have shown features of the proposed design method through numerical examples. Using obtained parameterizations, a design method for control systems will be discussed in another article.

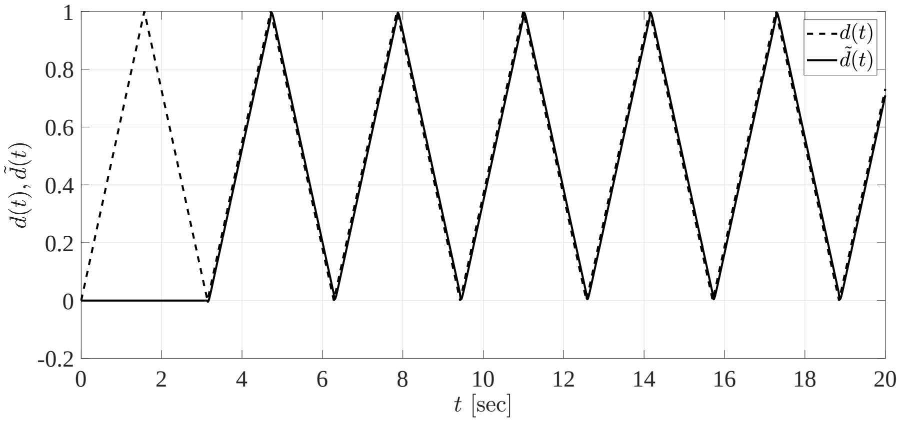

FIGURE 7. Response curves of the disturbance estimation

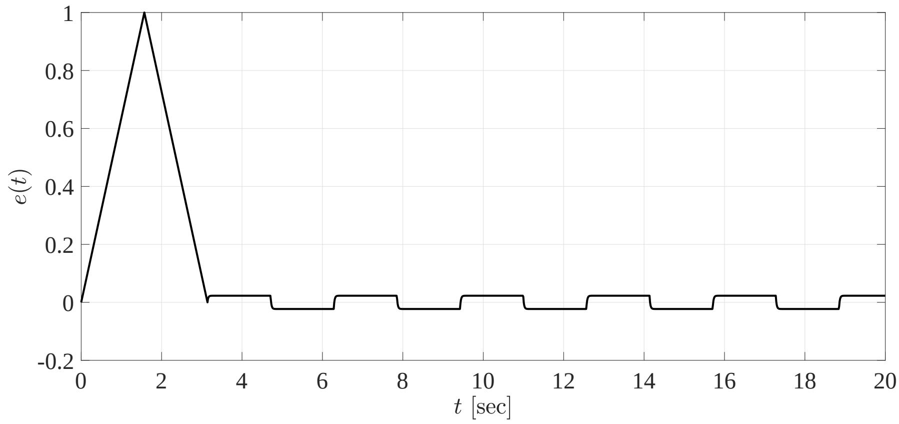

FIGURE 8. The response of the error $e(t)$ in (6)

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