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A STUDY ON BI-INTERIOR HYPERIDEALS IN HYPERSEMIGROUPS

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Abstract. *The concept of hypersemigroups is a generalization of semigroups with several real-world applications. The notions of hyperideals play an important role in studying hypersemigroups. This paper introduces the concept of bi-interior hyperideals, a combination of bi-hyperideals and interior hyperideals in hypersemigroups. We examine that bi-interior hyperideals generalize other hyperideals, especially bi-hyperideals and interior hyperideals. The intersection and union of bi-interior hyperideals are discussed. Furthermore, we provide the generating sets of bi-interior hyperideals. Finally, we apply the notion of bi-interior hyperideals to classifying regular and intra-regular hypersemigroups.*
Keywords: Hypersemigroup, Regular hypersemigroup, Intra-regular hypersemigroup, Bi-interior hyperideal

1. **Introduction.** The hyperstructure theory was first introduced by Marty [1] in 1934. After his introduction, hyperstructures have been used to study several branches of science, for example, biology, chemistry, and mathematics (see [2, 3, 4, 5, 6]). In classical algebraic structures, the composition of two elements is an element. By his introduction in 1934, Marty extended the composition of two elements in the classical algebraic structure to be a set. That is, the concept of hyperstructures can be seen as a generalization of the classical algebraic structures, such as groups, rings and semigroups, in the sense that the product of any two elements is assigned to be a singleton set. Therefore, many investigations of algebraic structure can be shifted to consider hyperstructures.

Hypergroups, a generalized concept of groups, were firstly considered by Marty in [1]. Recently, Tyr and Daher [7] used hypergroups to consider Jackson's inequalities. Leoreanu-Fotea et al. [8] discussed the center and centralizer elements of reversible regular hypergroups. Moreover, they also analyzed Rosenberg hypergroups. The proper left invariant metrics of hypergroups were introduced in [9]. The authors illustrated that any

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two proper left invariant metrics are coarsely equivalent. The solvable problems for hypergroups were classified in [10].

The concept of hyperrings was introduced by Krasner (see [11]). This concept is a generalization of rings. Omidi et al. [12] started the notion of n-hyperideals for commutative hyperrings and gave some interesting results. Yazarli et al. [13] defined the notion of generalized centroid to hyperrings and studied its properties. Furthermore, they provided the connection between hyperfields and hyperrings. The grade absorption of hyperideals in hyperrings was introduced in [14]. The characterizations of grade absorption of hyperideals were provided.

In this paper, we focus on the extended version of semigroups, the concept of hypersemigroups. The concept of hypersemigroup (semihypergroups, multisemigroups) was developed by many aspects of hypergroups. Therefore, different perspectives on hypersemigroups were established (see [15, 16, 17, 18, 19, 20]). In various ways, many problems in semigroups can be considered in terms of hypersemigroups. For the sake of the readers, let us discuss a few interesting works of literature. In 1996, Gutan [21] characterized hypersemigroups in which the relation β is transitive. This consideration addressed a problem given in [22]. Davvaz [23] discussed the concept of congruence in hypersemigroups and investigated its properties. Jafarabadi et al. [24] introduced the concept of simple hypersemigroups. They considered the product and the quotient of simple hypersemigroups if the resulting hypersemigroups are simple.

Hasankhani [25] introduced the notions of left (resp., right) hyperideals and examined their preliminary properties. In addition, the author studied Green's relations of hypersemigroups. It turns out that the notions of various kinds of hyperideals in hypersemigroups have been interesting to many researchers since hyperideals are essential in investigating Green's relations of hypersemigroups. Hila et al. [26] defined the notion of quasi-hyperideals as a generalized notion of left (resp., right) hyperideals. They considered the minimality of quasi-hyperideals and characterized them. The prime and semiprime properties of hyperideals were independently studied by Corsini et al. and Lekkoksung. Semisimple hyperideals were used to characterize a specific class of hypersemigroups known as semisimple hypersemigroups. Additionally, a specific case of semisimple hypersemigroups was examined using prime hyperideals (see [27, 28, 29]).

The other hyperideals in hypersemigroups were defined in a more general setting of hypersemigroups. For example, bi-hyperideals in ordered hypersemigroups were introduced by Changphas and Davvaz in [30]. They used bi-hyperideals to characterize intra-regular ordered hypersemigroups (see also [31]). The notion of (fuzzy) interior hyperideals in ordered semigroups was introduced independently in 2016. Tang et al. [32] considered the normality of fuzzy interior hyperideals and gave several related characterizations. Tipachot and Pibaljommee [33] applied fuzzy interior hyperideals to characterizing simple ordered hypersemigroups. Other tools for investigating ordered hypersemigroups are interval-valued intuitionistic fuzzy hyperideals and soft hyperideals (see [34, 35, 36]).

The conceptions of bi-hyperideals and interior hyperideals defined in ordered hypersemigroups with the equality relation are the notions in hypersemigroups since every ordered hypersemigroup with the equality relation can be regarded as a hypersemigroup. This means that hypersemigroups' notions of left (resp., right, quasi-, bi-, interior) hyperideals play important roles in exploring hypersemigroups. Recently, Lekkoksung et al. [37] defined more generalized notions of bi-hyperideals and interior hyperideals in hypersemigroups. They introduced the concept of (m, n)-hyperideals and n-interior hyperideals in hypersemigroups through the notion of ideal elements. Furthermore, these hyperideals were used to characterize many classes of hypersemigroups. The study of hyperideals and their extensions is not limited to just hypersemigroups and ordered hypersemigroups but also encompasses a wide range of hyperalgebraic structures. Research on hyperideals has been applied to investigating other mathematical hyperstructures, such as ordered Γ-hypersemigroups (see [38, 39]).

We can see the significance of hyperideals in hypersemigroups from the mentioned above. These implications motivate us to define a new hyperideal, which is a generalization of the earlier notions. Section 2 reminds us of some fundamental hypersemigroup knowledge and some hyperideals. Moreover, we introduce a new concept of hyperideals in hypersemigroups, so-called bi-interior hyperideals. In Section 3, the general properties of bi-interior hyperideals are given. We illustrate that bi-interior hyperideals are generalizations of bi-hyperideals and interior hyperideals. We give an example to show how we differentiated this concept from the earlier concepts. The relationships between bi-interior hyperideals and others are provided. We also provide certain conditions and classes of hypersemigroups to align the newly introduced concept of bi-interior hyperideals with other concepts of hyperideals. Section 4 investigates the intersection and union of bi-interior hyperideals in hypersemigroups. The minimality of bi-interior hyperideals is characterized by the minimality of their corresponding left and right hyperideals. The generating form of bi-interior hyperideals in hypersemigroups is also presented in this section. The characterizations of particular classes of hypersemigroups are discussed by bi-interior hyperideals in Section 5. Finally, conclusions are given in Section 6.

2. **Preliminaries.**

The fundamental definitions of the theory of hypersemigroups, which will be employed throughout the study, are recalled. Given the wide range of directions for the concept of hypersemigroups (semihypergroups, multisemigroups), in the current work, we use the one discussed by Kehayopulu (see [19, 40, 41, 42, 43]).

