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A NUMERICAL SCHEME FOR DEALING WITH FRACTIONAL INITIAL VALUE PROBLEM

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Abstract. *This paper aims to present a new numerical modification for solving fractional initial value problem. The proposed modification, which is named Modified Fractional Euler Method (MFEM), attempts to further improve the Fractional Euler Method (FEM) in terms of attaining more accuracy. The error bound generated by the proposed method is analysed and estimated by demonstrating a specific theoretical result. In order to validate the efficiency of the proposed method, several numerical comparisons are performed using MATLAB routines.*
Keywords: Fractional differential equations, Fractional Euler method, Caputo fractional derivative operator

1. Introduction.

In the past few years, the applications of fractional differential equations have been significantly implemented in widely different practical and engineering fields such as viscosity, signal processing, control and process modeling [1, 2, 3, 4, 5]. It is noteworthy that the analytical solutions of the nonlinear fractional differential equations are difficult to be obtained, and for this reason, resorting to approximate and numerical techniques has become a must [6, 7, 8, 9, 10]. In this regard, various numerical methods have been recently established and used to address this gap. Most of these methods have proved their accuracy in obtaining accurate approximate solutions when dealing with a lot of linear and nonlinear problems.

In [11], the authors proposed a numerical generalization to the classical Euler method called the Fractional Euler Method (FEM). This method played an active role in handling fractional initial value problems. The research context presented in this paper tends to develop upon the Modified Fractional Euler Method (MFEM) instead of the FEM for

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finding approximate numerical solutions for linear and nonlinear fractional differential equations. The proposed algorithm is characterized by the fact that it can provide more accuracy and efficiency than that of the FEM. This will be confirmed by performing several numerical comparisons via several illustrative examples. The error bound generated by the proposed method will be moreover analysed and estimated by demonstrating a specific theoretical result.

This paper is organized as follows. In Section 2, we present the basic definitions and theories that underpin the fundamentals of our research. In Section 3, we establish the MFEM based on FEM to solve the fractional initial value problem. In Section 4, we analyse and estimate the error bound generated by the MFEM by introducing some theoretical result. In Section 5, different numerical examples are presented showing the accuracy of the proposed method. Finally in Section 6, a quick conclusion of the research is summarized for completeness.

2. **Preliminaries.** This work aims to propose a numerical algorithm for introducing an approximate solution to the fractional initial value problem formulated in the sense of Caputo fractional differentiator. Such a problem has the form:

$$D_*^\alpha y(t) = f(t, y(t)), \tag{1}$$

with initial condition:

$$y(0) = y_0, \tag{2}$$

where $0 < \alpha \leq 1$. For this purpose, we recall next the most important definitions and concepts that will be useful for use throughout the paper.

Definition 2.1. [12] *The Riemann-Liouville integral operator of order α is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \tag{3}$$

where $x > 0$ and $0 < \alpha \leq 1$.

The Riemann-Liouville integrator satisfies the following properties [12]:

- $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$, $\alpha, \beta > 0$.
- $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$, $\alpha, \beta > 0$.
- $J^\alpha x^\varphi = \dfrac{\Gamma(\alpha+\varphi)}{\Gamma(\alpha+\varphi+1)} x^{\varphi+\alpha}$, $\varphi > -1$.

Definition 2.2. [12] *Suppose $m-1 < \alpha \leq m$ such that $m \in \mathbb{N}$ and $f \in C^m[0,b]$. The Caputo fractional differentiator is defined by*

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{(m-\alpha-1)} f^{(m)}(t) dt. \tag{4}$$

Lemma 2.1. [12] *If $f \in C^m[0,b]$, $x > 0$ and $m-1 < \alpha \leq m$ such that $m \in \mathbb{N}$, then we have*

$$D_*^\alpha J^\alpha f(x) = f(x), \tag{5}$$

$$J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=1}^{m-1} f^k(0^+) \frac{x^k}{k!}. \tag{6}$$

Definition 2.3. [12] *The Mittag-Leffler function of two parameters α and β is outlined by the following series:*

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta > 0$ *and* $t \in \mathbb{C}$.

Theorem 2.1. [Generalized Taylor's formula] [11] *Suppose that* $D_*^{k\alpha} f(x) \in C(0, b]$ *for* $k = 0, 1, 2, \ldots, n+1$, *where* $0 < \alpha \leq 1$. *Then we can expand the function* f *about the node* x_0 *as follows:*

$$f(x) = \sum_{i=0}^{n} \frac{(x - x_0)^{i\alpha}}{\Gamma(i\alpha + 1)} \left(D_*^{i\alpha} f\right)(x_0) + \frac{(x - x_0)^{(n+1)\alpha}}{\Gamma((n+1)\alpha + 1)} \left(D_*^{(n+1)\alpha} f\right)(\xi), \tag{7}$$

with $0 < \xi < x$, $\forall x \in (0, b]$.

