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# THE PARAMETERIZATION OF ALL EXTENDED SEMI-STRONGLY STABILIZING CONTROLLERS

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Abstract. *A strong stabilization is the control method to make control systems stable using stable controllers. Using this method, it is possible to construct a highly reliable control system because it has robustness. However, when a strongly stabilizing controller is used, steady-state error remains when step disturbances and uncertainties exist in the control system because the strongly stabilizing controllers do not have a pole on the origin. To solve this problem, Hoshikawa et al. define semi-strongly stabilizing controllers that have a pole on the origin and rest of the poles in the open left half plane. In addition, they clarify the parameterizations of all semi-strongly stabilizable plants and all semi-strongly stabilizing controllers for the strongly stabilizable plants. However, the method of Hoshikawa et al. fails to place the poles on the imaginary axis. Some control methods need to place poles on the imaginary axis; for example, the control systems follow sinusoidal signals and the self-repairing control systems with faulty sensors. To overcome this problem, we expand the results by Hoshikawa et al. and define extended semi-strongly stabilizing controllers. In addition, the parameterizations of all extended semi-strongly stabilizable plants and all extended semi-strongly stabilizing controllers for extended semi-strongly stabilizable plants are clarified.*
**Keywords:** Parameterization, Strongly stabilizing controller, Stability

1. **Introduction.** This paper concerns the parameterization of all extended semi-strongly stabilizing controllers. The parameterization is a method of finding all stabilizing controllers for a given plant [1, 2]. Using parametrization, the stability of the control system is guaranteed. Various papers have been published on parameterization problems, such as Propotional-Integral-Derivative (PID) control [3], two-degree-of-freedom stabilizing controller [13], disturbance observer [4], modified Smith predictor [5], and internally stabilizing controller [6]. However, the stability of the controller obtained in the parametrization is not considered. If the controller is unstable, the control system will be highly sensitive when parameters under control change [7].

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To be resilient to parameter changes, stable controllers should be used. Toward this problem, there exists a control method called strong stabilization, which stabilizes the control system using stable controllers. Using this method, there is no need to consider problems such as high sensitivity to disturbances and degradation of target tracking performance that occur when unstable controllers are used [8, 9]. However, for any plant, strongly stabilizing controllers do not necessarily exist. The condition that there exist strongly stabilizing controllers is known as the parity interlacing property condition [7, 10]. Wakaiki et al. examine the sensitivity reduction problem with stable controllers for the linear time-invariant multi-input/multi-output distributed parameter system [17, 18]. However, they do not clarify the class of strongly stabilizable plants. If the class of strongly stabilizable plants is clarified, we can obtain the parameterization of all stable stabilizing controllers. In addition, we can clarify the characteristic of strongly stabilizable plants. From this viewpoint, Hoshikawa et al. clarify the class of all strongly stabilizable plants [12]. In addition, Hoshikawa et al. [13] clarify the parameterization of all two-degree-of-freedom strongly stabilizing controllers.

Using strongly stabilizing controllers, when uncertainty in the plant or a step disturbance exists, the output of the control system cannot follow the step reference input without steady-state error. In many actual control systems, the output is required to follow the step reference input without steady-state error, even if the uncertainty in the plant or the step disturbance exists. To overcome this problem, an integrator must be introduced to offset elimination from a set point. From this viewpoint, Hoshikawa et al. extend the concept of strong stabilization and propose a concept of semi-strong stabilization, which is a stabilization by a controller that has a pole at the origin and rest of the poles in the open left half plane [14]. Then, a class of semi-strongly stabilizable control objects and a controller design method are proposed [15]. Using these controllers, a robust and reliable control system can be designed. However, the method by Hoshikawa et al. fails to place the poles on the imaginary axis. There exist control systems that need to have a pair of poles on the imaginary axis; for example, the control systems follow sinusoidal signals and the self-repairing control systems with faulty sensors [16]. Therefore, the controller that has a pair of poles on the imaginary axis and the others in the open left half plane is required. Furthermore, it becomes easy to design controllers having a pair of poles on the imaginary axis and the others in the open left half plane, if the parameterization of all these controllers is obtained. In addition, we can design such controllers systematically.

In this paper, we propose a concept of extended semi-strongly stabilizing controllers that have a pair of poles on the imaginary axis and the others in the open left half plane. The parameterization of all extended semi-strongly stabilizable plants is clarified. For the extended semi-strongly stabilizable plant, we propose the parameterization of all extended semi-strongly stabilizing controllers. This paper is organized as follows. In Section 2, the problem considered in this paper is explained. In Section 3, the class of all extended semi-strongly stabilizable plants is considered. That is, we clarify the parameterization of all extended semi-strongly stabilizable plants. In Section 4, for the extended semi-strongly stabilizable plants, the parameterization of all extended semi-strongly stabilizing controllers is clarified. In Section 5, we present a design procedure of extended semi-strongly stabilizing controllers. In Section 6, a numerical example is illustrated to show the effectiveness of the proposed method. Section 7 presents concluding remarks.

## 2. Problem Formulation.

Consider the control system:

$$\left\{ \begin{array}{l} y(s) = G(s)u(s) + d(s) \\ u(s) = C(s)(r(s) - y(s)) \end{array} \right. , \tag{1}$$

where $G(s) \in R(s)$ is the plant, $C(s) \in R(s)$ is the controller, $y(s) \in R$ is the output, $u(s) \in R$ is the control input, $r(s) \in R$ is the reference input, and $d(s) \in R$ is the disturbance.

Hoshikawa et al. define a semi-strongly stabilizing controller [15], which has a pole on the origin and no other unstable pole. There exist control systems that need to have a pair of poles on the imaginary axis; for example, the control systems follow the sinusoidal signal and the self-repairing control systems with faulty sensors [16]. To apply the concept of strong stabilization to self-repairing control systems with faulty sensors, we need to expand the concept of strong stabilization and define extended semi-strongly stabilizing controllers.