A *hyperoperation* $\circ$ on a nonempty set H is a function $\circ\colon H \times H \to \mathcal{P}^*(H)$, where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of H. A structure $(H;\circ)$ comprising a nonempty set H and a hyperoperation defined on H is called a *hypergroupoid*. A hyperoperation $\circ$ defined on the set H induces a binary operation $*$ defined on $\mathcal{P}^*(H)$ which is assigned by

$$A * B := \bigcup_{(a,b)\in A\times B} (a \circ b)$$

for all $A, B \in \mathcal{P}^*(H)$.

Definition 2.1. [19, 43] *A hypergroupoid $(H;\circ)$ is said to be a* hypersemigroup *if one of the following statements holds:*

(1) $\{a\} * (b \circ c) = (a \circ b) * \{c\}$ *for all* $a, b, c \in H$*;*
(2) $\{a\} * (\{b\} * \{c\}) = (\{a\} * \{b\}) * \{c\}$ *for all* $a, b, c \in H$*.*

For simplicity, we denote a (hypergroupoid) hypersemigroup $(H;\circ)$ by its carrier set as a boldface $\mathbf{H}$.

Some significant results of hypersemigroups were provided in [19, 43] as follows.

Lemma 2.1. [19, 43] *Let* $\mathbf{H}$ *be a hypersemigroup. The following statements hold.*

(1) $A * (B * C) = (A * B) * C$ *for all* $A, B, C \in \mathcal{P}^*(H)$*.*
(2) *If* $A_i, B \in \mathcal{P}^*(H)$ *for all* $i \in I$*, then*

$$\left(\bigcup_{i\in I} A_i\right) * B = \bigcup_{i\in I}(A_i * B) \text{ and } B * \left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I}(B * A_i).$$

(3) *If* $A_i, B \in \mathcal{P}^*(H)$ *for all* $i \in I$*, then*

$$\left(\bigcap_{i\in I} A_i\right) * B \subseteq \bigcap_{i\in I}(A_i * B) \text{ and } B * \left(\bigcap_{i\in I} A_i\right) = \bigcap_{i\in I}(B * A_i).$$

Let $\mathbf{H}$ be a hypersemigroup. For any natural number n and any nonempty subset A of H, we denote the n-product $A * A * \cdots * A$ of A by A^n. For more information on hypersemigroups, we can find in [19, 43].

Let $\mathbf{H}$ be a hypersemigroup. A nonempty subset A of H is called a *subhypersemigroup* of $\mathbf{H}$ if $A * A \subseteq A$. Many researchers have studied the concept of hypersemigroups by focusing on certain subhypersemigroups. Here are some subhypersemigroups that play an important part in studying hypersemigroups.

Definition 2.2. *Let $\mathbf{H}$ be a hypersemigroup. A nonempty subset A of H is said to be*

(1) *a* left hyperideal [25] *of $\mathbf{H}$ if $H * A \subseteq A$;*
(2) *a* right hyperideal [25] *of $\mathbf{H}$ if $A * H \subseteq A$;*
(3) *a* (two-sided) hyperideal [25] *of $\mathbf{H}$ if $H * A \subseteq A$ and $A * H \subseteq A$;*
(4) *a* quasi-hyperideal [26] *of $\mathbf{H}$ if $(A * H) \cap (H * A) \subseteq A$;*
(5) *a* bi-hyperideal [30] *of $\mathbf{H}$ if it is a subhypersemigroup and $A * H * A \subseteq A$;*
(6) *an* interior hyperideal [33] *of $\mathbf{H}$ if it is a subhypersemigroup and $H * A * H \subseteq A$.*

We can see that ordered hypersemigroups were the context in which the concepts of bi-hyperideals and interior hyperideals were developed. These concepts, however, are equally valid for hypersemigroups in the appropriate context.

Let $\mathbf{H}$ be a hypersemigroup. We denote by:

$\mathsf{L}(\mathbf{H})$ the set of all left hyperideals of $\mathbf{H}$;
$\mathsf{R}(\mathbf{H})$ the set of all right hyperideals of $\mathbf{H}$;
$\mathsf{J}(\mathbf{H})$ the set of all two-sided hyperideals of $\mathbf{H}$;
$\mathsf{Q}(\mathbf{H})$ the set of all quasi-hyperideals of $\mathbf{H}$;
$\mathsf{B}(\mathbf{H})$ the set of all bi-hyperideals of $\mathbf{H}$;
$\mathsf{I}(\mathbf{H})$ the set of all interior hyperideals of $\mathbf{H}$.

The relationships among hyperideals defined in Definition 2.2 are presented in Figure 1.

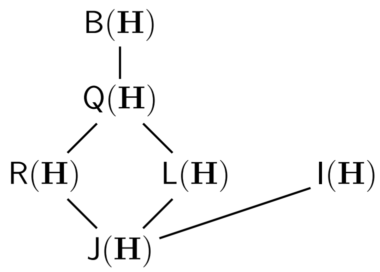

FIGURE 1. Relationships among hyperideals in Definition 2.2

Let $\mathbf{H}$ be a hypersemigroup and A a nonempty subset of H. We denote by

$(A)_{\mathrm{l}}$ the smallest left hyperideal of $\mathbf{H}$ containing A;
$(A)_{\mathrm{r}}$ the smallest right hyperideal of $\mathbf{H}$ containing A;
$(A)_{\mathrm{j}}$ the smallest two-sided hyperideal of $\mathbf{H}$ containing A;
$(A)_{\mathrm{q}}$ the smallest quasi-hyperideal of $\mathbf{H}$ containing A;
$(A)_{\mathrm{b}}$ the smallest bi-hyperideal of $\mathbf{H}$ containing A;
$(A)_{\mathrm{i}}$ the smallest interior hyperideal of $\mathbf{H}$ containing A.

They were illustrated in [19, 26, 40] that

(1) $(A)_{\mathrm{l}} = A \cup H * A$;

(2) $(A)_{\mathrm{r}} = A \cup A * H$;
(3) $(A)_{\mathrm{j}} = A \cup (H * A) \cup (A * H) \cup (H * A * H)$;
(4) $(A)_{\mathrm{q}} = A \cup [(A * H) \cap (H * A)]$;
(5) $(A)_{\mathrm{b}} = A \cup A^2 \cup (A * H * A)$;
(6) $(A)_{\mathrm{i}} = A \cup A^2 \cup (H * A * H)$.

We now introduce a new class of hyperideals that plays a crucial role in this paper.

Definition 2.3. *Let* $\mathbf{H}$ *be a hypersemigroup. A nonempty subset* A *of* H *is said to be a* bi-interior hyperideal *of* $\mathbf{H}$ *if* A *is a subhypersemigroup of* $\mathbf{H}$ *and* $(H * A * H) \cap (A * H * A) \subseteq A$. *We denote the set of all bi-interior hyperideals of* $\mathbf{H}$ *by* $\mathsf{BI}(\mathbf{H})$.

An example of bi-interior hyperideals is presented in Example 3.1 to demonstrate the concept.

The basic properties of bi-interior hyperideals in hypersemigroups are described in the following section. Bi-interior hyperideals and other varieties of hyperideals are also discussed in connection to one another.

3. Relationships of Bi-Hyperideals and Other Hyperideals.

As outlined in the Introduction, various types of hyperideals in hypersemigroups are instrumental in conducting detailed analyses of hypersemigroups. Additionally, they are used to classify hypersemigroups, providing a valuable framework for organizing and understanding the structural properties of these systems. Therefore, this section aims to explore the relationships between various types of hyperideals in hypersemigroups, including left (resp., right, two-sided, bi-, interior) hyperideals and bi-interior hyperideals, to gain a deeper understanding of hypersemigroups. We illustrate how the concept of bi-interior hyperideals is distinct from other notions presented in Definition 2.2. After examining the differences between bi-interior hyperideals and other types, we examine some of their fundamental characteristics. The smallest bi-interior hyperideal generated by a nonempty set is determined.