Actually, for more illustration, we can express the above expression of the function f as follows:

$$\begin{aligned} f(x) = f(x_0) + (D_*^{\alpha} f)(x_0) \frac{(x - x_0)^{\alpha}}{\Gamma(\alpha + 1)} + \left(D_*^{2\alpha} f\right)(x_0) \frac{(x - x_0)^{2\alpha}}{\Gamma(2\alpha + 1)} + \cdots \\ + \frac{(D_*^{n\alpha} f)(x_0)}{\Gamma(n\alpha + 1)} (x - x_0)^{n\alpha} + \frac{\left(D_*^{(n+1)\alpha} f\right)(\xi)}{\Gamma((n+1)\alpha + 1)} (x - x_0)^{(n+1)\alpha}. \end{aligned} \tag{8}$$

3. **Modified Fractional Euler Method (MFEM).** With the aim of handling the following initial value problem:

$$\frac{dy}{dx} = f(x, y), \tag{9}$$

subject to initial condition:

$$y(x_0) = y_0, \tag{10}$$

a small modification to the classical Euler Method (EM) was proposed in [13]. It was reported in [13, 14] that this method has confirmed its reliability and validity in dealing with such above classical initial value problem. In particular, it was declared in [13] that the numerical method that could be used to solve problem (9)-(10) applies on the following formula:

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right), \ n = 0, 1, 2, \ldots. \tag{11}$$

From this point of view and based on Formula (11), we intend in this section to propose the MFEM that would help one to solve the fractional initial value problem (1)-(2). The MFEM represents an improvement of the well-known FEM that typically applied to dealing with problem (1)-(2). For instance, to deal with the fractional initial value problem (1)-(2), we first suppose that $0 = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h < \cdots < t_n = t_0 + nh = b$ in which t_i are called the mesh points, and h is called the step size such that $h = \frac{b-a}{n}$, for $i = 1, 2, \ldots, n$. Now, by using the first three terms of the generalized Taylor theorem given in Theorem 2.1, we can expand $y(t)$ about $t = t_i$ as follows:

$$y(t) = y(t_i) + \frac{(D_*^{\alpha} y)(t_i)}{\Gamma(\alpha + 1)} (t - t_i)^{\alpha} + \frac{\left(D_*^{2\alpha} y\right)(\xi)}{\Gamma(2\alpha + 1)} (t - t_i)^{2\alpha},$$

where $\xi \in (0, t)$. If one substitutes t_{i+1} instead of t in the above equality, we get

$$y(t_{i+1}) = y(t_i) + \frac{(D_*^{\alpha} y)(t_i)}{\Gamma(\alpha + 1)} (t_{i+1} - t_i)^{\alpha} + \frac{\left(D_*^{2\alpha} y\right)(\xi)}{\Gamma(2\alpha + 1)} (t_{i+1} - t_i)^{2\alpha},$$

which immediately implies

$$y(t_{i+1}) = y(t_i) + \frac{h^\alpha}{\Gamma(\alpha+1)}(D_*^\alpha y)(t_i) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}\left(D_*^{2\alpha} y\right)(\xi).$$

Now, based on Formula (11), we can propose the following formula:

$$\begin{aligned} y(t_{i+1}) = y(t_i) &+ \frac{h^\alpha}{\Gamma(\alpha+1)} f\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, y_i + \frac{h^\alpha}{2\Gamma(\alpha+1)} f(t_i, y_i)\right) \\ &+ \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}\left(D_*^{2\alpha} y\right)(\xi). \end{aligned} \tag{12}$$

From now on, we may use $y(t_i)$ to denote the exact solution of problem (1)-(2) at t_i, and w_i to denote the numerical solution of the same problem at t_i such that

$$\begin{aligned} w_0 &= y_0 \\ w_{i+1} &= w_i + \frac{h^\alpha}{\Gamma(\alpha+1)} f\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, w_i + \frac{h^\alpha}{2\Gamma(\alpha+1)} f(t_i, w_i)\right), \end{aligned} \tag{13}$$

for $i = 1, 2, \ldots, n-1$.

4. **The Error Bound of MFEM.** Through this part, we aim to estimate the error bound of our proposed scheme established in Formula (13). This would be carried out by using the next lemma.