The definition of the extended semi-strongly stabilizing controller is summarized as follows.

**Definition 2.1.** *(Extended semi-strongly stabilizing controller)*
*The controller $C(s)$ is called an extended semi-strongly stabilizing controller if the following expressions hold:*
*1) $C(s)$ makes the control system in (1) internally stable;*
*2) $C(s)$ has a pair of complex conjugate poles on the imaginary axis and any other unstable poles.*
*That is, if $C(s)$ in (1) written as*

$$C(s) = \frac{Q_c(s)}{n_c(s)} \tag{2}$$

*stabilizes the control system in (1), we call $C(s)$ in (1) the extended semi-strongly stabilizing controller, where $n_c(s) \in RH_\infty$ is denoted by*

$$n_c(s) = \frac{s^2 + \omega^2}{n_{cd}(s)}, \tag{3}$$

*where $n_{cd}(s)$ is any Hurwitz polynomial of 2 degrees and $\omega \in R$, and $Q_c(s) \in RH_\infty$ satisfies*

$$Q_c(s)|_{s=\pm j\omega} \neq 0. \tag{4}$$

Note that any plants cannot be stabilized by extended semi-strongly stabilizing controllers in Definition 2.1. Therefore, we define plants stabilized by extended semi-strongly stabilizing controllers $C(s)$ in (2) per the following definition.

**Definition 2.2.** *(Extended semi-strongly stabilizable plant)*
*When the plant $G(s)$ in (1) can be stabilized by the extended semi-strongly stabilizing controller $C(s)$ in (2), the plant $G(s)$ is called an extended semi-strongly stabilizable plant.*

The problem considered in this paper is to clarify the parameterizations of all extended semi-strongly stabilizable plants and all extended semi-strongly stabilizing controllers for extended semi-strongly stabilizable plants.

## 3. **Extended Semi-Strongly Stabilizable Plants.**

In this section, we clarify the parameterization of all extended semi-strongly stabilizable plants.

The parameterization of all extended semi-strongly stabilizable plants $G(s)$ is summarized in the following theorem.

**Theorem 3.1.** *The plant $G(s)$ is an extended semi-strongly stabilizable plant if and only if the plant $G(s)$ takes the form of*

$$G(s) = \frac{n_c(s)Q_2(s) + n_b(s)}{\frac{1-n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)}, \tag{5}$$

*where* $n_b(s) \in RH_\infty$ *is an arbitrary function satisfying*

$$1 - n_b(s)Q_1(s)|_{s=\pm j\omega} = 0, \tag{6}$$

*and* $Q_1(s) \in RH_\infty$ *and* $Q_2(s) \in RH_\infty$ *are arbitrary functions satisfying*

$$Q_1(s)|_{s=\pm j\omega} \neq 0. \tag{7}$$

Proof of Theorem 3.1 requires the following lemma.

**Lemma 3.1.** [7] *Assume that* $A(s) \in RH_\infty^{m\times n}$, $B(s) \in RH_\infty^{q\times p}$, $C(s) \in RH_\infty^{m\times p}$, *and*

$$\operatorname{rank} \begin{bmatrix} A^T(s) & B^T(s) \end{bmatrix} = \gamma \tag{8}$$

*are satisfied. There exist* $X(s) \in RH_\infty$ *and* $Y(s) \in RH_\infty$ *satisfying*

$$X(s)A(s) + Y(s)B(s) = C(s) \tag{9}$$

*if and only if there exists* $U(s) \in \mathcal{U}$ *satisfying*

$$\begin{bmatrix} A(s) \\ B(s) \\ C(s) \end{bmatrix} = U(s) \begin{bmatrix} A(s) \\ B(s) \\ O \end{bmatrix}. \tag{10}$$

*When* $X_0(s) \in RH_\infty$ *and* $Y_0(s) \in RH_\infty$ *are solutions to (9), then all solutions to (9) are given by*

$$\begin{bmatrix} X(s) & Y(s) \end{bmatrix} = \begin{bmatrix} X_0(s) & Y_0(s) \end{bmatrix} + Q(s) \begin{bmatrix} W_1(s) & W_2(s) \end{bmatrix}, \tag{11}$$

*where* $W_1(s)$ *and* $W_2(s)$ *satisfy*

$$W_1(s)A(s) + W_2(s)B(s) = 0 \tag{12}$$

*and*

$$\operatorname{rank} \begin{bmatrix} W_1(s) & W_2(s) \end{bmatrix} = n + q - \gamma \tag{13}$$

*and* $Q(s) \in RH_\infty^{p\times(n+q-\gamma)}$ *is any function.*

Using Lemma 3.1, we show the proof of Theorem 3.1.

**Proof:** First, the necessity is shown. That is, we show that if the extended semi-strongly stabilizing controller $C(s)$ in (2) makes the control system in (1) stable, then the plant $G(s)$ takes the form of (5). Because the extended semi-strongly stabilizing controller $C(s)$ has the form of (2), the coprime factorization on $RH_\infty$ is given by the form

$$C(s) = \frac{Q_1(s)}{n_c(s)}, \tag{14}$$

where $Q_1(s) \in RH_\infty$ is an arbitrary function satisfying (7). From the assumption that $C(s)$ in (14) makes the control system (1) stable,

$$N(s)Q_1(s) + D(s)n_c(s) = 1 \tag{15}$$

holds true, where $N(s)$ and $D(s)$ are coprime factors of $G(s)$ on $RH_\infty$ satisfying

$$G(s) = \frac{N(s)}{D(s)}. \tag{16}$$

From Lemma 3.1, all solutions of $N(s)$ and $D(s)$ satisfying (15) are written as

$$N(s) = n_c(s)Q_2(s) + n_b(s) \tag{17}$$

and

$$D(s) = \frac{1 - n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s), \tag{18}$$

respectively, where $n_b(s) \in RH_\infty$ satisfies (6) and $Q_2(s) \in RH_\infty$ is any function. Substituting (17) and (18) for (16), we have (5). Thus, the necessity has been shown.

