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SEMIGROUPS OF AN INDUCTIVE COMPOSITION OF TREE LANGUAGES

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Abstract. *Let $W_\tau(X_n)$ denote the set of all n-ary terms of type τ. Each element of $\mathcal{P}(W_\tau(X_n))$ is called a tree language. Let $W_\tau^r(X_n)$ be a subset of $W_\tau(X_n)$ which contains all n-ary terms of type τ except all proper subterms of a fixed term r. With an r-inductive product, $W_\tau^r(X_n)$ forms a semigroup of an inductive composition of term. The generalization of such operation on tree languages is called an r-inductive product of tree languages. This operation is not associative on $\mathcal{P}(W_\tau(X_n))$, but on its subset $\mathcal{P}(W_\tau^r(X_n))$. In this paper, we define a generalization of an inductive composition of terms, called an inductive composition of tree languages and its corresponding operation, named an r-inductive product of tree languages, construct a semigroup of languages with such inductive product and study the algebraic structures of such semigroup including idempotent elements, regular elements, and Green's relations.*
Keywords: Tree languages, Inductive composition of tree languages, Inductive product of tree languages, Idempotent elements, Regular elements, Green's relations

1. **Introduction.** An algebra is a base set or carrier set with particular operations defined on it. A semigroup is an algebra with a binary operation satisfying associativity. It is one of the most interesting topics in studying algebra and other fields such as wording and automata theory (see e.g., [3, 5]). The related concepts which are usually studied in semigroup include idempotent elements and regular elements. In 1951, Green [6] introduced equivalent relations in semigroup theory. Later, these relations are well-known in developing semigroup theory and are called Green's relations. A recent study of semigroups can be found in [10, 11, 14].

The notion of terms is a significant concept in universal algebra. A set of terms is called a tree language. In 1960s, Büchi and Wright [15] were the first who studied about theory of tree languages. Then in 1984, Gécseg and Steinby [5] introduced some main operations on tree languages such as the z-product and z-product iteration. In 2006, Denecke and Glubudom [1] introduced the superposition $\hat{S}_g^n$ which was used to represent the z-product as $\cdot_{x_i}$ later by Denecke and Sarasit in 2011 [2]. Then Denecke and Sarasit discovered a semigroup of tree languages with the operation $\cdot_{x_i}$, characterized idempotent elements and regular elements, and also observed the Green's relations $\mathcal{L}$ and $\mathcal{R}$.

An inductive composition is an operation the set of terms, which was first introduced by Shtrakov [13] in 2007. In [9], Kitpratyakul and Pibaljommee defined inductive product, called r-inductive product using inductive composition of Shtrakov's. Furthermore, they constructed a semigroup of such product of terms, studied idempotent elements and

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regular elements, and gave some properties of all Green's relations. Recently, they also studied substructures and ideal characterization of such semigroup in [7, 8], respectively.

In this paper, we study the properties of a binary operation on tree languages generalized from the inductive composition on terms in [13] and its corresponding operation, an r-inductive product of tree languages, which is a generalization of the binary operation introduced in [2]. The power set of the set of all n-ary terms along with the binary operation in [2] is a semigroup. However, that set together with our new inductive product is just a groupoid. Therefore, we construct a new semigroup of languages with an inductive product of tree languages and finally, we investigate both idempotent elements and regular elements of the semigroup and give its characterization of all types of Green's relations.

2. **Preliminaries.** In this section, we provide some definitions and lemmas without proof, which can be generalized or applied in the later sections. Throughout this paper, we let $\mathbb{N}$ denote the set of all positive integers.

Definition 2.1. *Let $X_n = \{x_1, \ldots, x_n\}$ be an n-element set of variables and $(f_i)_{i\in I}$ be an indexed set of n_i-ary operation symbols of type $\tau = (n_i)_{i\in I}$, where $n, n_i \in \mathbb{N}$. The n-ary terms of type τ are inductively defined as follows:*

(i) Every variable $x_i \in X_n$ is an n-ary term.

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

We denote $W_\tau(X_n)$ be the set of all n-ary terms of type τ, which is the smallest set containing all variables in X_n and closed under finite operation of (ii). For a countably infinite set $X = \{x_1, x_2, \ldots\}$, $W_\tau(X)$ denotes the set of all terms of type τ, where $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$ (see [5]).

The notion of subterms of a given term t which was studied in [5, 12, 13] is required to understand an inductive composition of terms.

Definition 2.2. *Let $t \in W_\tau(X_n)$ be an n-ary term of type τ. The set $\operatorname{sub}(t)$ of all subterms of t is inductively defined as follows.*

(i) if $t \in X_n$, then $\operatorname{sub}(t) = \{t\}$;

(ii) if $f_i(t_1, \ldots, t_{n_i})$, then $\operatorname{sub}(t) = \{t\} \cup \operatorname{sub}(t_1) \cup \cdots \cup \operatorname{sub}(t_{n_i})$.

The next lemma is an important result from [9], which is used to reach our results.

Lemma 2.1. *Let $s, t \in W_\tau(X_n)$. Then $t \in \operatorname{sub}(s)$ if and only if $\operatorname{sub}(t) \subseteq \operatorname{sub}(s)$.*

Shtrakov [13] introduced the concept of inductive composition of terms as follows.

Definition 2.3. *Let $r, s, t \in W_\tau(X_n)$ be n-ary terms of type τ. The inductive composition $t(r \leftarrow s)$ is inductively defined as follows:*

(i) if $r \notin \operatorname{sub}(t)$, then $t(r \leftarrow s) = t$;

(ii) if $t = r$, then $t(r \leftarrow s) = s$;

(iii) if $t = f_i(t_1, \ldots, t_{n_i}) \neq r$ and $r \in \operatorname{sub}(t)$, then

$$t(r \leftarrow s) = f_i(t_1(r \leftarrow s), \ldots, t_{n_i}(r \leftarrow s)).$$

Terms can be demonstrated as several characteristics. Among them, a tree diagram is one of the displays of terms. Minimum depth is a well-known complexity measure of terms, which was defined in [4].

Definition 2.4. *The minimum depth of a term t, denoted by $\operatorname{mindepth}(t)$, is the length of the shortest path from the root to a vertex of a leaf in the tree, and is defined as follows:*

(i) *if* t *is a variable, then* $\mathrm{mindepth}(t) = 0$*;*
(ii) *if* $t = f_i(t_1, \ldots, t_{n_i})$*, then* $\mathrm{mindepth}(t) = 1 + \min\{\mathrm{mindepth}(t_j)|1 \leq j \leq n_i\}$*.*

The power set $\mathcal{P}(W_\tau(X_n))$ contains all subsets of $W_\tau(X_n)$. Each element of $\mathcal{P}(W_\tau(X_n))$ is called a tree language. In other words, a set of terms is a tree language.

It is a routine to extend the concepts on terms to the collection of terms, so called tree languages. Hence, we can generalize, in this work, the new notions on tree languages such as subterms of tree languages and an inductive composition of tree languages. For the measurement of terms, we introduce the notion involving with minimum depth of an arbitrary term, which is the shortest path from the root of a given term t to its subterm r and is called the minimum depth of a term t to a subterm r.

3. **The Semigroup $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$.** In this section, we construct a semigroup of tree languages using an inductive composition of tree languages. First, we characterize the inductive composition of tree languages and extend to a particular binary operation, called *r-inductive product*. In order to characterize the inductive composition of tree languages, we introduce the generalized subterms of tree languages, which is an important concept to characterize such operation.