Lemma 4.1. [15] *Suppose that δ and γ are two positive real numbers and $(a_i)_{i=0}^k$ is a sequence satisfying $a_0 \geq \frac{-\gamma}{\delta}$ and $a_{i+1} \leq (1+\delta)a_i + \gamma$ for each $i = 0, 1, 2, \ldots, k$. Then we have*

$$a_{i+1} \leq e^{(i+1)\delta}\left(a_0 + \frac{\gamma}{\delta}\right) - \frac{\gamma}{\delta}.$$

In what follows, we introduce a theoretical result that concerns with an estimation of the upper bound of the error generated by the proposed scheme MFEM.

Theorem 4.1. *Suppose that f is a continuous real-valued function satisfying Lipschetz condition with constant L on $D = [a, b] \times \mathbb{R}$, i.e.,*

$$|f(t, e_1) - f(t, e_2)| \leq L|e_1 - e_2|.$$

Suppose that a constant M exists with

$$|D_*^{n\alpha} y(t)| \leq M, \ \forall t \in [a, b].$$

Then, we have

$$|y(t_i) - w_i| \leq \frac{\gamma}{\delta}\left(e^{(i\delta)} - 1\right), \ \forall i = 0, 1, \ldots, n,$$

where $\gamma = \frac{h^{2\alpha} M}{\Gamma(2\alpha+1)}$, and $\delta = \left(\frac{2\Gamma(\alpha+1)h^\alpha + h^{2\alpha} L}{2(\Gamma(\alpha+1))^2}\right)$.

Proof: In order to prove this result, we first subtract (13) from (12) to get

$$\begin{aligned} y(t_{i+1}) - w_{i+1} = y(t_i) - w_i &+ \frac{h^\alpha}{\Gamma(\alpha+1)}\left\{f\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, y_i + \frac{h^\alpha}{2\Gamma(\alpha+1)} f(t_i, y_i)\right)\right. \\ &\left. - f\left(t_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}, w_i + \frac{h^\alpha}{2\Gamma(\alpha+1)}\right)\right\} + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}\left(D_*^{2\alpha} y\right)(\xi). \end{aligned}$$

This consequently gives

$$
\begin{aligned}
&|y(t_{i+1}) - w_{i+1}| \\
&\leq |y(t_i) - w_i| + \frac{h^\alpha}{\Gamma(\alpha+1)}\left|y(t_i) + \frac{h^\alpha}{2\Gamma(\alpha+1)}f(t_i, y_i) - w_i - \frac{h^{2\alpha}}{2\Gamma(\alpha+1)}f(t_i, w_i)\right| \\
&\quad + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}\left|\left(D_*^{2\alpha}y\right)(\xi)\right|,
\end{aligned}
$$

which implies

$$
\begin{aligned}
|y(t_{i+1}) - w_{i+1}| \leq\ & |y(t_i) - w_i| + \frac{h^\alpha}{\Gamma(\alpha+1)}|y(t_i) - w_i| \\
&+ \frac{h^\alpha}{\Gamma(\alpha+1)}\frac{h^\alpha}{2\Gamma(\alpha+1)}|f(t_i, y_i) - f(t_i, w_i)| + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}M.
\end{aligned}
$$

Hence, we have

$$
|y(t_{i+1}) - w_{i+1}| \leq |y(t_i) - w_i| + \frac{h^\alpha}{\Gamma(\alpha+1)}|y(t_i) - w_i| + \frac{h^{2\alpha}}{2(\Gamma(\alpha+1))^2}L|y_i - w_i| + \frac{h^{2\alpha}M}{\Gamma(2\alpha+1)},
$$

i.e.,

$$
|y(t_{i+1}) - w_{i+1}| \leq \left(1 + \frac{2\Gamma(\alpha+1)h^\alpha + h^{2\alpha}L}{2(\Gamma(\alpha+1))^2}\right)|y(t_i) - w_i| + \frac{h^{2\alpha}M}{\Gamma(2\alpha+1)}.
$$

Now, by letting $\delta = \left(\frac{2\Gamma(\alpha+1)h^\alpha + h^{2\alpha}L}{2(\Gamma(\alpha+1))^2}\right)$, $\gamma = \frac{h^{2\alpha}M}{\Gamma(2\alpha+1)}$ and $a_i = |y(t_i) - w_i|$, we obtain

$$
a_{i+1} \leq (1+\delta)a_i + \gamma, \text{ for } i = 0, 1, \ldots, k.
$$

Thus, by Lemma 4.1, we can have

$$
\begin{aligned}
|y(t_{i+1}) - w_{i+1}| &\leq e^{(i+1)\delta}\left(|y_0 - w_0| + \frac{\gamma}{\delta}\right) - \frac{\gamma}{\delta} \\
&= e^{(i+1)\delta}\frac{\gamma}{\delta} - \frac{\gamma}{\delta} \qquad (14)\\
&= \frac{\gamma}{\delta}\left(e^{(i+1)\delta} - 1\right),
\end{aligned}
$$

which implies directly the desired result.