Next, sufficiency is shown. That is, if $G(s)$ in (1) takes the form of (5), then there exists a semi-strongly stabilizing controller to make the control system in (1) stable. A controller is set as

$$C(s) = \frac{Q_1(s)}{n_c(s)}. \tag{19}$$

From simple manipulation and (19), transfer functions $C(s)G(s)/(1 + C(s)G(s))$, $C(s)/(1 + C(s)G(s))$, $G(s)/(1 + C(s)G(s))$, and $1/(1 + C(s)G(s))$ are rewritten, respectively, as

$$\frac{G(s)C(s)}{1 + G(s)C(s)} = \left(n_c(s)Q_2(s) + n_b(s)\right) Q_1(s), \tag{20}$$

$$\frac{C(s)}{1 + G(s)C(s)} = \left(\frac{1 - n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right) Q_1(s), \tag{21}$$

$$\frac{G(s)}{1 + G(s)C(s)} = n_c(s)\left(n_c(s)Q_2(s) + n_b(s)\right), \tag{22}$$

and

$$\frac{1}{1 + G(s)C(s)} = 1 - Q_1(s)\left(n_c(s)Q_2(s) + n_b(s)\right). \tag{23}$$

Because $Q_1(s) \in RH_\infty$, $Q_2(s) \in RH_\infty$, $n_b(s) \in RH_\infty$, $n_c(s) \in RH_\infty$ and (6), (20), (21), (22), (23) are stable, we have shown sufficiency.

We have thus proved Theorem 3.1. □

Thus, the parametrization of all extended semi-strongly stabilizable plants has been clarified. Note that the extended semi-strongly stabilizable plant is represented by (5). It is enough to design extended semi-strongly stabilizing controllers for plant $G(s)$ in (5). Using this, in the next section, we clarify the parameterization of all extended semi-strongly stabilizing controllers.

## 4. Extended Semi-Strongly Stabilizing Controllers.

In this section, the parameterization of all extended semi-strongly stabilizing controllers $C(s)$ for the extended semi-strongly stabilizable plant $G(s)$ in (5) is proposed.

The parameterization of all extended semi-strongly stabilizing controllers $C(s)$ for the extended semi-strongly stabilizable plant $G(s)$ in (5) is summarized in the following theorem.

**Theorem 4.1.** *For the plant $G(s)$ represented by (5), the controller $C(s)$ is an extended semi-strongly stabilizing controller if and only if $C(s)$ is written as*

$$C(s) = \frac{Q_1(s) + \left(\frac{1 - n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s)\right) P(s)}{n_c(s) - \left(n_b(s) + n_c(s)Q_2(s)\right) P(s)}, \tag{24}$$

*where $P(s) \in RH_\infty$ and $Q(s) \in RH_\infty$ are functions written as*

$$P(s) = n_c(s)Q(s) \tag{25}$$

*and*

$$Q(s) = \frac{1 - \hat{Q}(s)}{n_b(s) + n_c(s)Q_2(s)}, \tag{26}$$

*respectively.* $\hat{Q}(s) \in \mathcal{U}$ *is a unimodular function that makes* $Q(s)$ *proper and satisfies*

$$\frac{1}{(s-s_i)^{m_i-1}}\left\{1-\hat{Q}(s)\right\}\bigg|_{s=s_i} = 0 \ \forall i, \tag{27}$$

*where* $s_i \in R$ *is unstable zero of* $n_b(s) + n_c(s)Q_2(s)$ *and* $m_i$ *is its multiplicity.*

**Proof:** According to [7], the parameterization of all stabilizing controllers for $G(s)$ is given by

$$C(s) = \frac{X(s) + D(s)P(s)}{Y(s) - N(s)P(s)}, \tag{28}$$

where $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are the solutions of

$$N(s)X(s) + D(s)Y(s) = 1 \tag{29}$$

and $P(s) \in RH_\infty$ is any function.

Because the extended semi-strongly stabilizable plant $G(s)$ is represented by (5), coprime factors of $N(s)$ and $D(s)$ satisfying (16) are given by

$$\begin{cases} N(s) = n_b(s) + n_c(s)Q_2(s) \\ D(s) = \dfrac{1 - n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s) \end{cases} \tag{30}$$

Therefore, a pair of solution of (29) is given by

$$\begin{cases} X(s) = Q_1(s) \\ Y(s) = n_c(s) \end{cases} \tag{31}$$

Substituting (30) and (31) into (28), we have (24).

The rest is to show that the necessary and sufficient condition for $C(s)$ in (24) to be a semi-stabilizing controller is that in (24), $P(s)$ and $Q(s)$ are given by (25) and (26), respectively.

We show that if $C(s)$ is an extended semi-strongly stabilizing controller, then $P(s)$ and $Q(s)$ are written as (25) and (26), respectively, and $\hat{Q}(s)$ is a unimodular function that makes $Q(s)$ proper. From the assumption that $C(s)$ in (24) is a semi-strongly stabilizing controller, we have

$$n_c(s) - \{n_b(s) + n_c(s)Q_2(s)\}\, P(s)|_{s=\pm j\omega} = 0. \tag{32}$$

Because $n_b(s)|_{s=\pm j\omega} \neq 0$ from (6),

$$|P(s)|_{s=\pm j\omega} = 0 \tag{33}$$

is satisfied. This means that $P(s)$ is given by (25). Substituting (25) for (24), we have

$$C(s) = \frac{1}{n_c(s)}\left\{Q_1(s) + \frac{Q(s)}{1 - \{n_b(s) + n_c(s)Q_2(s)\}\, Q(s)}\right\}. \tag{34}$$

From the assumption that $C(s)$ in (34) is the extended semi-strongly stabilizable controller

$$n_c(s)C(s) = Q_1(s) + \frac{Q(s)}{1 - (n_b(s) + n_c(s)Q_2(s))\, Q(s)} \in RH_\infty. \tag{35}$$

Because $Q_1(s) \in RH_\infty$ and $Q(s) \in RH_\infty$,

$$\hat{Q}(s) = 1 - (n_b(s) + n_c(s)Q_2(s))\, Q(s) \in \mathcal{U} \tag{36}$$

is satisfied. This equation is rewritten in (26). Therefore, $\hat{Q}(s) \in \mathcal{U}$ in (36) is any function to make $Q(s)$ in (26) proper. It is obvious that $Q(s)$ in (26) is equivalent to (27). Thus, the necessity has been proven.