First, we present a connection between left (resp., right, two-sided) hyperideals and bi-interior hyperideals in hypersemigroups.

Proposition 3.1. *Let* $\mathbf{H}$ *be a hypersemigroup. Then, the following statements hold.*
(1) *Every left hyperideal of* $\mathbf{H}$ *is a bi-interior hyperideal of* $\mathbf{H}$.
(2) *Every right hyperideal of* $\mathbf{H}$ *is a bi-interior hyperideal of* $\mathbf{H}$.
(3) *Every two-sided hyperideal of* $\mathbf{H}$ *is a bi-interior hyperideal of* $\mathbf{H}$.

Proof: We prove only (1). For (2) and (3), we can prove similarly. Let A be a left hyperideal of $\mathbf{H}$. Then $A * A \subseteq H * A \subseteq A$. Moreover,

$$(H * A * H) \cap (A * H * A) \subseteq (A * H) \cap (A * A) \subseteq H \cap A = A.$$

Therefore, A is a bi-interior hyperideal of $\mathbf{H}$. □

Lemma 3.1. *The intersection of a left and a right hyperideal of* $\mathbf{H}$, *if it is nonempty, is a bi-interior hyperideal of* $\mathbf{H}$.

Proof: Let L and R be a left and a right hyperideal of $\mathbf{H}$, respectively. Suppose that $R \cap L \neq \emptyset$. Consider $(R \cap L) * (R \cap L) \subseteq R * S \subseteq R$ and $(R \cap L) * (R \cap L) \subseteq S * L \subseteq L$. Thus, we have $(R \cap L) * (R \cap L) \subseteq R \cap L$. Now, we consider

$$[(R \cap L) * H * (R \cap L)] \cap [H * (R \cap L) * H] \subseteq (R \cap L) * H * (R \cap L) \subseteq R * H * H \subseteq R$$

and

$$[(R \cap L) * H * (R \cap L)] \cap [H * (R \cap L) * H] \subseteq (R \cap L) * H * (R \cap L) \subseteq H * H * L \subseteq L.$$

Thus, we have $[(R \cap L) * H * (R \cap L)] \cap [H * (R \cap L) * H] \subseteq R \cap L$. Therefore, $R \cap L$ is a bi-interior hyperideal of $\mathbf{H}$. $\square$

The reverse of Proposition 3.1 does not generally hold, as we will see in Example 3.1. Thus, we can ask under which condition the converse of Proposition 3.1 is valid. A hypersemigroup $\mathbf{H}$ is said to be *left* (*resp., right*) *simple* if for every left (resp., right) hyperideal A of $\mathbf{H}$, we have $A = H$. A hypersemigroup $\mathbf{H}$ is called *simple* if $\mathbf{H}$ is both a left and a right simple hypersemigroup (see [42]).

Proposition 3.2. *Let $\mathbf{H}$ be a hypersemigroup. Then the following statements hold.*

(1) *If $\mathbf{H}$ is left simple, then every bi-interior hyperideal of $\mathbf{H}$ is a right hyperideal of $\mathbf{H}$.*
(2) *If $\mathbf{H}$ is right simple, then every bi-interior hyperideal of $\mathbf{H}$ is a left hyperideal of $\mathbf{H}$.*
(3) *If $\mathbf{H}$ is simple, then every bi-interior hyperideal of $\mathbf{H}$ is a two-sided hyperideal of $\mathbf{H}$.*

Proof: We prove only (1). For (2) and (3), we can prove similarly. Let A be a bi-interior hyperideal of a left simple hypersemigroup $\mathbf{H}$. It is not difficult to obtain that $H * A$ is a left hyperideal of $\mathbf{H}$. By hypothesis, we have that $H * A = H$ and then $A * H = A * (H * A) = A * H * A$. Then, we have

$$A * H = A * (H * A) = A * (H * A) * A \subseteq H * A * H.$$

Thus, we obtain that $A * H \subseteq (A * H * A) \cap (H * A * H) \subseteq A$. This shows that A is a right hyperideal of $\mathbf{H}$. $\square$

The following result illustrates that the concept of bi-interior hyperideals is a generalization of quasi-hyperideals.

Proposition 3.3. *Let $\mathbf{H}$ be a hypersemigroup. Then, every quasi-hyperideal of $\mathbf{H}$ is a bi-interior hyperideal of $\mathbf{H}$.*

Proof: Let A be a quasi-hyperideal of $\mathbf{H}$. Then, $A * A \subseteq A * H$ and $A * A \subseteq H * A$. This implies that $A * A \subseteq (A * H) \cap (H * A) \subseteq A$. Since

$$(H * A * H) \cap (A * H * A) \subseteq A * H * A \subseteq A * H$$

and

$$(H * A * H) \cap (A * H * A) \subseteq A * H * A \subseteq H * A,$$

we obtain

$$(H * A * H) \cap (A * H * A) \subseteq (A * H) \cap (H * A) \subseteq A.$$

Therefore, A is a bi-interior hyperideal of $\mathbf{H}$. $\square$

Example 3.1 shows that the opposite of the previous result is generally invalid. We propose a condition in which the concepts of quasi-hyperideals and bi-interior hyperideals are equivalent.

Let $\mathbf{H}$ be a hypersemigroup. An element e of H is called the *identity* [41] of $\mathbf{H}$ if $\{a\} * \{e\} = \{a\} = \{e\} * \{a\}$ for all $a \in H$. The following lemma is not difficult to obtain by the definition of the identity of hypersemigroup. Thus, the proof is omitted.

Lemma 3.2. *Let $\mathbf{H}$ be a hypersemigroup with the identity e. Then,*

$$A * \{e\} = A = \{e\} * A$$

for all nonempty subset A of H.

Applying Lemma 3.2, we obtain the converse of Proposition 3.3 as follows.

Proposition 3.4. *Let $\mathbf{H}$ be a hypersemigroup with the identity e. Suppose that $H = H * \{a\}$ for all $a \in H$. Then, every bi-interior hyperideal of $\mathbf{H}$ is a quasi-hyperideal of $\mathbf{H}$.*

Proof: Let B be a bi-interior hyperideal of $\mathbf{H}$. By our hypothesis, we have that $H = H * A$ for any nonempty subset A of H. This implies that $H = H * B$. By Lemma 3.2, we have $H * B = H * B * \{e\} \subseteq H * B * H$. Then, $(H * B) \cap (B * H) \subseteq (H * B * H) \cap (B * H * B) \subseteq B$. Therefore, B is a quasi-hyperideal of $\mathbf{H}$. □

Let us continue by determining how bi-hyperideals and bi-interior hyperideals relate to one another. Proposition 3.5 illustrates that any bi-hyperideal is a bi-interior hyperideal. That is, the concept of bi-interior hyperideals is a generalization of bi-hyperideals.