5. **Numerical Applications.** In this part, we provide three numerical examples that would demonstrate the simplicity of implementing the proposed scheme. In particular, the solutions produced by using the MFEM are shown for several values of α and h. In other words, the solutions' behaviors generated by the presented method appear to depend mainly on the values of α, and the accuracy of the approximation appears to be related to step size of h.

Example 5.1. *Consider the following linear fractional initial value problem* [11]*:*

$$
D_*^\alpha y(t) = -y(t), \ y(0) = 1, \ t > 0, \qquad (15)
$$

where $0 < \alpha \leq 1$*. Note that the exact solution of the above problem is* $y(t) = E_{\alpha,1}(-t^\alpha)$*. However, to deal with such problem, we apply on Formula (13). This would produce Figure 1 that includes a numerical comparison between the numerical solution of problem (15) gained by using MFEM and FEM by considering* $\alpha = 0.9$ *and* $h = 0.1$*.*

For more illustration, we plot below Figure 2 and produce Table 1 that show a strong contrast in absolute error values for the numerical solutions between the two schemes in favor of our proposed schema.

Based on the previous numerical simulations, it can be noticed that the MFEM's solution is closer to the exact solution than that of the FEM's solution.

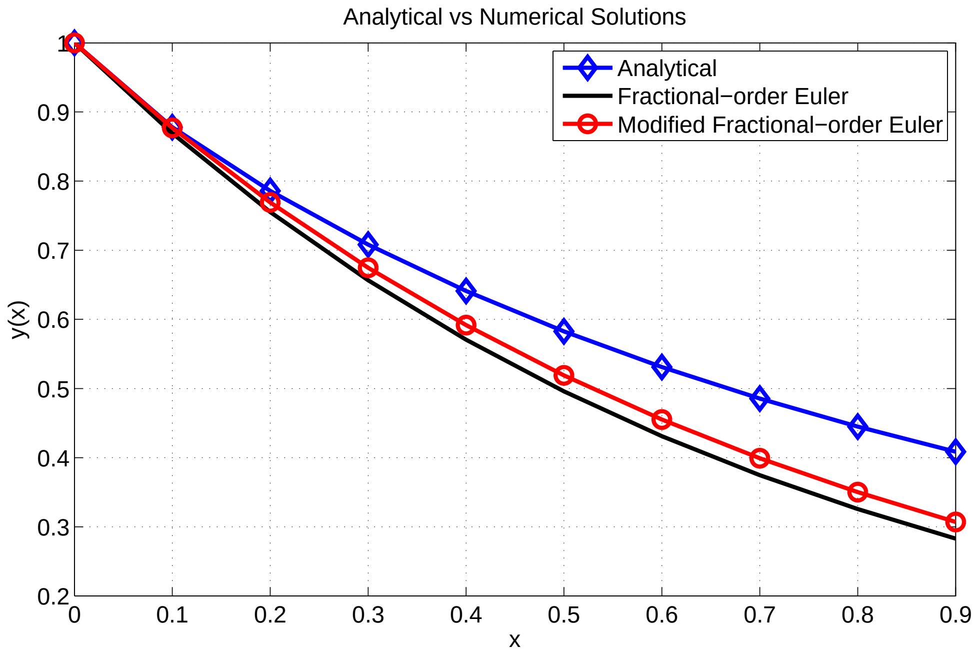

FIGURE 1. Numerical solution of problem (15) using MFEM and FEM when $h = 0.1$ and $\alpha = 0.9$