Definition 3.1. *For a tree language* $A \in \mathcal{P}(W_\tau(X_n))$*, the set of all subterms of* A*, denoted by* $\mathrm{Sub}(A)$*, is defined by* $\mathrm{Sub}(A) = \bigcup_{t\in A} \mathrm{sub}(t)$*.*

We observe that $\mathrm{Sub}(\{t\}) = \mathrm{sub}(t)$ and from Lemma 2.1, we have that $t \in \mathrm{Sub}(A)$ if and only if $\mathrm{Sub}(\{t\}) \subseteq \mathrm{Sub}(A)$.

We can extend the notion of inductive composition on terms to tree languages as the following definition.

Definition 3.2. *Let* $r, t \in W_\tau(X_n)$ *be n-ary terms of type* τ *and* $A, B \in \mathcal{P}(W_\tau(X_n))$ *be tree languages. The inductive composition* $A(r \leftarrow B)$ *is inductively defined as follows:*
(i) *if* $A = \{t\}$ *and* $r \notin \mathrm{sub}(t)$*, then* $A(r \leftarrow B) = A$*;*
(ii) *if* $A = \{t\}$ *and* $t = r$*, then* $A(r \leftarrow B) = B$*;*
(iii) *if* $A = \{t\}$*,* $t = f_i(t_1, \ldots, t_{n_i})$*,* $r \in \mathrm{sub}(t)$ *and* $t \neq r$*, we inductively assume that* $\{t_j\}(r \leftarrow B)$ *for each* $1 \leq j \leq n_i$ *are already defined, then*

$$A(r \leftarrow B) = \{f_i(s_1, \ldots, s_{n_i})|s_j \in \{t_j\}(r \leftarrow B), 1 \leq j \leq n_i\};$$

(iv) *if* A *is any arbitrary nonempty subset of* $W_\tau(X_n)$*, then* $A(r \leftarrow B) = \bigcup_{t\in A}\{t\}(r \leftarrow B)$*. If* A *or* B *is an empty set, we define* $A(r \leftarrow B) = \emptyset$*.*

By the definition of the inductive composition of tree languages, the language $A(r \leftarrow B)$ is obtained by replacing, for each term belonging to A, every occurrence of the subterm r by some terms in B.

Letting a fixed term be replaced, we directly use inductive composition of tree languages to define a new binary operation called an *r-inductive product* of tree languages.

Definition 3.3. *Let* $r \in W_\tau(X_n)$ *be a fixed n-ary term of type* τ *and* $A, B \in \mathcal{P}(W_\tau(X_n))$ *be tree languages. An r-inductive product of tree languages, denoted by* $\hat{\cdot}_r$*, is a mapping on* $\mathcal{P}(W_\tau(X_n))$ *defined by* $A \hat{\cdot}_r B = A(r \leftarrow B)$*.*

If $r = x_i \in X_n$, then we obtain the x_i-product, $\cdot_{x_i}$, which was defined in [2, 5]. Hence, the operation $\hat{\cdot}_r$ is a generalization of $\cdot_{x_i}$ on $\mathcal{P}(W_\tau(X_n))$. By the definition of $\hat{\cdot}_r$, the language $A \hat{\cdot}_r B$ is obtained by simultaneously replacing, for each term belonging to A, every occurrence of the subterm r by some terms in B. The next example shows such approach.

Example 3.1. *Let $\tau = (2, 1)$ with a binary operation f and a unary operation g. Let $r = f(x_1, x_2)$, $A = \{f(g(x_1), f(x_1, x_2)), g(f(x_1, x_2))\}$ and $B = \{g(x_2), f(x_2, x_3)\}$. Then*

$$\begin{aligned} A\,\hat{\cdot}_r B &= A(r \leftarrow B) \\ &= \{f(g(x_1), f(x_1, x_2))\}(f(x_1, x_2) \leftarrow B) \cup \{g(f(x_1, x_2))\}(f(x_1, x_2) \leftarrow B) \\ &= \{f(g(x_1), g(x_2)), f(g(x_1), f(x_2, x_3))\} \cup \{g(g(x_2)), g(f(x_2, x_3))\} \\ &= \{f(g(x_1), g(x_2)), f(g(x_1), f(x_2, x_3)), g(g(x_2)), g(f(x_2, x_3))\}. \end{aligned}$$

The following lemma of an r-inductive products provide some common properties of an r-inductive product on the power set $\mathcal{P}(W_\tau(X_n))$ which are beneficial in the later sections.

Lemma 3.1. *Let $r \in W_\tau(X_n)$ be a fixed term and $A, B \in \mathcal{P}(W_\tau(X_n))$ be tree languages.*

(i) If $r \in A$, then $B \subseteq A\,\hat{\cdot}_r B$.
(ii) If $s \in \{t\}\,\hat{\cdot}_r B$ and $r \in \mathrm{Sub}(\{t\})$, then there exists $b \in B$ such that $b \in \mathrm{Sub}(\{s\})$.
(iii) If $r \in \mathrm{Sub}(A)$, then $\mathrm{Sub}(B) \subseteq \mathrm{Sub}(A\,\hat{\cdot}_r B)$.
(iv) If $r \in \mathrm{Sub}(A)$ and $r \in \mathrm{Sub}(B)$, then $r \in \mathrm{Sub}(A\,\hat{\cdot}_r B)$. The converse is true if $r \in X_n$.

Proof:

(i) Assume that $r \in A$. Then $B = \{r\}\,\hat{\cdot}_r B \subseteq \bigcup_{t\in A}\{t\}\,\hat{\cdot}_r B = A\,\hat{\cdot}_r B$. Therefore, $B \subseteq A\,\hat{\cdot}_r B$.

(ii) Assume that $s \in \{t\}\,\hat{\cdot}_r B$ and $r \in \mathrm{Sub}(\{t\})$. If $t = r$, then $\{t\}\,\hat{\cdot}_r B = B$. It follows that $s \in B$ and so, $s = b$ for some $b \in B$. Thus, $b \in \mathrm{Sub}(\{s\})$. For $t = f_i(t_1, \ldots, t_{n_i}) \neq r$ and $r \in \mathrm{sub}(t)$, we inductively assume that if $s' \in \{t_j\}\,\hat{\cdot}_r B$ and $r \in \mathrm{Sub}(\{t_j\})$ for each $1 \leq j \leq n_i$, then there exists $b' \in B$ such that $b' \in \mathrm{Sub}(\{s'\})$. Since

$$r \in \mathrm{Sub}(\{t\}) = \{t\} \cup \mathrm{sub}(t_1) \cup \cdots \cup \mathrm{sub}(t_{n_i}),$$

we obtain that $r \in \mathrm{sub}(t_j)$ for some $j \in \{1, \ldots, n_i\}$. Then $r \in \mathrm{Sub}(\{t_j\})$ for some $j \in \{1, \ldots, n_i\}$. Since $s \in \{t\}\,\hat{\cdot}_r B = \{f_i(s_1, \ldots, s_{n_i}) | s_j \in \{t_j\}\,\hat{\cdot}_r B, 1 \leq j \leq n_i\}$, we have $s = f_i(s_1, \ldots, s_{n_i})$ for some $s_j \in \{t_j\}\,\hat{\cdot}_r B$, $1 \leq j \leq n_i$. Since $s_j \in \{t_j\}\,\hat{\cdot}_r B$ and $r \in \mathrm{Sub}(\{t_j\})$, by assumption, there exists $b_j \in B$ such that

$$b_j \in \mathrm{Sub}(\{s_j\}) \subseteq \mathrm{sub}(s) = \mathrm{Sub}(\{s\}).$$