Proposition 3.5. *Let* $\mathbf{H}$ *be a hypersemigroup. Then, every bi-hyperideal of* $\mathbf{H}$ *is a bi-interior hyperideal of* $\mathbf{H}$*.*

Proof: Let A be a bi-hyperideal of $\mathbf{H}$. By hypothesis, we have that $A * H * A \subseteq A$. Then, $(H * A * H) \cap (A * H * A) \subseteq A * H * A \subseteq A$. Therefore, A is a bi-interior hyperideal of $\mathbf{H}$. □

The converse of Proposition 3.5, in general, is not true, as shown by Example 3.1. However, under certain conditions, the concepts of bi-hyperideals and bi-interior hyperideals coincide.

Proposition 3.6. *Let* $\mathbf{H}$ *be a simple hypersemigroup. Then, every bi-interior hyperideal of* $\mathbf{H}$ *is a bi-hyperideal of* $\mathbf{H}$*.*

Proof: Let A be a bi-interior hyperideal of $\mathbf{H}$. It is not difficult to illustrate that $H * A * H$ is a hyperideal of $\mathbf{H}$. Since $\mathbf{H}$ is a simple hypersemigroup, we have $H = H * A * H$. Then, $A * H * A = H \cap (A * H * A) = (H * A * H) \cap (A * H * A) \subseteq A$. Therefore, A is a bi-hyperideal of $\mathbf{H}$. □

Proposition 3.6 establishes that in simple hypersemigroups, bi-interior hyperideals and bi-hyperideals are equivalent. Finally, a relationship between interior hyperideals and bi-interior hyperideals is established. The result demonstrates that bi-interior hyperideals encompass a broader range of possibilities than interior hyperideals and can be considered a generalization of the latter.

Proposition 3.7. *Let* $\mathbf{H}$ *be an hypersemigroup. Then, every interior hyperideal of* $\mathbf{H}$ *is a bi-interior hyperideal of* $\mathbf{H}$*.*

Proof: Let A be an interior hyperideal of $\mathbf{H}$. Then, $(H * A * H) \cap (A * H * A) \subseteq H * A * H \subseteq A$. Therefore, A is a bi-interior hyperideal of $\mathbf{H}$. □

The above result illustrates that the notion of bi-interior hyperideals is an extension of interior hyperideals. The converse of Proposition 3.7, in general, is not true as presented in Example 3.1.

Now, we give the condition that the concepts of bi-interior hyperideals and interior hyperideals coincide.

Proposition 3.8. *Let* $\mathbf{H}$ *be a hypersemigroup with the identity* e*. Then, every bi-interior hyperideal of* $\mathbf{H}$ *is an interior hyperideal of* $\mathbf{H}$*.*

Proof: Let A be a bi-interior hyperideal of $\mathbf{H}$. By Lemma 3.2, we have that $A = \{e\} * A * \{e\} \subseteq H * A * H$. Then, $A * H * A \subseteq (H * A * H) * H * A \subseteq H * A * H$. Thus, $A * H * A = (A * H * A) \cap (A * H * A) \subseteq (H * A * H) \cap (A * H * A) \subseteq A$. Therefore, A is an interior hyperideal of $\mathbf{H}$. □

The following theorem can summarize all the above propositions.

Theorem 3.1. *Let* $\mathbf{H}$ *be a hypersemigroup. Then, we obtain the following statements.*

(1) *If* $\mathbf{H}$ *is left simple, then* $\mathsf{BI}(\mathbf{H}) = \mathsf{R}(\mathbf{H})$*.*
(2) *If* $\mathbf{H}$ *is right simple, then* $\mathsf{BI}(\mathbf{H}) = \mathsf{L}(\mathbf{H})$*.*

(3) *If* $\mathbf{H}$ *is simple, then* $\mathsf{BI}(\mathbf{H}) = \mathsf{J}(\mathbf{H}) = \mathsf{R}(\mathbf{H}) = \mathsf{L}(\mathbf{H}) = \mathsf{Q}(\mathbf{H}) = \mathsf{B}(\mathbf{H})$.
(4) *If* $\mathbf{H}$ *has the identity* e *with* $H = H * \{a\}$ *for all* $a \in H$*, then* $\mathsf{BI}(\mathbf{H}) = \mathsf{Q}(\mathbf{H}) = \mathsf{I}(\mathbf{H})$.
(5) *If* $\mathbf{H}$ *has the identity* e*, then* $\mathsf{BI}(\mathbf{H}) = \mathsf{I}(\mathbf{H})$.

The distinction between bi-interior hyperideals and other concepts is explored in the following example. This helps us achieve one of our goals in developing a new hyperideal that is more general than the others.

Example 3.1. *Let* $H = \{1, 2, 3, 4, 5, 6, 7\}$*. Define a hyperoperation* $\circ$ *on* H *by the following table.*

$\circ$	1	2	3	4	5	6	7
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
3	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 2\}$
5	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 3\}$	$\{1\}$
6	$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 3\}$	$\{1\}$	$\{1\}$	$\{5\}$
7	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1, 2\}$	$\{1, 3, 4\}$	$\{1\}$

We can carefully calculate that $\mathbf{H} := (H; \circ)$ *is a hypersemigroup. Let* $A = \{1, 7\}$*. Then*
(1) $A * A = \{1\}$*;*
(2) $A * H * A = \{1, 2\}$*;*
(3) $H * A * H = \{1, 3\}$*;*
(4) $(A * H * A) \cap (H * A * H) = \{1\}$.
This means that A *is a bi-interior hyperideal of* $\mathbf{H}$*. We can also see that* A *is neither a bi-hyperideal nor an interior hyperideal of* $\mathbf{H}$*. Moreover,* A *is not a left (resp., right, two-sided, quasi-) hyperideal of* $\mathbf{H}$*. In this case, we conclude that* $\mathsf{B}(\mathbf{H}) \subset \mathsf{BI}(\mathbf{H})$ *and* $\mathsf{I}(\mathbf{H}) \subset \mathsf{BI}(\mathbf{H})$.

We summarize all relationships in Figure 2.

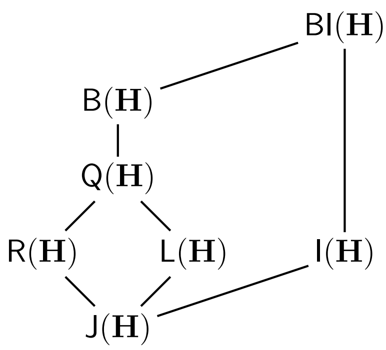

Figure 2. Relationships among hyperideals

This section comes to a close with a description of bi-interior hyperideals. In a particular class of hypersemigroups, the concept of bi-interior hyperideals can be characterized by the notions of left hyperideals and right hyperideals, as shown by the following theorem.

A hypersemigroup $\mathbf{H}$ is said to be *regular* if for every nonempty subset A of H, we have $A \subseteq A * H * A$ (see [42]).

Lemma 3.3. *Let* $\mathbf{H}$ *be a regular hypersemigroup. Then* $A \subseteq (H * A * H) \cap (A * H * A)$ *for any subset* A *of* H.

Proof: Let A be a subset of H. The proof is clear if A is an empty set. Suppose that A is nonempty. By the regularity of $\mathbf{H}$, we obtain $A \subseteq A * H * A$, and $A \subseteq A * H * A \subseteq A * H * (A * H * A) \subseteq H * A * H$. Thus, $A \subseteq (H * A * H) \cap (A * H * A)$. Therefore, we complete the proof. □

The following gives a characterization of bi-interior hyperideals in regular hypersemigroups.