Next, we show the sufficiency, that is, if $C(s)$ is written as (24), then $C(s)$ is an extended semi-strongly stabilizing controller. Substituting $P(s)$ in (25) into (24), we have

$$C(s) = \frac{1}{n_c(s)} \left\{ Q_1(s) + \frac{Q(s)}{\hat{Q}(s)} \right\}. \tag{37}$$

Because $Q_1(s) \in RH_\infty$, $Q(s) \in RH_\infty$, and $\hat{Q}(s) \in \mathcal{U}$, if $C(s)$ in (37) stabilizes the control system in (1), $C(s)$ is an extended semi-strongly stabilizing controller.

The rest is to show that $C(s)$ in (37) stabilizes the control system in (1). From simple manipulation, we have

$$\frac{G(s)C(s)}{1+G(s)C(s)} = (n_c(s)Q_2(s) + n_b(s)) \left( Q_1(s)\hat{Q}(s) + Q(s) \right), \tag{38}$$

$$\frac{C(s)}{1+G(s)C(s)} = \left( \frac{1 - n_b(s)Q_1(s)}{n_c(s)} - Q_1(s)Q_2(s) \right) \left( Q_1(s)\hat{Q}(s) + Q(s) \right), \tag{39}$$

$$\frac{G(s)}{1+G(s)C(s)} = (n_b(s) + n_c(s)Q_2(s))\, n_c(s)\hat{Q}(s) \tag{40}$$

and

$$\frac{1}{1+G(s)C(s)} = (1 - n_b(s)Q_1(s) - n_c(s)Q_1(s)Q_2(s))\, \hat{Q}(s). \tag{41}$$

Because $n_c(s) \in RH_\infty$, $n_b(s) \in RH_\infty$, $Q_1(s) \in RH_\infty$, $Q_2(s) \in RH_\infty$, $Q(s) \in RH_\infty$, $\hat{Q}(s) \in \mathcal{U}$ and (6), transfer functions in (38), (39), (40), and (41) are stable. Thus, the control system in (1) is stable. Therefore, sufficiency is shown.

Consequently, Theorem 4.1 has been proven. □

Thus, the parameterization of all extended semi-strongly stabilizing controllers has been clarified. The parameterization of all extended semi-strongly stabilizing controllers allows the designing of an extended semi-strongly stabilizing controller with the stability of the control system guaranteed by setting the free parameter. In the next section, we present a design procedure for extended semi-strongly stabilizing controllers.

5. **Design Procedure.** In this section, a design procedure for extended semi-strongly stabilizing controllers is presented. We extend the design procedure proposed by Hoshikawa et al., which cannot design an extended semi-strongly stabilizing controller, and present a design procedure for extended semi-strongly stabilizing controllers. From Theorem 4.1, to design an extended semi-strongly stabilizing controller $C(s)$, $\hat{Q}(s)$ in (26) need to be $\hat{Q}(s) \in \mathcal{U}$ satisfying (27) and to make $Q(s)$ in (26) proper.

A design method for an extended semi-strongly stabilizing controller $C(s)$ in (24) with poles on the imaginary axis is summarized as follows.

1) We factorize:

$$\tilde{Q}(s) = n_b(s) + n_c(s)Q_2(s) \in RH_\infty \tag{42}$$

as

$$n_b(s) + n_c(s)Q_2(s) = \tilde{Q}_i(s)\tilde{Q}_o(s), \tag{43}$$

where $\tilde{Q}_i(s) \in RH_\infty$ is an inner function satisfying $\tilde{Q}_i(0) = 1$ and $\tilde{Q}_o(s) \in RH_\infty$ is an outer function.

2) Using $\tilde{Q}_o(s)$, $\bar{Q}(s) \in RH_\infty$ is settled by

$$\bar{Q}(s) = \frac{q(s)}{\tilde{Q}_o(s)}, \tag{44}$$

where

$$q(s) = \frac{k}{(\rho s + 1)^m}, \tag{45}$$

where $\rho \in R$ is an arbitrary positive number, $m$ is an arbitrary positive integer to make $\bar{Q}(s)$ in (44) proper, and $k \in R$ is a real number satisfying $0 < k < 1$ and

$$k \simeq 1. \tag{46}$$

3) Using $\bar{Q}(s)$ in (44), $\hat{Q}(s) \in \mathcal{U}$ is set as

$$\hat{Q}(s) = 1 - (n_b(s) + n_c(s)Q_2(s))\,\bar{Q}(s). \tag{47}$$

4) Substituting (47) into (26), $Q(s)$ is obtained. An extended semi-strongly stabilizing controller with poles on the imaginary axis $C(s)$ is given by (24).

Next, using the presented procedure, we show that $\hat{Q}(s)$ in (47) satisfies $\hat{Q}(s) \in \mathcal{U}$ and (27) and makes $Q(s)$ in (26) proper. First, we show that $\hat{Q}(s)$ in (47) satisfies $\hat{Q}(s) \in \mathcal{U}$, (27) and makes $\bar{Q}(s)$ in (44) proper. Substituting (42), (44), and (45) for (47), $\hat{Q}(s)$ in (47) is written as

$$\hat{Q}(s) = 1 - \tilde{Q}_i(s)q(s). \tag{48}$$

Because $\tilde{Q}_i(s)$ is an inner function, $\tilde{Q}_i(s)$ is biproper. That is, $\tilde{Q}_i(s)q(s)$ is strictly proper. In addition, from (45) and $0 < k < 1$,

$$\left\| \tilde{Q}_i(s)q(s) \right\|_\infty < 1 \tag{49}$$

is satisfied. This implies that $\hat{Q}(s) \in \mathcal{U}$.