(iii) Assume that $r \in \mathrm{Sub}(A)$. It is obvious if B is empty. We then prove by induction on A. If $A = \{t\}$ and $t = r$, then $A\,\hat{\cdot}_r B = B$. Therefore, $\mathrm{Sub}(B) \subseteq \mathrm{Sub}(B) = \mathrm{Sub}(A\,\hat{\cdot}_r B)$. For $A = \{t\}$, $t = f_i(t_1, \ldots, t_{n_i}) \neq r$, we inductively assume that if $r \in \mathrm{Sub}(\{t_j\})$, then $\mathrm{Sub}(B) \subseteq \mathrm{Sub}(\{t_j\}\,\hat{\cdot}_r B)$ for each $1 \leq j \leq n_i$. Since

$$r \in \mathrm{sub}(t) = \{t\} \cup \mathrm{sub}(t_1) \cup \cdots \cup \mathrm{sub}(t_{n_i}),$$

we have $r \in \mathrm{sub}(t_j) = \mathrm{Sub}(\{t_j\})$ for some $1 \leq j \leq n_i$. Note that

$$A\,\hat{\cdot}_r B = \{f_i(s_1, \ldots, s_{n_i}) | s_j \in \{t_j\}\,\hat{\cdot}_r B, 1 \leq j \leq n_i\}.$$

Then $\mathrm{Sub}(B) \subseteq \mathrm{Sub}(\{t_j\}\,\hat{\cdot}_r B) \subseteq \mathrm{Sub}(A\,\hat{\cdot}_r B)$. For any arbitrary nonempty subset A of $W_\tau(X_n)$, $\mathrm{Sub}(A) = \bigcup_{t\in A}\mathrm{Sub}(\{t\})$. Then $r \in \mathrm{Sub}(\{t_0\})$ for some $t_0 \in A$. It follows that $\mathrm{Sub}(B) \subseteq \mathrm{Sub}(\{t\}\,\hat{\cdot}_r B) \subseteq \bigcup_{t\in A}\mathrm{Sub}(\{t\}\,\hat{\cdot}_r B) = \mathrm{Sub}(A\,\hat{\cdot}_r B)$.

(iv) Assume that $r \notin \mathrm{Sub}(A\,\hat{\cdot}_r B)$ and $r \in \mathrm{Sub}(A)$. We show that $r \notin \mathrm{Sub}(B)$. By (iii), we obtain that $\mathrm{Sub}(B) \subseteq \mathrm{Sub}(A\,\hat{\cdot}_r B)$. Hence, $r \notin \mathrm{Sub}(B)$. Conversely, assume that $r \in X_n$ such that $r \notin \mathrm{Sub}(A)$ or $r \notin \mathrm{Sub}(B)$. We prove that $r \notin \mathrm{Sub}(A\,\hat{\cdot}_r B)$. It is clear whenever A or B is empty. Now, we additionally assume that A and B are nonempty. If $r \notin \mathrm{Sub}(A)$, then $A\,\hat{\cdot}_r B = A$. Thus, $r \notin \mathrm{Sub}(A\,\hat{\cdot}_r B)$. For $r \notin \mathrm{Sub}(B)$, we prove by induction on A. If $A = \{t\}$ and $t = r$, then $A\,\hat{\cdot}_r B = B$. Thus, $r \notin \mathrm{Sub}(A\,\hat{\cdot}_r B)$. For $A = \{t\}$, $t = f_i(t_1, \ldots, t_{n_i}) \neq r$ and $r \in \mathrm{sub}(t)$, we inductively assume that $r \notin \mathrm{Sub}(\{t_j\}\,\hat{\cdot}_r B)$ for each $j \in \{1, \ldots, n_i\}$. Note that

$A\hat{\cdot}_r B = \{f_i(s_1,\ldots,s_{n_i})|s_j \in \{t_j\}\hat{\cdot}_r B, 1 \le j \le n_i\}$. Then $r \notin \text{sub}(s_j)$ for each $j \in \{1,\ldots,n_i\}$. If $r \in \text{sub}(f_i(s_1,\ldots,s_{n_i}))$ for some $f_i(s_1,\ldots,s_{n_i}) \in A\hat{\cdot}_r B$, then $r = f_i(s_1,\ldots,s_{n_i})$, which contradict to $r \in X_n$. Therefore, $r \notin \text{sub}(f_i(s_1,\ldots,s_{n_i}))$ for each $f_i(s_1,\ldots,s_{n_i}) \in A\hat{\cdot}_r B$. Hence, $r \notin \text{Sub}(A\hat{\cdot}_r B)$. For any arbitrary nonempty subset A of $W_\tau(X_n)$, we have $\text{Sub}(A\hat{\cdot}_r B) = \bigcup_{t\in A}\text{Sub}(\{t\}\hat{\cdot}_r B)$. Suppose that $r \in \text{Sub}(A\hat{\cdot}_r B)$. Then $r \in \text{Sub}(\{t_0\}\hat{\cdot}_r B)$ for some $t_0 \in A$. It follows that $r \in \text{Sub}(\{t_0\})$ and $r \in \text{Sub}(B)$. Therefore, $r \in \bigcup_{t\in A}\text{Sub}(\{t\}) = \text{sub}(A)$ and $r \in \text{Sub}(B)$. □

It was discovered in [2] that $(\mathcal{P}(W_\tau(X_n)), \cdot_{x_i})$ is a semigroup of the product of tree languages. For the operation $\hat{\cdot}_r$, the algebra $(\mathcal{P}(W_\tau(X_n)), \hat{\cdot}_r)$ is not necessary to be a semigroup as $\hat{\cdot}_r$ is not associative on such base set. Next example illustrates such aspect.

Example 3.2. *Let $\tau = (2,1)$ with a binary operation f and a unary operation g. Let $r = f(x_1, g(x_3))$, $A = \{g(f(x_1,x_2)), f(x_1, f(x_1, g(x_3)))\}$, $B = \{g(x_3)\}$ and $C = \{f(x_1,x_2)\}$. Then*

$$\begin{aligned}(A\hat{\cdot}_r B)\hat{\cdot}_r C &= (\{g(f(x_1,x_2)), f(x_1,f(x_1,g(x_3)))\}\hat{\cdot}_r\{g(x_3)\})\hat{\cdot}_r\{f(x_1,x_2)\}\\ &= (\{g(f(x_1,x_2)), f(x_1,g(x_3))\})\hat{\cdot}_r\{f(x_1,x_2)\}\\ &= \{g(f(x_1,x_2)), f(x_1,x_2)\}.\end{aligned}$$

However,

$$\begin{aligned}A\hat{\cdot}_r(B\hat{\cdot}_r C) &= \{g(f(x_1,x_2)), f(x_1,f(x_1,g(x_3)))\}\hat{\cdot}_r(\{g(x_3)\}\hat{\cdot}_r\{f(x_1,x_2)\})\\ &= \{g(f(x_1,x_2)), f(x_1,f(x_1,g(x_3)))\}\hat{\cdot}_r\{g(x_3)\}\\ &= \{g(f(x_1,x_2)), f(x_1,g(x_3))\}.\end{aligned}$$

As a result, $A\hat{\cdot}_r(B\hat{\cdot}_r C) \ne A\hat{\cdot}_r(B\hat{\cdot}_r C)$.