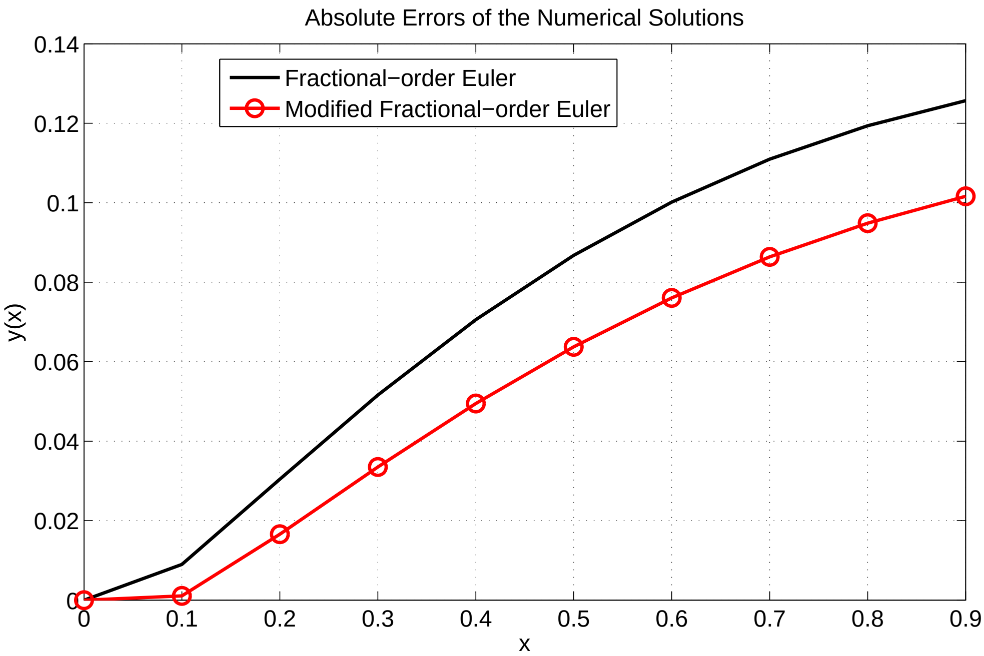

FIGURE 2. Absolute error between the exact and numerical solution of problem (15)

Example 5.2. *Consider the following nonlinear fractional initial value problem* [11]*:*

$$D_*^\alpha y(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - y(t) + t^2 - t, \ y(0) = 0, \ t > 0, \quad (16)$$

where $0 < \alpha \leq 1$. *The exact solution when* $\alpha = 1$ *of the above problem is* $y(t) = t^2 - t$. *Herein, we apply on Formula (13) to solve problem (16), and hence generate Figure 3.*

TABLE 1. Absolute error between the exact and numerical solution of problem (15)

t	Fractional Euler Method	Modified Fractional Euler Method
0	0.000000	0.000000
0.1	0.008993	0.001069
0.2	0.030423	0.016586
0.3	0.051607	0.033486
0.4	0.070561	0.049466
0.5	0.086758	0.063735
0.6	0.100164	0.076044
0.7	0.110951	0.086382
0.8	0.119368	0.094852
0.9	0.125691	0.101610

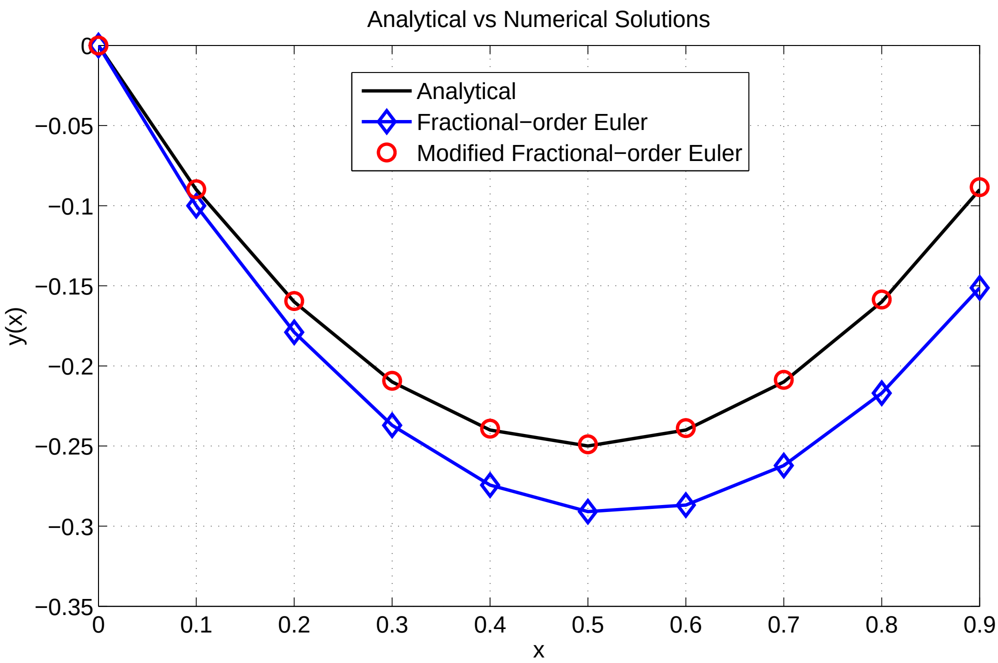

FIGURE 3. Numerical solution of problem (16) using MFEM and FEM when $h = 0.1$ and $\alpha = 1$

Such a figure illustrates the numerical solution of problem (16) gained by using MFEM and FEM by considering $\alpha = 1$ and $h = 0.1$.