Next, we show that (27) holds true. $s_i \in R$ is unstable zero of $n_b(s) + n_c(s)Q_2(s)$ and $m_i$ is its multiplicity. Because $\tilde{Q}_i$ is an inner function of $n_b(s) + n_c(s)Q_2(s)$, (27) holds true, and

$$\left. \frac{1}{(s - s_i)^{m_i - 1}} \tilde{Q}_i(s) \right|_{s = s_i} = 0 \ \forall i \tag{50}$$

holds true. From this equation and (45),

$$\left. \frac{1}{(s - s_i)^{m_i - 1}} \tilde{Q}_i(s)q(s) \right|_{s = s_i} = 0 \ \forall i \tag{51}$$

is satisfied.

## 6. **Numerical Example.**

In this section, we show a numerical example to illustrate the effectiveness of the proposed method.

Consider the problem of designing an extended semi-strongly stabilizing controller that has a pole on $\pm j$ for the plant $G(s)$ [15] written as

$$G(s) = \frac{90900}{(s + 0.117)\left(s^2 + 3.97s + 2020\right)}. \tag{52}$$

Because $G(s)$ in (52) is rewritten as (5), where

$$n_c(s) = \frac{s^2 + 1}{1.01s^2 + 2s + 1.01}, \tag{53}$$

$$n_b(s) = \frac{s^2 + 200s + 1}{1.01s^2 + 2s + 1.01}, \tag{54}$$

$$Q_1(s) = 0.01 \tag{55}$$

and

$$Q_2(s) = \frac{-(s + 197.7)(s - 0.7893)(s + 0.3697)\left(s^2 + 6.766s + 1587\right)}{(s + 0.3142)\left(s^2 + 0.2564s + 1.821\right)\left(s^2 + 3.516s + 2018\right)}, \tag{56}$$

$G(s)$ in (52) is an extended semi-strongly stabilizable plant.

Using the design procedure described in Section 5, we design an extended semi-strongly stabilizing controller $C(s)$. $\tilde{Q}(s)$ in (42) is factorized as (43), where

$$\tilde{Q}_i(s) = 1 \tag{57}$$

and

$$\tilde{Q}_o(s) = \tilde{Q}(s). \tag{58}$$

Using (58), $\bar{Q}(s)$ is settled by (44), where

$$q(s) = \frac{0.99}{(s + 1)^3}. \tag{59}$$

$\hat{Q}(s)$ is set by (47) and written as

$$\hat{Q}(s) = \frac{s^3 + 3s^2 + 3s + 0.01}{(s + 1)^3}. \tag{60}$$

Using the presented parameters, an extended semi-strongly stabilizing controller $C(s)$ is obtained as

$$C(s) = \frac{0.01011(s + 0.2333)\left(s^2 - 0.613s + 0.691\right)\left(s^2 + 4.133s + 7.775\right)}{\left(s^2 + 1\right)(s + 1.997)\left(s^2 + 1.003s + 0.9967\right)}. \tag{61}$$

It is obvious that $C(s)$ in (61) has a pair of complex conjugate poles at $\pm j\omega$, which are located on the imaginary axis, and other poles in the open left half plane. Therefore, if $C(s)$ in (61) stabilizes the control system in (1), $C(s)$ in (61) is an extended semi-strongly stabilizing controller for $G(s)$ in (52).

Using this extended semi-strongly stabilizing controller $C(s)$ in (61), the response of the output $y(t)$ for the sinusoidal disturbance

$$d(t) = \sin t \tag{62}$$

is shown in Figure 1. Figure 1 shows that the control system in (1) is stable. Therefore, $C(s)$ in (61) is an extended semi-strongly stabilizing controller for $G(s)$ in (52). As shown, we can easily design extended semi-strongly stabilizing controllers using the parameterization of all extended semi-strongly stabilizing controllers. In addition, we find that the sinusoidal disturbance $d(t) = \sin t$ is attenuated effectively because the extended semi-strongly stabilizing controller has poles at $\pm j$ on the imaginary axis.

Next, we show the response for the reference input. The response of the error $e(t) = r(t) - y(t)$ for the reference input

$$r(t) = \sin t \tag{63}$$

is shown in Figure 2. Figure 2 shows that the output $y(t)$ follows the reference input $r(t)$ without steady-state error because the extended semi-strongly stabilizing controller has poles at $\pm j$ on the imaginary axis.