We denote $\mathcal{P}(W_\tau^r(X_n))$ to be the power set of $W_\tau^r(X_n)$, where $W_\tau^r(X_n) := W_\tau(X_n)\setminus(\text{sub}(r)\setminus\{r\})$, defined in [9]. It is natural to find a restriction to establish a semigroup from the algebra $(\mathcal{P}(W_\tau(X_n)), \hat{\cdot}_r)$. So, we eliminate some elements from the former base set which lacks associativity to the new one, that is $\mathcal{P}(W_\tau^r(X_n))$. We then present some essential properties of our binary operation on the power set $\mathcal{P}(W_\tau^r(X_n))$ in next lemma. These will be applied in getting some required conditions in semigroup construction.

Lemma 3.2. *Let $r \in W_\tau(X_n)$ be a fixed term and $A, B \in \mathcal{P}(W_\tau^r(X_n))$ be tree languages.*
(i) If $r \in A\hat{\cdot}_r B$, then $r \in A$ and $r \in B$.
(ii) $r \in \text{Sub}(A)$ and $r \in \text{Sub}(B)$ if and only if $r \in \text{Sub}(A\hat{\cdot}_r B)$.

Proof:

(i) Suppose that $r \in A\hat{\cdot}_r B$. It is obvious that A and B are nonempty. We prove that $r \in A$ and $r \in B$ by induction on A. If $A = \{t\}$ and $r \notin \text{sub}(t)$, then $r \in A\hat{\cdot}_r B = A$. It follows that $t = r$, which is a contradiction. If $A = \{t\}$ and $t = r$, then $r \in A$ and $r \in B$. For $A = \{t\}$, $t = f_i(t_1,\ldots,t_{n_i}) \ne r$ and $r \in \text{sub}(t)$, we inductively assume that if $r \in \{t_j\}\hat{\cdot}_r B$, then $r \in \{t_j\}$ and $r \in B$ for each $1 \le j \le n_i$. Note that $r \in A\hat{\cdot}_r B = \{f_i(s_1,\ldots,s_{n_i})|s_j \in \{t_j\}\hat{\cdot}_r B, 1 \le j \le n_i\}$. Since $r \in \text{sub}(t)$ and $t \ne r$, we have $r \in \text{sub}(t_j)$ for some $1 \le j \le n_i$. Then $r \in \text{Sub}(\{t_j\})$. Thus, $r = f_i(s_1,\ldots,s_{n_i})$ for some $f_i(s_1,\ldots,s_{n_i}) \in A\hat{\cdot}_r B$. By Lemma 3.1 (ii), there exists $b \in B$ such that $b \in \text{Sub}(\{s_j\})$. Then $\text{sub}(b) \subseteq \text{sub}(s_j) \subset \text{sub}(r)$. Therefore, $b \notin W_\tau^r(X_n)$, which is a contradiction. For any arbitrary nonempty subset A of $W_\tau^r(X_n)$, $A\hat{\cdot}_r B = \bigcup_{t\in A}\{t\}\hat{\cdot}_r B$. Since $r \in A\hat{\cdot}_r B$, we have $r \in \{t\}\hat{\cdot}_r B$ for some $t \in A$. It follows that $r \in \{t\}$ and $r \in B$. Therefore, $r \in A$ and $r \in B$.

(ii) Suppose that $r \notin \mathrm{Sub}(A)$ or $r \notin \mathrm{Sub}(B)$. We show that $r \notin \mathrm{Sub}(A \hat{\cdot}_r B)$. This is immediate when A or B is empty. We now assume that A and B are nonempty. If $r \notin \mathrm{Sub}(A)$, then $A \hat{\cdot}_r B = A$. Thus, $r \notin \mathrm{Sub}(A \hat{\cdot}_r B)$. For $r \notin \mathrm{Sub}(B)$, we prove by induction on A. If $A = \{t\}$ and $t = r$, then $A \hat{\cdot}_r B = B$. Thus, $r \notin \mathrm{Sub}(A \hat{\cdot}_r B)$. For $A = \{t\}$, $t = f_i(t_1, \ldots, t_{n_i}) \neq r$ and $r \in \mathrm{sub}(t)$, we inductively assume that $r \notin \mathrm{Sub}(\{t_j\} \hat{\cdot}_r B)$ for each $j \in \{1, \ldots, n_i\}$. Note that

$$A \hat{\cdot}_r B = \{f_i(s_1, \ldots, s_{n_i}) | s_j \in \{t_j\} \hat{\cdot}_r B, 1 \leq j \leq n_i\}.$$

Since $r \notin \mathrm{Sub}(\{t_j\} \hat{\cdot}_r B)$, we have $r \notin \mathrm{sub}(s_j)$ for each $j \in \{1, \ldots, n_i\}$. Suppose that $r \in \mathrm{sub}(f_i(s_1, \ldots, s_{n_i}))$ for some $f_i(s_1, \ldots, s_{n_i}) \in A \hat{\cdot}_r B$. Then $r = f_i(s_1, \ldots, s_{n_i})$ and so, $r \in A \hat{\cdot}_r B = \{t\} \hat{\cdot}_r B$. Since $r \in \mathrm{sub}(t) = \mathrm{Sub}(\{t\})$, Lemma 3.1 (ii) implies $b \in \mathrm{Sub}(\{r\}) = \mathrm{sub}(r)$ for some $b \in B$. Since $r \notin \mathrm{Sub}(B)$, we get $b \neq r$ and so, $b \in \mathrm{sub}(r) \setminus \{r\}$, which is a contradiction. Thus, $r \notin \mathrm{sub}(f_i(s_1, \ldots, s_{n_i}))$ for each $f_i(s_1, \ldots, s_{n_i}) \in A \hat{\cdot}_r B$. Hence, $r \notin \mathrm{Sub}(A \hat{\cdot}_r B)$. For any arbitrary nonempty subset A of $W_\tau^r(X_n)$, we have $\mathrm{Sub}(A \hat{\cdot}_r B) = \bigcup_{t \in A} \mathrm{Sub}(\{t\} \hat{\cdot}_r B)$. Suppose that $r \in \mathrm{Sub}(A \hat{\cdot}_r B)$. Then $r \in \mathrm{Sub}(\{t_0\} \hat{\cdot}_r B)$ for some $t_0 \in A$. It follows that $r \in \mathrm{Sub}(\{t_0\})$ and $r \in \mathrm{Sub}(B)$. Hence, $r \in \bigcup_{t \in A} \mathrm{Sub}(\{t\}) = \mathrm{sub}(A)$ and $r \in \mathrm{Sub}(B)$. The converse is directly true by Lemma 3.1 (iv). □

The following lemma shows that the operation $\hat{\cdot}_r$ is associative on $\mathcal{P}(W_\tau^r(X_n))$.