Theorem 3.2. *Let* $\mathbf{H}$ *be a regular hypersemigroup. Then the following statements are equivalent.*

(1) *A is a bi-interior hyperideal of* $\mathbf{H}$.
(2) *$A = R * L$ for some right hyperideal R and left hyperideal L of* $\mathbf{H}$.

Proof: (1) ⇒ (2). By the proof of Lemma 3.3, it is not difficult to see that $A * H * H * A \subseteq (H * A * H) \cap (A * H * A)$. We consider

$$\begin{aligned} A &\subseteq A * H * A \\ &\subseteq (A * H) * (A * H * A) && \text{(since } \mathbf{H} \text{ is regular)} \\ &\subseteq A * H * H * A \\ &\subseteq (H * A * H) \cap (A * H * A) \\ &\subseteq A. \end{aligned}$$

This means that $A = (A * H) * (H * A)$. Since $A * H$ and $H * A$ is a right hyperideal and a left hyperideal of $\mathbf{H}$, we obtain (2).

(2) ⇒ (1). We consider $A * A = (R * L) * (R * L) \subseteq R * L = A$ and

$$\begin{aligned} (A * H * A) \cap (H * A * H) &= [(R * L) * H * (R * L)] \cap [H * (R * L) * H] \\ &\subseteq (R * L) * H * (R * L) \\ &= R * (L * H * R * L) \\ &\subseteq R * L && \text{(since } L \text{ is a left hyperideal of } \mathbf{H}) \\ &= A. \end{aligned}$$

This shows that A is a bi-interior hyperideal of $\mathbf{H}$. □

In the following section, we examine the minimality of bi-interior hyperideals and the generating system of bi-interior hyperideals in hypersemigroups.

4. The Minimality and the Generating Systems of Bi-Interior Hyperideals.

The minimality of bi-interior hyperideals in hypersemigroups is discussed in this section. We apply the minimalities of left and right hyperideals to describing the minimal bi-interior hyperideals. In our last contribution, we provide a generating form for bi-interior hyperideals.

Let $\mathbf{H}$ be a hypersemigroup. A left (resp., right, bi-interior) hyperideal A of $\mathbf{H}$ is said to be *minimal* if there is no other left (resp., right, bi-interior) hyperideal B of $\mathbf{H}$ such that $B \subset A$. Equivalently, if B is a left (resp., right, bi-interior) hyperideal of $\mathbf{H}$ such that $B \subseteq A$, then $A = B$. More result about the minimality of left (resp., right, bi-) hyperideals in hypersemigroups can be found in [44].

The following theorem illustrates that the product of any minimal left and right hyperideals is a minimal bi-interior hyperideal.

Theorem 4.1. *Let* $\mathbf{H}$ *be a hypersemigroup. Suppose that R and L are a minimal right and a minimal left hyperideal of* $\mathbf{H}$*, respectively. We have that $R * L$ is a minimal bi-interior hyperideal of* $\mathbf{H}$.

Proof: By the definition of the operation $*$ defined on $\mathcal{P}^*(H)$, it is clear that $R*L \neq \emptyset$. We let $A := R*L$. Consider $A*A = (R*L)*(R*L) \subseteq R*L = A$ and

$$A*H*A = (R*L)*H*(R*L) \subseteq R*H*L \subseteq R*L = A.$$

Thus, A is a bi-hyperideal of $\mathbf{H}$. By Proposition 3.5, A is a bi-interior hyperideal of $\mathbf{H}$. Now, we let Z be a bi-interior hyperideal of $\mathbf{H}$ such that $Z \subseteq A$. It is not difficult to illustrate that $H*Z$ and $Z*H$ is a left hyperideal and a right hyperideal of $\mathbf{H}$, respectively. Since R and L are minimal, we obtain $H*Z = L$ and $Z*H = R$. Then,

$$A = R*L = (Z*H)*(H*Z) \subseteq Z*H*Z$$

and

$$A = R*L = R*(L*H) = (Z*H)*(H*Z)*H \subseteq H*Z*H.$$

Thus, $A \subseteq (H*Z*H) \cap (Z*H*Z) \subseteq Z$. This implies that $Z = A$. Therefore, A is a minimal bi-interior hyperideal of $\mathbf{H}$. □

The following result examines that the intersection of any bi-interior hyperideals in hypersemigroups is also a bi-interior hyperideal.

Proposition 4.1. *Let* $\mathbf{H}$ *be a hypersemigroup, and* $\mathcal{A} := \{A_i : i \in I\}$ *is a nonempty collection of bi-interior hyperideals of* $\mathbf{H}$. *If* $\bigcap \mathcal{A} \neq \emptyset$, *then* $\bigcap \mathcal{A}$ *is a bi-interior hyperideal of* $\mathbf{H}$.

Proof: Since $\bigcap \mathcal{A} \subseteq A_i$ for all $i \in I$, we have

$$\left(\bigcap \mathcal{A}\right) * \left(\bigcap \mathcal{A}\right) \subseteq A_i * A_i \subseteq A_i$$

and

$$\left(\left(\bigcap \mathcal{A}\right) * H * \left(\bigcap \mathcal{A}\right)\right) \cap \left(H * \left(\bigcap \mathcal{A}\right) * H\right) \subseteq (A_i * H * A_i) \cap (H * A_i * H) \subseteq A_i$$

for all $i \in I$. Therefore, $\bigcap \mathcal{A}$ is a bi-interior hyperideal of $\mathbf{H}$. □

The above proposition does not hold for the union, as illustrated by the following example.

Example 4.1. *By Example 3.1, we can illustrate that* $B = \{1, 6\}$ *is also a bi-interior hyperideal of* $\mathbf{H}$. *However,* $C := A \cup B = \{1, 6, 7\}$ *is not a bi-interior hyperideal of* $\mathbf{H}$ *since* $C * C = \{1, 3, 4, 5\} \not\subseteq C$.

Let $\mathbf{H}$ be a hypersemigroup. For any given nonempty subset A of H, we know by Proposition 4.1 that any intersection of bi-interior hyperideals of $\mathbf{H}$ containing A is also a bi-interior hyperideal of $\mathbf{H}$ containing A. Hence, we denote the intersection of all bi-interior hyperideals of $\mathbf{H}$ containing A by $(A)_{\mathrm{bn}}$. The set $(A)_{\mathrm{bn}}$ is called the *bi-interior hyperideal of* $\mathbf{H}$ *generated by* A. Now, we can ask for the construction of $(A)_{\mathrm{bn}}$.