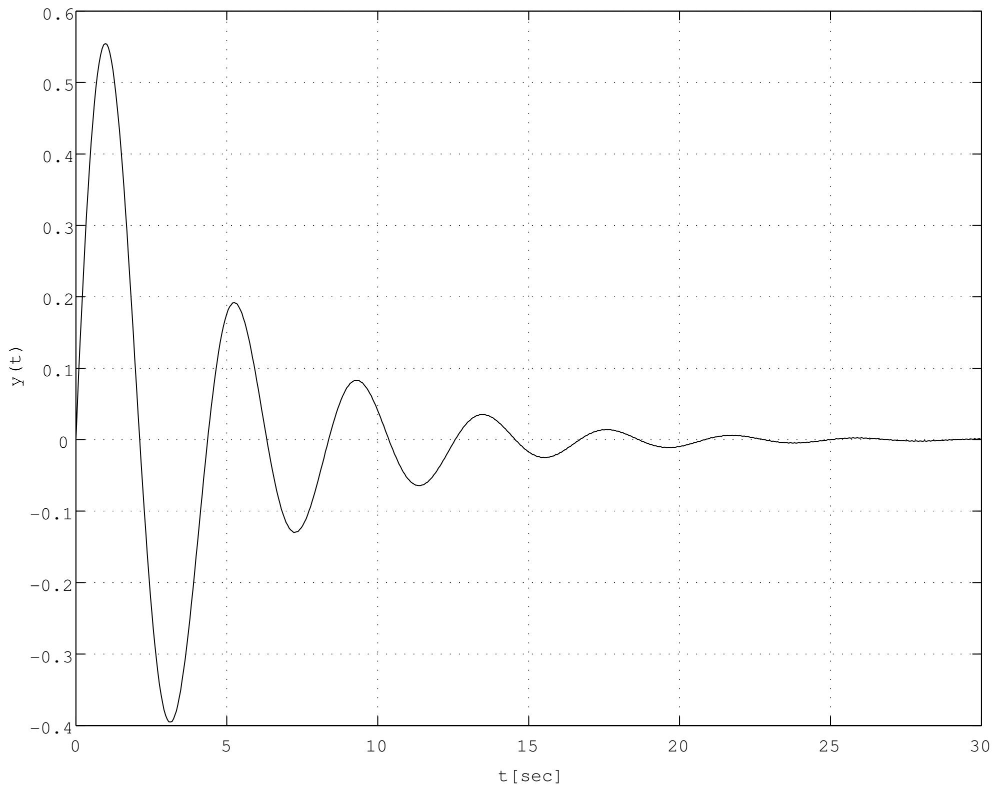


FIGURE 1. Response of the output $y(t)$ for the disturbance $d(t)$ in (62)

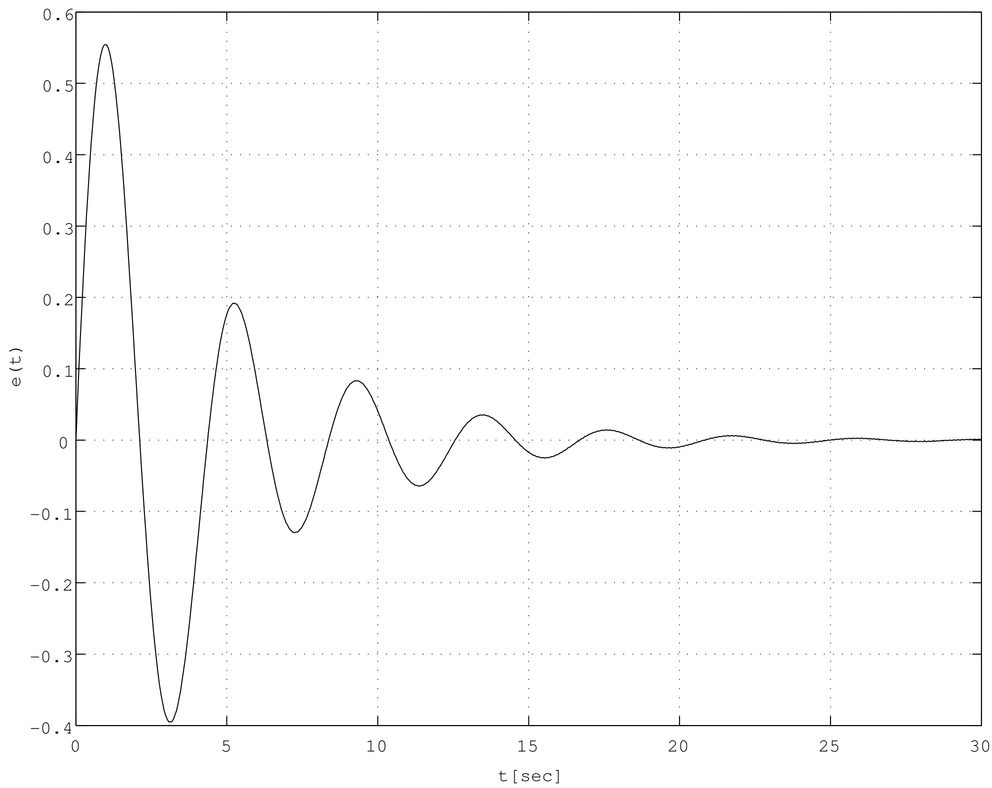


FIGURE 2. Response of the error $e(t)$ for the reference input $r(t)$ in (63)

Next, we compare the extended semi-strongly stabilizing controller proposed in this paper with the semi-strongly stabilizing controller proposed by Hoshikawa et al. [15]. In [15], for the plant $G(s)$ in (52), the semi-strongly stabilizing controller is designed by

$$C(s) = \frac{1.36\left(s^2 + 0.241s + 0.118\right)\left(s^2 + 4.12s + 2.0310^3\right)}{s(s + 0.167)\left(s^2 + 1.5010^2 s + 7.4810^3\right)^3}. \tag{64}$$

When $r(t)$ is given in (63), for the reference input $r(t)$ in (63), responses of the error $e(t) = r(t) - y(t)$ of the extended semi-strongly stabilizing controller and the semi-strongly stabilizing controller are shown in Figure 3. In Figure 3, the solid line shows the response of the extended semi-strongly stabilizing controller, and the dashed line shows that of the semi-strongly stabilizing controller. The semi-strongly stabilized controller could not attenuate sinusoidal signals because it did not have poles on the imaginary axis. However, the extended semi-strongly stabilizing controller can attenuate sinusoidal signals because it has poles at $\pm j$ on the imaginary axis.