Lemma 3.3. *Let $r \in W_\tau(X_n)$ be a fixed term and $A, B, C \in \mathcal{P}(W_\tau^r(X_n))$ be tree languages. Then $(A \hat{\cdot}_r B) \hat{\cdot}_r C = A \hat{\cdot}_r (B \hat{\cdot}_r C)$.*

Proof: Let $A, B, C \in \mathcal{P}(W_\tau^r(X_n))$. If one of A, B or C is empty, then we are done. Now, suppose that A, B and C are nonempty. We prove by induction on A. If $A = \{t\}$ and $r \notin \mathrm{sub}(t)$, then $(A \hat{\cdot}_r B) \hat{\cdot}_r C = A \hat{\cdot}_r C = A = A \hat{\cdot}_r (B \hat{\cdot}_r C)$. If $A = \{t\}$ and $t = r$, then $(A \hat{\cdot}_r B) \hat{\cdot}_r C = B \hat{\cdot}_r C = A \hat{\cdot}_r (B \hat{\cdot}_r C)$. For $A = \{t\}$, $t = f_i(t_1, \ldots, t_{n_i}) \neq r$ and $r \in \mathrm{sub}(t)$, we inductively assume that $(\{t_j\} \hat{\cdot}_r B) \hat{\cdot}_r C = \{t_j\} \hat{\cdot}_r (B \hat{\cdot}_r C)$ for each $j \in \{1, \ldots, n_i\}$. Note that $A \hat{\cdot}_r B = \{f_i(s_1, \ldots, s_{n_i}) | s_j \in \{t_j\} \hat{\cdot}_r B, 1 \leq j \leq n_i\}$. Since $t \neq r$ and $A = \{t\}$, Lemma 3.2 (i) implies that $r \notin A \hat{\cdot}_r B$. Let $u = f_i(s_1, \ldots, s_{n_i})$ for some $f_i(s_1, \ldots, s_{n_i}) \in A \hat{\cdot}_r B$. If $r \notin \mathrm{sub}(u)$, then $\{u\} \hat{\cdot}_r C = \{u\}$. Since $r \notin \mathrm{sub}(u)$, there exists $\{b_1, \ldots, b_k\} \subseteq B$ such that $r \notin \mathrm{Sub}(\{b_1, \ldots, b_k\})$ and thus, $\{b_1, \ldots, b_k\} \hat{\cdot}_r C = \{b_1, \ldots, b_k\}$. Then we have $u \in A \hat{\cdot}_r \{b_1, \ldots, b_k\} = A \hat{\cdot}_r (\{b_1, \ldots, b_k\} \hat{\cdot}_r C) \subseteq A \hat{\cdot}_r (B \hat{\cdot}_r C)$. If $r \in \mathrm{sub}(u)$, then

$$\begin{aligned}
\{u\} \hat{\cdot}_r C &= \{f_i(v_1, \ldots, v_{n_i}) | v_j \in \{s_j\} \hat{\cdot}_r C, 1 \leq j \leq n_i\} \\
&\subseteq \{f_i(v_1, \ldots, v_{n_i}) | v_j \in (\{t_j\} \hat{\cdot}_r B) \hat{\cdot}_r C, 1 \leq j \leq n_i\} \\
&= \{f_i(v_1, \ldots, v_{n_i}) | v_j \in \{t_j\} \hat{\cdot}_r (B \hat{\cdot}_r C), 1 \leq j \leq n_i\} \\
&= A \hat{\cdot}_r (B \hat{\cdot}_r C).
\end{aligned}$$

Thus, $(A \hat{\cdot}_r B) \hat{\cdot}_r C \subseteq A \hat{\cdot}_r (B \hat{\cdot}_r C)$. Now, consider

$$\begin{aligned}
A \hat{\cdot}_r (B \hat{\cdot}_r C) &= \{f_i(v_1, \ldots, v_{n_i}) | v_j \in \{t_j\} \hat{\cdot}_r (B \hat{\cdot}_r C), 1 \leq j \leq n_i\} \\
&= \{f_i(v_1, \ldots, v_{n_i}) | v_j \in (\{t_j\} \hat{\cdot}_r B) \hat{\cdot}_r C, 1 \leq j \leq n_i\} \\
&\subseteq (A \hat{\cdot}_r B) \hat{\cdot}_r C.
\end{aligned}$$

Hence, $(A \hat{\cdot}_r B) \hat{\cdot}_r C = A \hat{\cdot}_r (B \hat{\cdot}_r C)$. For any arbitrary nonempty subset A of $W_\tau^r(X_n)$,

$$(A \hat{\cdot}_r B) \hat{\cdot}_r C = \left(\bigcup_{t \in A} \{t\} \hat{\cdot}_r B \right) \hat{\cdot}_r C = \bigcup_{t \in A} (\{t\} \hat{\cdot}_r B) \hat{\cdot}_r C = \bigcup_{t \in A} \{t\} \hat{\cdot}_r (B \hat{\cdot}_r C) = A \hat{\cdot}_r (B \hat{\cdot}_r C).$$

□

To construct a semigroup, associativity and closeness are significantly needed. The above lemma is mainly used to exhibit associativity of our main operation on the power set $\mathcal{P}(W_\tau^r(X_n))$. Hence, to notify that the algebra $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$ is a semigroup, we remain to prove that our carrier set, $\mathcal{P}(W_\tau^r(X_n))$ is closed under the operation $\hat{\cdot}_r$. The next theorem shall play an important role in assisting the algebra $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$ to be actually a semigroup.

Theorem 3.1. *Let $r \in W_\tau(X_n)$ be a fixed term. Then the power set $\mathcal{P}(W_\tau^r(X_n))$ is a subset of $\mathcal{P}(W_\tau(X_n))$ and is closed under the operation $\hat{\cdot}_r$.*

Proof: Let $A, B \in \mathcal{P}(W_\tau^r(X_n))$. If one of A or B is empty, then $A \hat{\cdot}_r B = \emptyset \in \mathcal{P}(W_\tau^r(X_n))$. Next, assume that A and B are nonempty. We show that $A \hat{\cdot}_r B \in \mathcal{P}(W_\tau^r(X_n))$ by induction on A. If $A = \{t\}$ and $r \notin \text{sub}(t)$, then $A \hat{\cdot}_r B = A \in \mathcal{P}(W_\tau^r(X_n))$. If $A = \{t\}$ and $t = r$, then $A \hat{\cdot}_r B = B \in \mathcal{P}(W_\tau^r(X_n))$. For $A = \{t\}$, $t = f_i(t_1, \ldots, t_{n_i})$, $r \in \text{sub}(t)$ and $t \neq r$, we inductively assume that $\{t_j\} \hat{\cdot}_r B$ for each $1 \leq j \leq n_i$ are already defined in $\mathcal{P}(W_\tau^r(X_n))$. Then $A \hat{\cdot}_r B = \{f_i(s_1, \ldots, s_{n_i}) | s_j \in \{t_j\} \hat{\cdot}_r B, 1 \leq j \leq n_i\}$. Now, suppose that $f_i(s_1, \ldots, s_{n_i}) \in \text{sub}(r) \setminus \{r\}$ for some $f_i(s_1, \ldots, s_{n_i}) \in A \hat{\cdot}_r B$. Then $\text{sub}(f_i(s_1, \ldots, s_{n_i})) \subseteq \text{sub}(r) \setminus \{r\}$ for each $1 \leq j \leq n_i$. Thus, for each $1 \leq j \leq n_i$, $s_j \in \text{sub}(r) \setminus \{r\}$, which is a contradiction, since for each $1 \leq j \leq n_i$, $s_j \in W_\tau^r(X_n)$. Hence, $A \hat{\cdot}_r B \in \mathcal{P}(W_\tau^r(X_n))$. If A is an arbitrary nonempty subset of $W_\tau^r(X_n)$, then $A \hat{\cdot}_r B = \bigcup_{t \in A} \{t\} \hat{\cdot}_r B \in \mathcal{P}(W_\tau^r(X_n))$. □

Eventually, we see that the operation $\hat{\cdot}_r$ is associative on the power set $\mathcal{P}(W_\tau^r(X_n))$ by Lemma 3.3 and by Theorem 3.1, a semigroup $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$ is constructed. Moreover, we observe that it is a monoid where $\{r\} \in \mathcal{P}(W_\tau^r(X_n))$ works as an identity, that is $\{r\} \hat{\cdot}_r A = \{r\} = A \hat{\cdot}_r \{r\}$ for each $A \in \mathcal{P}(W_\tau^r(X_n))$. Hence, we are now ready to investigate the algebraic properties of the semigroup which we have just discovered in the next sections.