In order to take a look at the absolute error values for the numerical solutions generated by the MFEM and FEM, we plot Figure 4 and produce Table 2.

In view of the previous numerical results, one can clearly observe that the accuracy of the proposed method is better than that of the FEM.

Example 5.3. *Consider the following nonlinear fractional initial value problem* [16]*:*

$$D_*^\alpha y(t) + 2(y(t))^2 = \Gamma(\alpha + 2)t + 2\left(t^{\alpha+1}\right)^2, \ y(0) = 0, \ t > 0, \tag{17}$$

where $0 < \alpha \le 1$. The exact solution of the above problem is given by $y(t) = t^{\alpha+1}$. In a similar manner to the previous two examples, we apply also Formula (13) to solving this

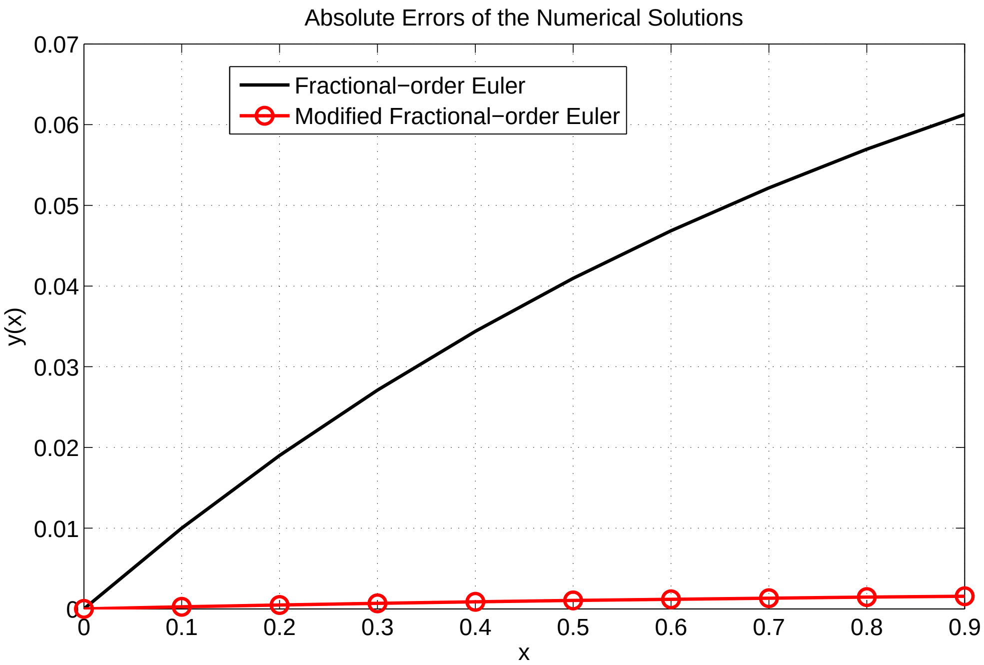

FIGURE 4. Absolute error between the exact and numerical solution of problem (16)

TABLE 2. Absolute error between the exact and numerical solution of problem (16)

t	Fractional Euler Method	Modified Fractional Euler Method
0	0.000000	0.000000
0.1	0.010000	0.000250
0.2	0.019000	0.000476
0.3	0.027100	0.000681
0.4	0.034390	0.000866
0.5	0.040951	0.001034
0.6	0.046856	0.001186
0.7	0.052170	0.001323
0.8	0.056953	0.001447
0.9	0.061258	0.001560

problem. As a result, Figure 5 is then generated that illustrates the numerical solutions of such a problem using MFEM and FEM with $\alpha = 0.5$ *and* $h = 0.1$.

In the same regard, we plot Figure 6 and produce Table 3 for the purpose of highlighting the absolute error values for the numerical solutions generated by the MFEM and FEM.