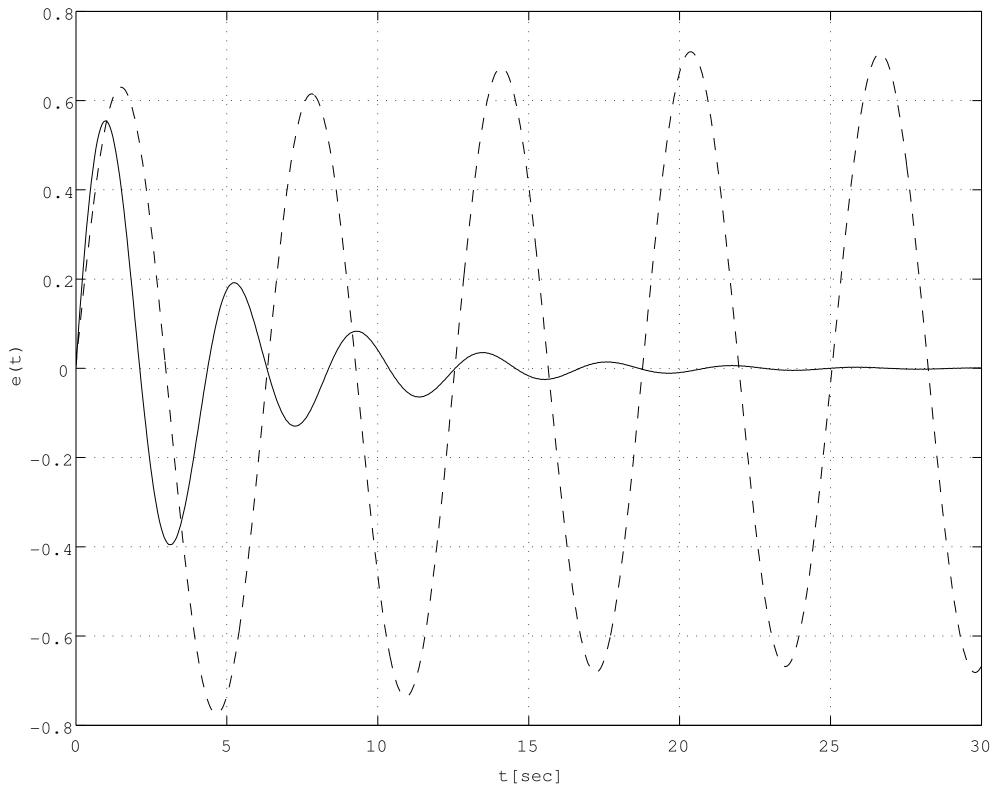


FIGURE 3. Response of the error $e(t)$ for the reference input $r(t)$ in (63) of the extended semi-strongly stabilizing controller and the semi-strongly stabilizing controller

Next, the robustness of the extended semi-strongly stabilizing controller is shown. When $\omega = 1$ in (3), which is used for controller design, is different from that of the reference input $r(t)$ as

$$r(t) = \sin\omega(t) \quad \omega = 1, 1.1, 1.2, \tag{65}$$

the response of the error $e(t) = r(t) - y(t)$ is shown in Figure 4. In Figure 4, the solid line shows the response for $\omega = 1$, the dashed line shows that for $\omega = 1.1$, and the dotted line shows that for $\omega = 1.2$. The extended semi-strongly stabilizing controller designed in (61) has poles at $\pm j$ on the imaginary axis and thus attenuates the sinusoidal wave when

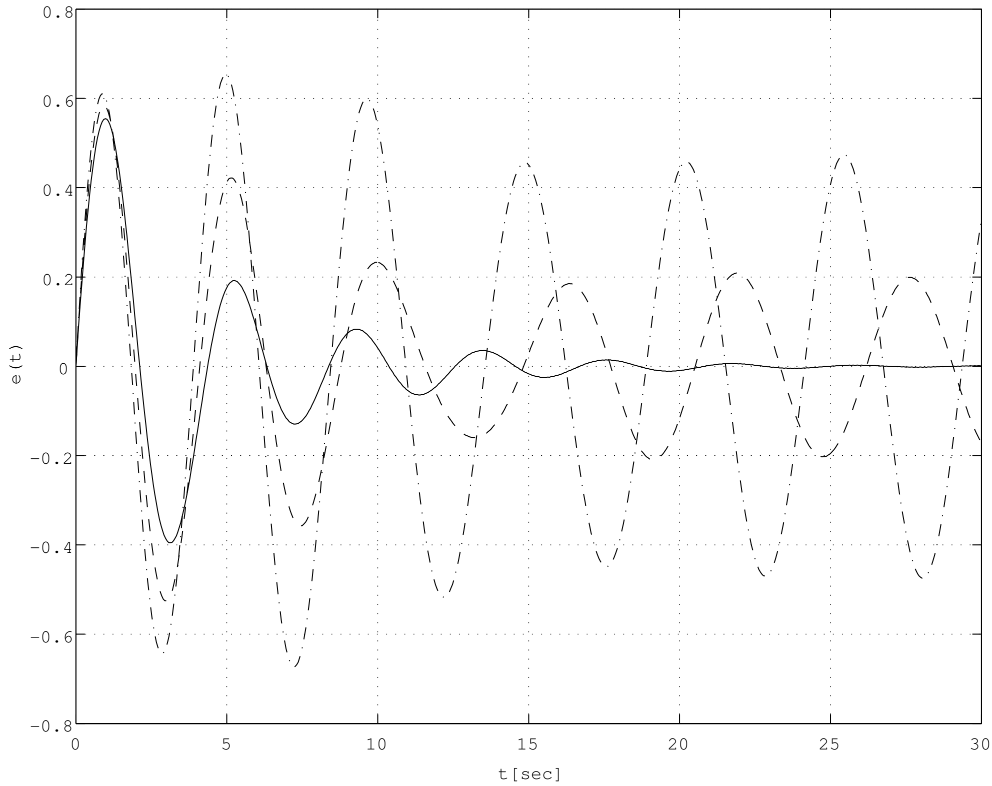


FIGURE 4. Response of the error $e(t)$ for $r(t) = \sin \omega t$, $\omega = 1, 1.1, 1.2$

$\omega = 1$. However, it can be seen that as the frequency $\omega$ increases from 1, it is no longer able to follow the sinusoidal wave.

Next, we show responses when uncertainty exists in the plant. Consider the problem of designing an extended semi-strongly stabilizing controller that has a pole on $\pm j$ for the plant $G(s)$ written as

$$G(s) = \frac{-1.5\left(s^2 + 1.667s + 2.667\right)}{(s+3)(s+2)}. \tag{66}$$

Because $G(s)$ in (66) is rewritten as (5), where

$$n_c(s) = \frac{s^2+1}{(s+3)(s+2)}, \tag{67}$$

$$n_b(s) = 1, \tag{68}$$

$$Q_1(s) = 1 \tag{69}$$

and

$$Q_2(s) = 2, \tag{70}$$

$G(s)$ in (66) is an extended semi-strongly stabilizable plant.