4. **Idempotent and Regular Elements of $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$.** In this section, we characterize idempotent elements and regular elements of the semigroup $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$. An element $A \in (\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$ is called *idempotent* provided that $A \hat{\cdot}_r A = A$ and A is called *regular* provided that there is $B \in (\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$ such that $A \hat{\cdot}_r B \hat{\cdot}_r A = A$.

Firstly, we introduce the notion of the shortest path from the root of a term t to a subterm r, which is called a *minimum depth* of a term t to a subterm r and is denoted by $\text{mindepth}_r(t)$. We denote ∞ for which $n < \infty$, for each $n \in \mathbb{N}$.

Definition 4.1. *Let $r, t \in W_\tau(X_n)$. The minimum depth of a term t to a subterm r, $\text{mindepth}_r(t)$, is inductively defined as follows:*

(i) if $r \notin \text{sub}(t)$, then $\text{mindepth}_r(t) = \infty$;
(ii) if $t = r$, then $\text{mindepth}_r(t) = 0$;
(iii) if $t = f_i(t_1, \ldots, t_{n_i}) \neq r$ and $r \in \text{sub}(t)$, then

$$\text{mindepth}_r(t) = 1 + \min\{\text{mindepth}_r(t_j) | 1 \leq j \leq n_i\}.$$

In order to prove the next result, we require the following sets. For each $A \in \mathcal{P}(W_\tau^r(X_n))$, we define

$\bar{A} := \{a \in A | r \in \text{Sub}(\{a\})\}$;
$A^0 := \{a \in A | r \notin \text{Sub}(\{a\})\}$;
$A_k := \{a \in A | \text{mindepth}_r(a) = k\}$.

We remark that $A = \bar{A} \cup A^0$ and for each $k \in \mathbb{N}$, every element $a \in A_k$ satisfies $r \in \text{Sub}(\{a\})$.

Lemma 4.1. *Let $r \in W_\tau(X_n)$, $t, b \in W_\tau^r(X_n)$ and $A \in \mathcal{P}(W_\tau^r(X_n))$. If $t \in \{b\} \hat{\cdot}_r A$, then*

$$\text{mindepth}_r(t) \geq \text{mindepth}_r(b) + \text{mindepth}_r(a)$$

for some $a \in A$.

Proof: It is clear that A is nonempty. We prove by induction on the structure of b. If $r \notin \mathrm{sub}(b)$, then $\mathrm{mindepth}_r(b) = \infty$ and $\{b\} \hat{\cdot}_r A = \{b\}$. Thus, $t = b$ and $\mathrm{mindepth}_r(t) \geq \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a)$ for some $a \in A$. If $b = r$, $\mathrm{mindepth}_r(b) = 0$ and $\{b\} \hat{\cdot}_r A = A$. Thus, $t = a$ for some $a \in A$. It follows that $\mathrm{mindepth}_r(t) = \mathrm{mindepth}_r(a)$ for some $a \in A$. Hence, $\mathrm{mindepth}_r(t) = \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a)$ for some $a \in A$. For $b = f_i(b_1, \ldots, b_{n_i}) \neq r$ and $r \in \mathrm{sub}(b)$, we inductively assume that for each $1 \leq j \leq n_i$, if $c_j \in \{b_j\} \hat{\cdot}_r A$, then $\mathrm{mindepth}_r(c_j) \geq \mathrm{mindepth}_r(b_j) + \mathrm{mindepth}_r(a_j)$ for some $a_j \in A$. Since $b \neq r$, $\mathrm{mindepth}_r(b) = 1 + \min\{\mathrm{mindepth}_r(b_j) | 1 \leq j \leq n_i\}$. Note that $\{b\} \hat{\cdot}_r A = \{f_i(c_1, \ldots, c_{n_i}) | c_k \in \{b_k\} \hat{\cdot}_r A, 1 \leq k \leq n_i\}$. Then $t = f_i(c_1, \ldots, c_{n_i})$ for some $f_i(c_1, \ldots, c_{n_i}) \in \{b\} \hat{\cdot}_r A$. Hence, $\mathrm{mindepth}_r(t) = 1 + \mathrm{mindepth}_r(c_k)$ for some $c_k \in \{b_k\} \hat{\cdot}_r A$, $1 \leq k \leq n_i$. Then

$$\begin{aligned}\mathrm{mindepth}_r(t) &\geq 1 + \mathrm{mindepth}_r(b_k) + \mathrm{mindepth}_r(a_k) \\ &\geq 1 + \min\{\mathrm{mindepth}_r(b_j), 1 \leq j \leq n_i\} + \mathrm{mindepth}_r(a) \\ &= \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a),\end{aligned}$$

where $a \in \{a_j \mid 1 \leq j \leq n_i\}$ with $\mathrm{mindepth}_r(a) = \min\{\mathrm{mindepth}_r(a_j) \mid 1 \leq j \leq n_i\}$. □

In particular, when t from the previous lemma satisfies $r \in \mathrm{sub}(t)$, we get the following corollary.

Corollary 4.1. *If* $t \in \{b\} \hat{\cdot}_r A$ *and* $r \in \mathrm{sub}(t)$, *then* $\mathrm{mindepth}_r(t) \geq \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a)$ *for some* $a \in \bar{A}$.

Proof: Assume that $t \in \{b\} \hat{\cdot}_r A$ and $r \in \mathrm{sub}(t)$. Then $\mathrm{mindepth}_r(t) < \infty$. By Lemma 4.1, we have that $\mathrm{mindepth}_r(t) \geq \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a)$ for some $a \in A$. It follows that $\mathrm{mindepth}_r(a) < \infty$. This means that $r \in \mathrm{sub}(a)$, i.e., $a \in \bar{A}$. □

We observe that the language $\{r\}$ is trivially idempotent and so is A whenever $r \notin \mathrm{Sub}(A)$ for any $A \in \mathcal{P}(W^r_\tau(X_n))$. In addition, if A is idempotent and $r \in \mathrm{Sub}(A)$, we obtain that $r \in A$. Next lemma demonstrates a more general result.

Lemma 4.2. *Let* $r \in W_\tau(X_n)$ *be a fixed term and* $A, B \in (\mathcal{P}(W^r_\tau(X_n))$ *be tree languages with* $r \in \mathrm{Sub}(A)$. *If* $A = B \hat{\cdot}_r A$ *or* $A = A \hat{\cdot}_r B$, *then* $r \in B$.