Theorem 4.2. *Let* A *be a nonempty subset of a hypersemigroup* $\mathbf{H}$. *Then,*

$$(A)_{\mathrm{bn}} = A \cup A^2 \cup [(A*H*A) \cap (H*A*H)].$$

Proof: Let $B = A \cup A^2 \cup [(A*H*A) \cap (H*A*H)]$. We show that B is the smallest bi-interior hyperideal of $\mathbf{H}$ containing A. Consider

$$\begin{aligned} B*B &\subseteq A^2 \cup A^3 \cup A^4 \cup (A*H*A) \\ &\subseteq A^2 \cup (A*H*A) \cup (A*H*A) \cup (A*H*A) \\ &= A^2 \cup (A*H*A) \end{aligned}$$

and

$$\begin{aligned} B * B &\subseteq A^2 \cup A^3 \cup A^4 \cup (H * A * H) \\ &\subseteq A^2 \cup (H * A * H) \cup (H * A * H) \cup (H * A * H) \\ &= A^2 \cup (H * A * H). \end{aligned}$$

Thus, we have

$$\begin{aligned} B * B &\subseteq \left[A^2 \cup (A * H * A)\right] \cap \left[A^2 \cup (H * A * H)\right] \\ &= A^2 \cup [(A * H * A) \cap (H * A * H)] \\ &\subseteq B. \end{aligned}$$

This shows that B is a subhypersemigroup of $\mathbf{H}$. Consider

$$B * H * B \subseteq \left(A \cup A^2 \cup (A * H * A)\right) * H * \left(A \cup A^2 \cup (A * H * A)\right) \subseteq A * H * A$$

and

$$B * H * B \subseteq \left(A \cup A^2 \cup (H * A * H)\right) * H * \left(A \cup A^2 \cup (H * A * H)\right) \subseteq H * A * H.$$

Thus, we have $B * H * B \subseteq (A * H * A) \cap (H * A * H) \subseteq B$. These illustrate that B is a bi-interior hyperideal of $\mathbf{H}$.

Now, let C be a bi-interior hyperideal of $\mathbf{H}$ such that $A \subseteq C$. Then,

$$\begin{aligned} B &= A \cup A^2 \cup [(A * H * A) \cap (H * A * H)] \\ &\subseteq C \cup C^2 \cup [(C * H * C) \cap (H * C * H)] \\ &\subseteq C \cup C \cup C \\ &= C. \end{aligned}$$

This means that B is the smallest bi-interior hyperideal of $\mathbf{H}$.

By the definition of $(A)_{\mathrm{bn}}$ and the smallest property of B, we obtain $(A)_{\mathrm{bn}} \subseteq B \subseteq (A)_{\mathrm{bn}}$. Therefore, we have $(A)_{\mathrm{bn}} = A \cup A^2 \cup [(A * H * A) \cap (H * A * H)]$. □

5. **An Application of Bi-Interior Hyperideals.** In the final section, we combine bi-interior hyperideals with other hyperideals to characterize particular classes of hypersemigroups. Using bi-interior hyperideals, left hyperideals, and right hyperideals, we first discuss regular hypersemigroups.

We recall that a hypersemigroup $\mathbf{H}$ is *regular* if for every nonempty subset A of H, we have $A \subseteq A * H * A$. The following lemma is a helpful tool for characterizing regular hypersemigroups.

Lemma 5.1. [42, Theorem 2.1] *Let* $\mathbf{H}$ *be a hypersemigroup. Then, the following conditions are equivalent.*

(1) $\mathbf{H}$ *is regular.*
(2) $R \cap L \subseteq R * L$ *for every right hyperideal* R *and every left hyperideal* L *of* $\mathbf{H}$.

A characterization of regular hypersemigroups by bi-interior hyperideals and other hyperideals is provided as follows.

Theorem 5.1. *Let* $\mathbf{H}$ *be hypersemigroup. Then, the following conditions are equivalent.*

(1) $\mathbf{H}$ *is regular.*
(2) $A = (A * H * A) \cap (H * A * H)$ *for every bi-interior hyperideal* A *of* $\mathbf{H}$.
(3) $A \cap L \subseteq A * L$ *for every bi-interior hyperideal* A *and every left hyperideal* L *of* $\mathbf{H}$.
(4) $R \cap A \subseteq R * A$ *for every right hyperideal* R *and every bi-interior hyperideal* A *of* $\mathbf{H}$.

Proof: (1) $\Rightarrow$ (2). Let A be a bi-interior hyperideal of $\mathbf{H}$. By Lemma 3.3, we have

$$A \subseteq (A * H * A) \cap (H * A * H) \subseteq A.$$

This implies that $A = (A * H * A) \cap (H * A * H)$.

(2) $\Rightarrow$ (1). We prove this direction by using Lemma 5.1. Let R and L be a right hyperideal and left hyperideal of $\mathbf{H}$, respectively. If $R \cap L = \emptyset$, then, by Lemma 5.1, we complete the proof. Suppose that $R \cap L \neq \emptyset$. By Lemma 3.1, $R \cap L$ is a bi-interior hyperideal of $\mathbf{H}$. This implies that

$$\begin{aligned} R \cap L &= [(R \cap L) * H * (R \cap L)] \cap [H * (R \cap L) * H] && \text{(by our presumption)} \\ &\subseteq (R \cap L) * H * (R \cap L) \\ &\subseteq R * H * L \\ &\subseteq R * L. \end{aligned}$$

By Lemma 5.1, $\mathbf{H}$ is regular.

(1) $\Rightarrow$ (3). Let A and L be a bi-interior hyperideal and a left hyperideal of $\mathbf{H}$, respectively. Since $\mathbf{H}$ is regular, we obtain

$$B \cap L \subseteq (B \cap L) * H * (B \cap L) \subseteq B * H * L \subseteq B * L.$$

(3) $\Rightarrow$ (1). Let A be a nonempty subset of H. Then,

$$\begin{aligned} A &\subseteq (A)_{\mathrm{bn}} \cap (A)_{\mathrm{l}} \\ &\subseteq (A)_{\mathrm{bn}} * (A)_{\mathrm{l}} && \text{(by our presumption)} \\ &= \big(A \cup A^2 \cup [(A * H * A) \cap (H * A * H)]\big) * [A \cup (H * A)] \\ &\subseteq \big(A \cup A^2 \cup (A * H * A)\big) * [A \cup (H * A)] \\ &\subseteq A^2 \cup A * H * A. \end{aligned}$$

The proof is done if $A \subseteq A * H * A$. Suppose that $A \subseteq A^2$. Then, $A \subseteq A * A \subseteq A * (A * A) \subseteq A * H * A$. Therefore, $\mathbf{H}$ is regular.

(1) $\Leftrightarrow$ (4). The proof of this equivalence can be done similarly to (1) $\Leftrightarrow$ (3). $\square$

A hypersemigroup $\mathbf{H}$ is said to be *intra-regular* [42, 43] if for every nonempty subset A of H, we have $A \subseteq H * A^2 * H$. We apply bi-interior hyperideals, left hyperideals, and right hyperideals to describing intra-regular hypersemigroups as a result of the following theorem.

Theorem 5.2. *Let* $\mathbf{H}$ *be a hypersemigroup. Then the following conditions are equivalent.*

(1) $\mathbf{H}$ *is intra-regular.*
(2) $A \cap L \subseteq L * A * H$ *for every bi-interior hyperideal* A *and every left hyperideal* L *of* $\mathbf{H}$.
(3) $R \cap A \subseteq H * A * R$ *for every right hyperideal* R *and every bi-interior hyperideal* A *of* $\mathbf{H}$.
(4) $A \cap Q \subseteq H * A * Q * H$ *for every bi-interior hyperideal* A *and every quasi-hyperideal* Q *of* $\mathbf{H}$.
(5) $Q \cap A \subseteq H * A * Q * H$ *for every quasi-hyperideal* Q *and every bi-interior hyperideal* A *of* $\mathbf{H}$.