Clearly, one can note that the numerical solution generated by the presented method is closer to the exact solution than that of the numerical solution generated by the FEM. Thus, we conclude the importance of the presented method in gaining a reasonable accuracy when dealing with fractional initial value problems.

6. **Conclusion.** The intended objective of this work has been to create a numerical scheme for finding numerical solutions to linear and nonlinear fractional initial value problems based on the Fractional Euler Method (FEM). This scheme, which is named

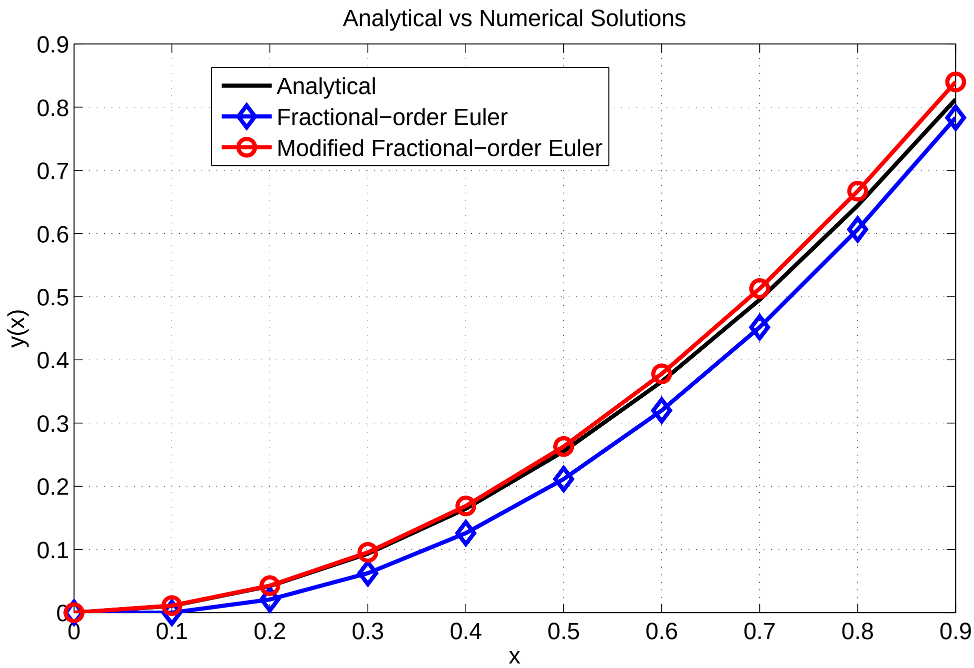

FIGURE 5. Numerical solution of problem (17) using MFEM and FEM when $h = 0.1$ and $\alpha = 0.5$

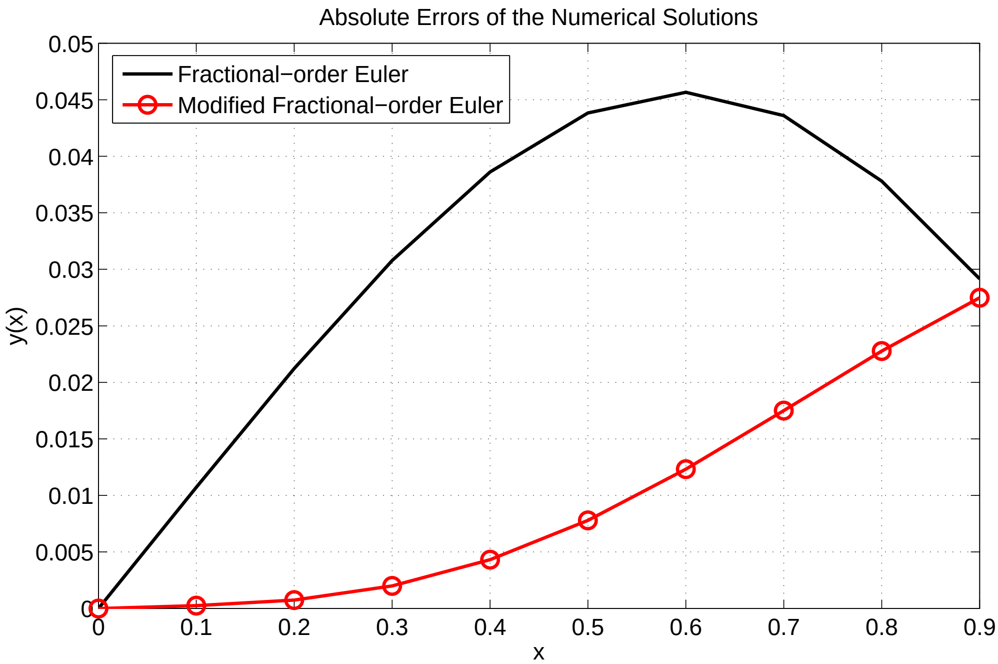

FIGURE 6. Absolute error between the exact and numerical solution of problem (17)

Modified Fractional Euler Method (MFEM), has confirmed its efficiency in providing a more accurate approximate solution than that of the approximate solution provided by the FEM. The error bound generated by the proposed method has been discussed and estimated. In the near future, we will attempt to find more accurate and comprehensive numerical methods to solve fractional initial value problems.