Proof: Assume that $A = B \hat{\cdot}_r A$. If $r \notin \mathrm{Sub}(B)$, then $A = B \hat{\cdot}_r A = B$. It follows that $r \notin \mathrm{Sub}(B) = \mathrm{Sub}(A)$, which is a contradiction. Therefore, $r \in \mathrm{Sub}(B)$ and $\bar{B} \neq \emptyset$. Since $r \in \mathrm{Sub}(A)$, we have $\bar{A} \neq \emptyset$. If $r \in A = B \hat{\cdot}_r A$, then $r \in B$ by Lemma 3.2 (i). Additionally assume that $r \notin A$, then $\mathrm{mindepth}_r(t) \geq 1$ for all $a \in \bar{A}$. Let s be the least natural number such that $A_s \neq \emptyset$. Suppose that $r \notin B$. Then $\mathrm{mindepth}_r(b) \geq 1$ for all $b \in \bar{B}$. Consider $t \in A_s$. Since $t \in A = B \hat{\cdot}_r A$, we have $t \in \{b\} \hat{\cdot}_r A$ for some $b \in B$. Then by Corollary 4.1,

$$\begin{aligned}\mathrm{mindepth}_r(t) &\geq \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a) \text{ for some } a \in \bar{A} \\ &> \mathrm{mindepth}_r(a) \\ &\geq \mathrm{mindepth}_r(t),\end{aligned}$$

which is a contradiction. Hence, $r \in B$.

Assume that $A = A \hat{\cdot}_r B$. Since $r \in \mathrm{Sub}(A)$, we have $\bar{A} \neq \emptyset$. If $r \in A = A \hat{\cdot}_r B$, then $r \in B$ by Lemma 3.2 (i). Next, we assume that $r \notin A$, then $\mathrm{mindepth}_r(t) \geq 1$ for all $a \in \bar{A}$. Let s be the least natural number such that $A_s \neq \emptyset$. Suppose that $r \notin B$. Then $\mathrm{mindepth}_r(b) \geq 1$ for all $b \in \bar{B}$. Consider $t \in A_s$. Since $t \in A = A \hat{\cdot}_r B$, we have $t \in \{a\} \hat{\cdot}_r B$ for some $a \in A$. Since $r \in \mathrm{sub}(t)$, we get $a \in \bar{A}$. Then by Corollary 4.1,

$$\begin{aligned}\mathrm{mindepth}_r(t) &\geq \mathrm{mindepth}_r(a) + \mathrm{mindepth}_r(b) \text{ for some } b \in \bar{B} \\ &> \mathrm{mindepth}_r(a)\end{aligned}$$

$$\geq \text{mindepth}_r(t),$$

which is a contradiction. Hence, $r \in B$. □

As a result, we immediately get the following corollary.

Corollary 4.2. *Let $r \in W_\tau(X_n)$ be a fixed term and $A \in \mathcal{P}(W^r_\tau(X_n))$ be a tree language with $r \in \text{Sub}(A)$. If A is an idempotent element of $(\mathcal{P}(W^r_\tau(X_n)), \hat{\cdot}_r)$, then $r \in A$.*

The following lemma presents a certain condition to make A be infinite under the same condition as Lemma 4.2.

Lemma 4.3. *Let $A, B \in \mathcal{P}(W^r_\tau(X_n))$ be tree languages with $r \in \text{Sub}(A)$. If $A = A \hat{\cdot}_r B$ and there exists a natural number $s \geq 1$ such that $B_s \neq \emptyset$, then A is infinite.*

Proof: Suppose that A is finite. Then $\bar{A}$ is finite. Since $r \in \text{Sub}(A)$, $\bar{A} \neq \emptyset$. It follows that there exists a term $t \in \bar{A}$ which has maximum mindepth_r. Since $A = A \hat{\cdot}_r B$ and $r \in \text{Sub}(A)$, Lemma 4.2 implies that $r \in B$. For $b \in B_s$, we have $\text{mindepth}_r(b) \geq 1$. Let $h \in \{t\} \hat{\cdot}_r B_s$. Then by Corollary 4.1, we have

$$\begin{aligned}\text{mindepth}_r(h) &\geq \text{mindepth}_r(t) + \text{mindepth}_r(b) \text{ for some } b \in \bar{B}_s\\ &\geq \text{mindepth}_r(t) + 1\\ &> \text{mindepth}_r(t).\end{aligned}$$

Since $r \in \text{Sub}(B)$ and $r \in \text{sub}(t)$, we obtain that $r \in \text{sub}(h)$. Therefore, $h \in \bar{A}$ and $\text{mindepth}_r(h) > \text{mindepth}_r(t)$, which is a contradiction. Therefore, A is infinite. □

Consequently, if A is an idempotent element with $A_s \neq \emptyset$ for some $s \geq 1$, then A is infinite. It is obvious that $\{r\}$ is both an idempotent element and a regular element. In this sense, we extend to a more general outcome. Next theorem proves that idempotent elements are actually regular.

Theorem 4.1. *A is idempotent of $(\mathcal{P}(W^r_\tau(X_n)), \hat{\cdot}_r)$ if and only if A is regular.*

Proof: Let A be regular. The result is clear when A is empty. Now, consider the case that A is nonempty. Then there exists a subset B of $W^r_\tau(X_n)$ such that $A = A \hat{\cdot}_r B \hat{\cdot}_r A$. If $r \notin \text{Sub}(A)$, then $A \hat{\cdot}_r B = A$ and so, $A = A \hat{\cdot}_r B \hat{\cdot}_r A = A \hat{\cdot}_r A$. Hence, A is idempotent. For $r \in \text{Sub}(A)$, we have $r \in A \hat{\cdot}_r B$ by Lemma 4.2. By Lemma 3.2 (i), we have $r \in A$ and $r \in B$. Then Lemma 3.1 (i) implies that $A \subseteq B \hat{\cdot}_r A \subseteq A \hat{\cdot}_r B \hat{\cdot}_r A = A$. Hence, $A = B \hat{\cdot}_r A$ and $A \hat{\cdot}_r A = A$. Therefore, A is an idempotent. The converse is obvious. □

Hence, idempotence and regularity of the elements in the semigroup $(\mathcal{P}(W^r_\tau(X_n)), \hat{\cdot}_r)$ are coincident. By the definition of regular elements, we have $A = A \hat{\cdot}_r B \hat{\cdot}_r A$ for some $B \in \mathcal{P}(W^r_\tau(X_n))$. In the case that $r \in \text{Sub}(A)$, we prove that $r \in B$ by applying coincidence of idempotent and regular elements in our semigroup in Theorem 4.1 and using an additional condition.

Lemma 4.4. *Let $A \in \mathcal{P}(W^r_\tau(X_n))$ with $r \in \text{Sub}(A)$ be a regular (an idempotent) element of $(\mathcal{P}(W^r_\tau(X_n)), \hat{\cdot}_r)$. Then for all $\emptyset \neq B \subseteq A$, we have $r \in B$ if and only if $A = A \hat{\cdot}_r B \hat{\cdot}_r A$.*

Proof: Since A is idempotent, we have $r \in A$. Assume that $r \in B$. Then by Lemma 3.1 (i), $A \subseteq B \hat{\cdot}_r A \subseteq A \hat{\cdot}_r B \hat{\cdot}_r A \subseteq A \hat{\cdot}_r A \hat{\cdot}_r A = A$. Hence, $A = A \hat{\cdot}_r B \hat{\cdot}_r A$. Conversely, assume that $A = A \hat{\cdot}_r B \hat{\cdot}_r A$. Since $r \in A$, Lemma 3.2 (i) gives $r \in B$. □

Example 4.1. *To check idempotence and regularity of tree languages, we consider the following: Given $\tau = (2, 1)$ with a binary operation f and a unary operation g. Let $r = f(x_1, x_2)$. We examine the following tree languages.*

(1) $A = \{f(x_1, g(x_2))\}$. We see that $r \notin \text{Sub}(A)$. Then A is exactly idempotent and by Theorem 4.1, A is regular.