Using the design procedure described in Section 5, we design an extended semi-strongly stabilizing controller $C(s)$. $\tilde{Q}(s)$ in (42) is factorized as (43), where

$$\tilde{Q}_i(s) = 1 \tag{71}$$

and

$$\tilde{Q}_o(s) = \tilde{Q}(s). \tag{72}$$

Using (72), $\bar{Q}(s)$ is settled by (44), where

$$q(s) = \frac{0.99}{(0.2s+1)^3}. \tag{73}$$

$\hat{Q}(s)$ is set by (47) and written as

$$\hat{Q}(s) = \frac{(s+0.01672)\left(s^2+14.98s+74.75\right)}{(s+5)^3}. \tag{74}$$

Using the presented parameters, an extended semi-strongly stabilizing controller $C(s)$ is obtained as

$$C(s) = \frac{(s+3)(s+2)(s+0.8453)\left(s^2-0.9871s+4.213\right)\left(s^2+14.83s+70.44\right)}{\left(s^2+1\right)(s+0.01673)\left(s^2+1.667s+2.667\right)\left(s^2+14.98s+74.75\right)}. \tag{75}$$

If the uncertainty exists in the plant $G(s)$, the real plant $G_{\text{true}}(s)$ is written as

$$G_{\text{true}}(s) = (1+\Delta(s))G(s) \tag{76}$$

where $\Delta(s)$ is an uncertainty. The response of the error $e(t)$ for the reference input $r(t)$ in (63) when no uncertainty exists in real plant $G_{\text{true}}(s)$ and the uncertainty

$$\Delta(s) = \frac{1+3s}{1+2s} \tag{77}$$

exists in real plant $G_{\text{true}}(s)$ is shown in Figure 5. In Figure 5, the solid line shows the response when no uncertainty exists and the dashed line shows the response when the uncertainty $\Delta(s)$ in (77) exists. The extended semi-strongly stabilizing controller has poles

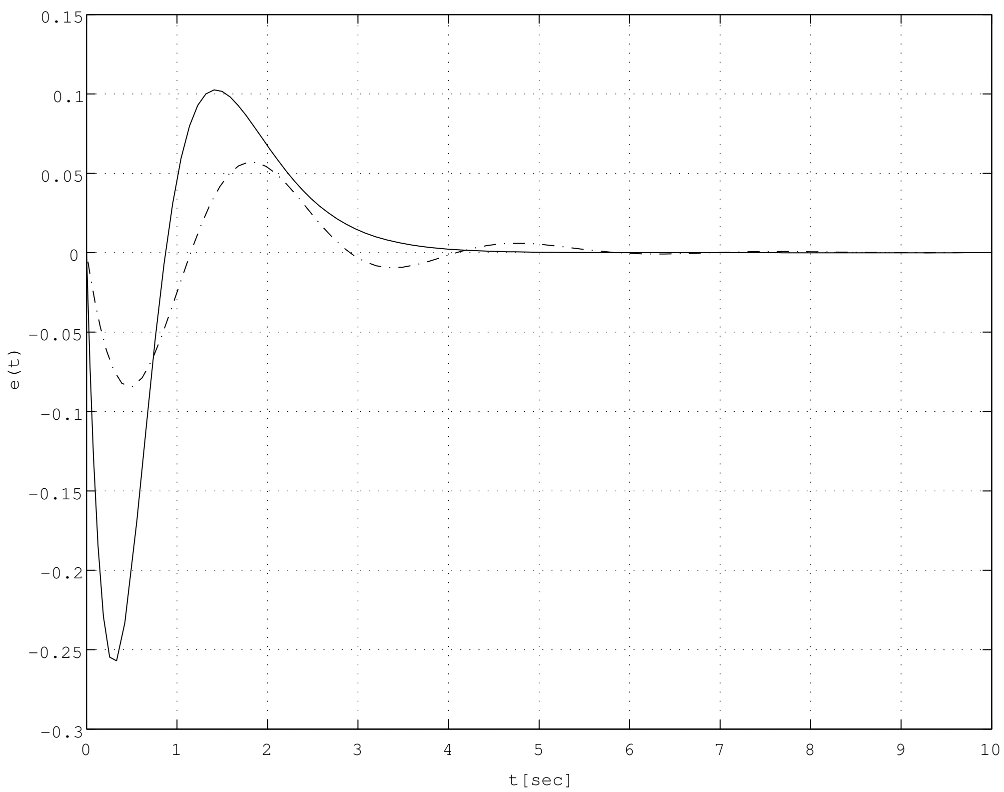


FIGURE 5. Response of disturbance $e(t)$ for $r(t)$ in (63) when no uncertainty exists and the uncertainty $\Delta(s)$ in (77) exists in real plant $G_{\text{true}}(s)$

that are stable except for the pole on the imaginary axis, so it is resilient to parameter changes. Therefore, as can be seen in Figure 5, it maintains stability even when the plant parameter changes.

7. **Conclusions.** In this paper, we expand the concept of the semi-strongly stabilizing controller proposed in [15] and define the extended semi-strongly stabilizing controllers. Next, parameterizations of all extended semi-strongly stabilizable plants and all extended semi-strongly stabilizing controllers for the extended semi-strongly stabilizable plants are proposed. A design procedure of the extended semi-strongly stabilizing controllers is presented. A numerical example is illustrated to show the effectiveness of the proposed method. The failure detection method in feedback control systems with poles on the imaginary axis will be discussed in another article. In addition, there is a possibility that the proposed semi-strongly stabilizing controller can be applied to a pressure tank control system, which is controlled by the recurrent cerebellar model articulation control system [19]. This problem is one of our future studies.

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