Proof: (1) $\Rightarrow$ (2). Let A and L be a bi-interior hyperideal and a left hyperideal of $\mathbf{H}$, respectively. Then, we have

$$A \cap L \subseteq H * (A \cap L)^2 * H = H * (A \cap L) * (A \cap L) * H \subseteq H * L * A * H \subseteq L * A * H.$$

(2) $\Rightarrow$ (1). Let $A \subseteq H$ such that $A \neq \emptyset$. Then,

$$\begin{aligned} A &\subseteq (A)_{\mathrm{bn}} \cap (A)_{\mathrm{l}} \\ &\subseteq (A)_{\mathrm{l}} * (A)_{\mathrm{bn}} * H \qquad \text{(by our presumption)} \\ &= [A \cup (H * A)] * \big[A \cup A^2 \cup [(H * A * H) \cap (A * H * A)]\big] * H \\ &\subseteq [A \cup (H * A)] * \big[A \cup A^2 * (A * H * A)\big] * H \\ &\subseteq \big(A^2 * H\big) \cup \big(H * A^2 * H\big). \end{aligned}$$

The proof is done if $A \subseteq H * A^2 * H$. Suppose that $A \subseteq A^2 * H$. Then,

$$A \subseteq A^2 * H \subseteq A * \big(A^2 * H\big) * H \subseteq H * A^2 * H.$$

Therefore, $\mathbf{H}$ is intra-regular.

(1) $\Leftrightarrow$ (3). The proof of this equivalence can be done similarly to (1) $\Leftrightarrow$ (2).

(1) $\Rightarrow$ (4). Let A and Q be a bi-interior hyperideal and a quasi-hyperideal of $\mathbf{H}$, respectively. Then, we have

$$A \cap Q \subseteq H * (A \cap Q)^2 * H = H * (A \cap Q) * (A \cap Q) * H \subseteq H * A * Q * H.$$

(4) $\Rightarrow$ (1). Let A be a nonempty subset of H. Then,

$$\begin{aligned} A &\subseteq (A)_{\mathrm{bn}} \cap (A)_{\mathrm{q}} \\ &\subseteq H * (A)_{\mathrm{bn}} * (A)_{\mathrm{q}} * H \qquad \text{(by our presumption)} \\ &= H * \big[A \cup A^2 \cup [(A * H * A) \cap (H * A * H)]\big] * [A \cup [(A * H) \cap (H * A)]] * H \\ &\subseteq H * \big[A \cup A^2 \cup (A * H * A)\big] * [A \cup (A * H)] * H \\ &= \big(H * A^2 * H\big) \cup \big(H * A^2 * H^2\big) \cup \big(H * A^3 * H\big) \cup \big(H * A^3 * H^2\big) \\ &\quad \cup \big(H * A * H * A^2 * H\big) \cup \big(H * A * H * A^2 * H^2\big) \\ &\subseteq H * A^2 * H. \end{aligned}$$

Therefore, $\mathbf{H}$ is intra-regular.

(1) $\Leftrightarrow$ (5). The proof of this equivalence can be done similarly to (1) $\Leftrightarrow$ (4). $\square$

To demonstrate the utility of Theorem 5.1, let us consider the following example.

Example 5.1. *Let $H = \{1, 2, 3, 4, 5, 6, 7\}$. Define a hyperoperation $\circ$ on H by the following table.*

$\circ$	1	2	3	4	5	6	7
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{4\}$	$\{5\}$	$\{5\}$	$\{4\}$
2	$\{1\}$	$\{1,2\}$	$\{1\}$	$\{4\}$	$\{5\}$	$\{5\}$	$\{4,7\}$
3	$\{1\}$	$\{1\}$	$\{1,3\}$	$\{4\}$	$\{5\}$	$\{5,6\}$	$\{4\}$
4	$\{4\}$	$\{4\}$	$\{4\}$	$\{5\}$	$\{1\}$	$\{1\}$	$\{5\}$
5	$\{5\}$	$\{5\}$	$\{5\}$	$\{1\}$	$\{4\}$	$\{4\}$	$\{1\}$
6	$\{5\}$	$\{5,6\}$	$\{5\}$	$\{1\}$	$\{4\}$	$\{4\}$	$\{1,3\}$
7	$\{4\}$	$\{4\}$	$\{4,7\}$	$\{5\}$	$\{1\}$	$\{1,2\}$	$\{5\}$

We can carefully calculate that $\mathbf{H} := (H; \circ)$ is a hypersemigroup. Then,

$$\begin{aligned} \mathsf{BI}(\mathbf{H}) = \{&\{0,3,4\}, \{0,1,3,4\}, \{0,2,3,4\}, \{0,3,4,5\}, \{0,3,4,6\}, \{0,1,3,4,5\}, \\ &\{0,1,3,4,6\}, \{0,2,3,4,5\}, \{0,2,3,4,6\}, H\} \end{aligned}$$

It is not difficult to obtain that $A = (A * H * A) \cap (H * A * H)$ for any $A \in \mathsf{BI}(\mathbf{H})$. By Theorem 5.1, we have that $\mathbf{H}$ is regular.

6. **Conclusion.** In this study, the concept of bi-interior hyperideals in hypersemigroups is introduced. An example from the paper demonstrates how the idea of bi-interior hyperideals differs from other hyperideals. This example is the special feature of this paper. We uncover several relationships between bi-interior hyperideals and other hyperideals, providing insight into the generality of various types of hyperideals. This understanding aids in classifying regular and intra-regular hypersemigroups, shedding light on their structural properties, as shown by Theorems 5.1 and 5.2. In these theorems, bi-interior hyperideals serve as a key tool in characterizing regular and intra-regular hypersemigroups. The generating systems of bi-interior hyperideals, as constructed in Theorem 4.2, are used in each characterization, highlighting their importance in analyzing these classes of hypersemigroups. In addition, the fundamental attributes of bi-interior hyperideals, their intersection and union are examined. In our future study, we can investigate if other classes of hypersemigroups or other hyperalgebraic systems can be described by bi-interior hyperideals.

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REFERENCES

[1] F. Marty, On a generalization of the notion of groups, *The 8th Congress Math. Scandinaves*, Stockholm, pp.45-49, 1934.

[2] M. Al-Tahan and B. Davvaz, Chemical hyperstructures for elements with four oxidation states, *Iranian J. Math. Chem.*, vol.13, no.2, pp.85-97, 2022.

[3] R. B. Tajvar and M. Latifi, On the path hyperoperation and its connections with hypergraph theory, *J. Algebr. Syst.*, vol.10, no.2, pp.309-321, 2023.

[4] P. Corsini and V. Leoreanu-Fotea, *Applications of Hyperstructure Theory. Advances in Mathematics*, Kluwer Academic Publishers, Dordrecht, 2003.

[5] D. Preethi, J. Vimala, S. Rajareega and M. Al-Tahan, Fuzzy hyperlattice ordered δ-group and its application on ABO blood group system, *J. Intell. Fuzzy Syst.*, vol.41, no.5, pp.5309-5315, 2021.

[6] E. H. Sadrabadi and B. Davvaz, A new relationship between population genetics and fuzzy sets, *Comput. Appl.*, vol.41, no.6, pp.1-13, 2022.

[7] O. Tyr and R. Daher, Jackson's inequalities in Laguerre hypergroup, *J. Pseudo-Differ. Oper. Appl.*, vol.13, no.4, pp.1-19, 2022.

[8] V. Leoreanu-Fotea, P. Corsini, A. Sonea and D. Heidari, Complete parts and subhypergroups in reversible regular hypergroups, *An. Şt. Univ. Ovidius Constanţa*, vol.30, no.1, pp.219-230, 2022.