TABLE 3. Absolute error between the exact and numerical solution of problem (17)

t	Fractional Euler Method	Modified Fractional Euler Method
0	0.000000	0.000000
0.1	0.010715	0.000256
0.2	0.021222	0.000738
0.3	0.030780	0.001994
0.4	0.038606	0.004313
0.5	0.043831	0.007796
0.6	0.045654	0.012316
0.7	0.043600	0.017500
0.8	0.037798	0.022773
0.9	0.029162	0.027490

REFERENCES

[1] M. A. Hammad, Conformable fractional martingales and some convergence theorems, *Mathematics*, vol.10, no.1, 6, 2022.

[2] I. M. Batiha, S. Alshorm, A. Ouannas, S. Momani, O. Y. Ababneh and M. Albdareen, Modified three-point fractional formulas with Richardson extrapolation, *Mathematics*, vol.10, no.19, 3489, 2022.

[3] R. B. Albadarneh, I. M. Batiha and M. Zurigat, Numerical solutions for linear fractional differential equations of order $1 < \alpha < 2$ using finite difference method, *Int. J. Math. Comput. Sci.*, vol.16, no.1, pp.103-111, 2016.

[4] I. M. Batiha, R. B. Albadarneh, S. Momani and I. H. Jebril, Dynamics analysis of fractional-order Hopfield neural networks, *International Journal of Biomathematics*, vol.13, no.8, 2050083, 2020.

[5] K. H. Hussain, Some new existence and uniqueness results of Caputo fractional integro-differential equations, *International Journal of Innovative Computing, Information and Control*, vol.17, no.6, pp.2071-2079, 2021.

[6] R. B. Albadarneh, I. M. Batiha, A. Adwai, N. Tahat and A. K. Alomari, Numerical approach of Riemann-Liouville fractional derivative operator, *International Journal of Electrical and Computer Engineering*, vol.11, no.6, pp.5367-5378, 2021.

[7] R. B. Albadarneh, I. M. Batiha, N. Tahat and A. K. N. Alomari, Analytical solutions of linear and non-linear incommensurate fractional-order coupled systems, *Indonesian Journal of Electrical Engineering and Computer Science*, vol.21, no.2, pp.776-790, 2021.

[8] R. B. Albadarneh, I. Batiha, A. K. Alomari and N. Tahat, Numerical approach for approximating the Caputo fractional-order derivative operator, *AIMS Mathematics*, vol.6, no.11, pp.12743-12756, 2021.

[9] R. B. Albadarneh, M. Zerqat and I. M. Batiha, Numerical solutions for linear and non-linear fractional differential equations, *Int. J. Pure Appl. Math*, vol.106, no.3, pp.859-871, 2016.

[10] Z. Dahmani, A. Anber and I. Jebril, Solving conformable evolution equations by an extended numerical method, *Jordan Journal of Mathematics and Statistics*, vol.15, no.2, pp.363-380, 2022.

[11] Z. M. Odibat and S. Momani, An algorithm for the numerical solution of differential equations of fractional order, *Journal of Applied Mathematics & Informatics*, vol.26, nos.1-2, pp.15-27, 2008.

[12] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, Academic Press, San Diego, 1999.

[13] A. B. M. Hamed, I. Yuosif, I. A. Alrhaman and I. Sani, The accuracy of Euler and modified Euler technique for first order ordinary differential equations with initial condition, *American Journal of Engineering Research*, vol.6, no.9, pp.334-338, 2017.

[14] A. H. Workie, Small modification on modified Euler method for solving initial value problems, *Abstract and Applied Analysis*, vol.2021, 9951815, 2021.

[15] R. L. Burden and J. D. Faires, *Numerical Analysis*, 9th Edition, Brookscole, Boston, 2011.

[16] D. Baleanu, J. A. T. Machado and A. C. Luo, *Fractional Dynamics and Control*, Springer, New York, 2012.

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