[9] M. Amini, A. A. Amjadi and S. M. Tabatabaie, Coarse structure of metrizable hypergroups, *Complex Anal. Oper. Theory*, vol.16, no.1, pp.1-12, 2022.

[10] A. V. Vasil'ev and P. H. Zieschang, Solvable hypergroups and a generalization of Hall's theorems on finite solvable groups to association schemes, *J. Algebra*, vol.594, pp.733-750, 2022.

[11] M. Krasner, On the approximation of the complete valued fields of characteristic p by those of characteristic zero, *Higher Algebra Colloquium (Brussels, December 1956)*, CBRM, Brussels, 1957.

[12] S. Omidi, B. Davvaz and J. Zhan, Some properties of n-hyperideals in commutative hyperrings, *J. Algebr. Hyperstrucres Log. Algebr.*, vol.1, no.2, pp.23-30, 2020.

[13] H. Yazarli, B. Davvaz and D. Yilmaz, Generalized centroid of hyperrings, *Discuss. Math. – Gen. Algebra Appl.*, vol.42, no.1, pp.5-16, 2022.

[14] P. Ghiasvand, M. Raeisi, S. Mirvakili and F. Farzalipour, On graded absorbing hyperideals in graded Krasner hyperrings, *J. Algebr. Hyperstrucres Log. Algebr.*, vol.3, no.3, pp.65-84, 2022.

[15] P. Corsini, Sur les semi-hypergroupes [On semi-hypergroups], *Atti Soc. Pelorit. Sci. Fis. Math. Nat.*, vol.26, pp.363-372, 1980 (in French).

[16] P. Corsini, Sur les semi-hypergroupes complets et les groupoides [Complete semi-hypergroups and groupoids], *Atti Soc. Pelorit. Sci. Fis. Mat. Natur.*, vol.26, no.4, pp.391-398, 1980 (in French).

[17] M. De Salvo and D. Freni, Cyclic semihypergroups and hypergroups, *Atti Sem. Mat. Fis. Univ. Modena*, vol.30, no.1, pp.44-59, 1981 (in Italian).

[18] P. Bonansinga and P. Corsini, On semihypergroup and hypergroup homomorphisms, *Boll. Un. Mat. Ital. B.*, vol.6, no.1, pp.717-727, 1982 (in Italian).

[19] N. Kehayopulu, On hypersemigroups, *Pure Math. Appl.*, vol.25, no.2, pp.151-156, 2015.

[20] G. Kudryavtseva and V. Mazorchuk, On multisemigroups, *Port. Math.*, vol.72, no.1, pp.47-80, 2015.

[21] M. Gutan, On the transitivity of the relation β in semihypergroups, *Rend. Circ. Mat. Palermo*, vol.45, no.2, pp.189-200, 1996.

[22] D. Freni, A note on the core of a hypergroup and on the β^* transitive closure of β, *Ann. Mat. Pura Appl.*, vol.8, pp.153-156, 1991.

[23] B. Davvaz, Some results on congruences on semihypergroups, *Bull. Malays. Math. Sci. Soc.*, vol.23, no.1, pp.53-58, 2000.

[24] H. M. Jafarabadi, N. H. Sarmin and M. R. Molaei, Completely simple and regular semihypergroups, *Bull. Malays. Math. Sci. Soc.*, vol.35, no.2, pp.335-343, 2012.

[25] A. Hasankhani, Ideals in a semihypergroup and Green's relations, *Ratio Mathematica*, vol.13, no.1, pp.29-36, 1999.

[26] K. Hila, B. Davvaz and K. Naka, On quasi-hyperideals in semihypergroups, *Commun. Algebra*, vol.39, no.11, pp.4183-4194, 2011.

[27] P. Corsini, M. Shabir and T. Mahmood, Semisimple semihypergroups in terms of hyperideals and fuzzy hyperideals, *Iran. J. Fuzzy Syst.*, vol.8, no.1, pp.95-111, 2011.

[28] S. Lekkoksung, On weakly semi-prime hyperideals in semihypergroups, *Int. J. Algebra*, vol.6, nos.13-16, pp.613-616, 2012.

[29] S. Lekkoksung, On left, right weakly prime hyperideals on semihypergroups, *Int. J. Contemp. Math. Sci.*, vol.7, nos.21-24, pp.1193-1197, 2012.

[30] T. Changphas and B. Davvaz, Bi-hyperideals and quasi-hyperideals in ordered semihypergroups, *Ital. J. Pure Appl. Math.*, vol.35, pp.493-508, 2015.

[31] B. Pibaljommee and B. Davvaz, Characterizations of (fuzzy) bi-hyperideals in ordered semihypergroups, *J. Intell. Fuzzy Syst.*, vol.28, no.5, pp.2141-2148, 2015.

[32] J. Tang, B. Davvaz and Y. F. Luo, A study on fuzzy interior hyperideals in ordered semihypergroups, *Ital. J. Pure Appl. Math.*, vol.36, pp.125-146, 2016.

[33] N. Tipachot and B. Pibaljommee, Fuzzy interior hyperideals in ordered semihypergroups, *Ital. J. Pure Appl. Math.*, vol.36, pp.859-870, 2016.

[34] S. Lekkoksung and N. Lekkoksung, On interval valued intuitionistic fuzzy hyperideals of ordered semihypergroups, *Korean J. Math.*, vol.28, no.4, pp.753-774, 2020.

[35] J. Mekwian, S. Lekkoksung and N. Lekkoksung, Characterizing ordered semihypergroups by using generalized int-soft hyperideals, *Int. J. Math. Comput. Sci.*, vol.17, no.1, pp.143-161, 2022.

[36] J. Mekwian, S. Lekkoksung and N. Lekkoksung, Characterizations of regularities in ordered semihypergroups in terms of some generalized union soft hyperideals, *J. Math. Computer Sci.*, vol.26, no.3, pp.309-321, 2022.

[37] S. Lekkoksung, A. Iampan and N. Lekkoksung, On ideal elements of partially ordered semigroups with the greatest element, *International Journal of Innovative Computing, Information and Control*, vol.18, no.6, pp.1941-1955, 2022.

[38] M. Kondo and N. Lekkoksung, On intra-regular ordered Γ-semihypergroups, *Int. J. Math. Anal.*, vol.7, no.28, pp.1379-1386, 2013.

[39] H. Sanpan, N. Lekkoksung, S. Lekkoksung and W. Samormob, Characterizations of some regularities of ordered Γ-semihypergroups in terms of interval-valued Q-fuzzy Γ-hyperideals, *International Journal of Innovative Computing, Information and Control*, vol.17, no.4, pp.1391-1400, 2021.

[40] N. Kehayopulu, On hypersemigroups, *Turk. J. Math.*, vol.46, no.4, pp.1580-1618, 2022.

[41] N. Kehayopulu, Adjunction identity to hypersemigroups, *Turk. J. Math.*, vol.46, no.7, pp.2834-2853, 2022.

[42] N. Kehayopulu, Hypersemigroups and fuzzy hypersemigroups, *Eur. J. Pure Appl. Math.*, vol.10, no.5, pp.929-945, 2017.

[43] N. Kehayopulu, How we pass from semigroups to hypersemigroups, *Lobachevskii J. Math.*, vol.39, no.1, pp.121-128, 2018.

[44] S. Hobanthad, On 0-minimal $(0, 2)$-bi-hyperideal of semihypergroups, *Thai J. Math.*, vol.19, no.2, pp.399-405, 2021.

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