(2) $B = \{g(f(x_1, x_2)), f(g(x_4), x_4)\}$ *and* $C = \{x_3, f(x_1, g(x_2))\}$. *We see that* $r \in \mathrm{Sub}(B)$ *and* $r \notin C$. *Then Lemma 4.2 implies that* $B \hat{\cdot}_r C \neq B$ *and* $C \hat{\cdot}_r B \neq B$.

(3) $D = \{x_5, g(f(x_1, x_2))\}$. *Clearly,* $r \in \mathrm{Sub}(D)$ *and* $r \notin D$. *Thus,* D *is not idempotent by Corollary 4.2.*

(4) $E = \{g(x_2), f(x_1, x_2), f(x_3, x_5), g(f(x_1, x_2))\}$. *We see that* $r \in \mathrm{Sub}(E)$ *and* $E_1 \neq \emptyset$. *Since* E *is finite, Lemma 4.3 implies that* E *is not idempotent.*

5. **Green's Relations.** Green's relations play an important role in studying semigroups. In this section, we characterize all Green's relations for the semigroup $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$. First, we recall the definitions of $\mathcal{L}$ and $\mathcal{R}$.

Definition 5.1. *Let* $A, B \in \mathcal{P}(W_\tau^r(X_n))$. *Green's relations* $\mathcal{L}$ *and* $\mathcal{R}$ *are defined as follows.*

(i) $A\mathcal{L}B$ *if and only if there are* $C, D \in \mathcal{P}(W_\tau^r(X_n))$ *such that* $C \hat{\cdot}_r A = B$ *and* $D \hat{\cdot}_r B = A$.

(ii) $A\mathcal{R}B$ *if and only if there are* $E, F \in \mathcal{P}(W_\tau^r(X_n))$ *such that* $A \hat{\cdot}_r E = B$ *and* $B \hat{\cdot}_r F = A$.

The following theorems characterize $\mathcal{L}$ and $\mathcal{R}$ for both $r \notin \mathrm{Sub}(A)$ and $r \in \mathrm{Sub}(A)$.

Theorem 5.1. *Let* $r \in W_\tau(X_n)$ *be a fixed term and* $A, B \in \mathcal{P}(W_\tau^r(X_n))$.

(i) Let $r \notin \mathrm{sub}(A)$. *Then* $A\mathcal{L}B$ *if and only if* $r \notin \mathrm{Sub}(B)$.

(ii) Let $r \in \mathrm{sub}(A)$. *Then* $A\mathcal{L}B$ *if and only if* $A = B$, *i.e.,* $\mathcal{L}$ *is the diagonal* $\Delta_{\mathcal{P}(W_\tau^r(X_n))}$.

Proof:

(i) The result is immediate whenever A is empty. Now, we assume that $A\mathcal{L}B$ and A is nonempty. Then $D \hat{\cdot}_r A = B$ for some $D \in \mathcal{P}(W_\tau^r(X_n)) \setminus \{\emptyset\}$. Since $r \notin \mathrm{Sub}(A)$, Lemma 3.2 (ii) implies that $r \notin \mathrm{Sub}(D \hat{\cdot}_r A) = \mathrm{Sub}(B)$. Conversely, assume that $r \notin \mathrm{Sub}(B)$. If A or B is empty, then the result is obvious. We additionally assume that A and B are nonempty. Then $B \hat{\cdot}_r A = B$. Since $r \notin \mathrm{Sub}(A)$, we have $A \hat{\cdot}_r B = A$. Therefore, $A\mathcal{L}B$.

(ii) Assume that $A\mathcal{L}B$. Then $C \hat{\cdot}_r A = B$ and $D \hat{\cdot}_r B = A$ for some $C, D \in \mathcal{P}(W_\tau^r(X_n))$. It follows that $C \hat{\cdot}_r D \hat{\cdot}_r B = B$ and $D \hat{\cdot}_r C \hat{\cdot}_r A = A$. By Lemma 4.2, $r \in D \hat{\cdot}_r C$. Then Lemma 3.2 (i) implies that $r \in D$ and $r \in C$. By Lemma 3.1 (ii), we have $C \subseteq D \hat{\cdot}_r C$ and $D \subseteq C \hat{\cdot}_r D$. Therefore, we obtain that $B = C \hat{\cdot}_r A \subseteq D \hat{\cdot}_r C \hat{\cdot}_r A = A$ and $A = D \hat{\cdot}_r B \subseteq C \hat{\cdot}_r D \hat{\cdot}_r B = B$. Therefore, $A = B$. The converse is clear by the definition.

Altogether, $\mathcal{L} = \Delta_{\mathcal{P}(W_\tau^r(X_n))} \cup \{(A, B) | r \notin \mathrm{Sub}(A) \text{ and } r \notin \mathrm{Sub}(B)\}$. □

Theorem 5.2. *Let* $r \in W_\tau(X_n)$ *be a fixed term and* $A, B \in \mathcal{P}(W_\tau^r(X_n))$.

(i) Let $r \notin \mathrm{Sub}(A)$. *Then* $A\mathcal{R}B$ *if and only if* $A = B$.

(ii) Let $r \in \mathrm{Sub}(A)$. *If* $A\mathcal{R}B$, *then* $r \in \mathrm{Sub}(B)$ *and*

$$\{a \in A | r \notin \mathrm{Sub}(\{a\})\} = \{b \in B | r \notin \mathrm{Sub}(\{b\})\}.$$

(iii) Let $r \in \mathrm{Sub}(A)$. *Then* $A\mathcal{R}B$ *if and only if* $A = B$.

Proof:

(i) The result is obvious whenever A is empty. Now, we assume that $A\mathcal{R}B$ and A is nonempty. Then $A \hat{\cdot}_r D = B$ for some $D \in \mathcal{P}(W_\tau^r(X_n)) \setminus \{\emptyset\}$. Since $r \notin \mathrm{Sub}(A)$, we have $A = A \hat{\cdot}_r D = B$. The converse is obvious.

(ii) Assume that $A\mathcal{R}B$. Then $A = B \hat{\cdot}_r D$ and $B = A \hat{\cdot}_r C$ for some $C, D \in \mathcal{P}(W_\tau^r(X_n))$. Obviously, C and D are nonempty. Since $r \in \mathrm{Sub}(A) = \mathrm{Sub}(B \hat{\cdot}_r D)$, Lemma 3.2 (ii) implies that $r \in \mathrm{Sub}(B)$. Let $a \in A$ such that $r \notin \mathrm{Sub}(\{a\})$. Then we have $\{a\} = \{a\} \hat{\cdot}_r C \subseteq A \hat{\cdot}_r C = B$. Therefore, $a \in \{b \in B | r \notin \mathrm{Sub}(\{b\})\}$. Similarly, for all

$b \in B$ such that $r \notin \mathrm{Sub}(\{b\})$, we can show that $b \in \{a \in A | r \notin \mathrm{Sub}(\{a\})\}$. Hence, $\{a \in A | r \notin \mathrm{Sub}(\{a\})\} = \{b \in B | r \notin \mathrm{Sub}(\{b\})\}$.

(iii) Assume that $A\mathcal{R}B$. Then $A \hat{\cdot}_r C = B$ and $B \hat{\cdot}_r D = A$ for some $C, D \in \mathcal{P}(W_\tau^r(X_n))$. Clearly, C and D are nonempty. It follows that $A \hat{\cdot}_r C \hat{\cdot}_r D = A$. By Lemma 4.2, we have $r \in C \hat{\cdot}_r D$. By Lemma 3.2 (i), we have $r \in C$ and $r \in D$. Now, we have $A = A \hat{\cdot}_r \{r\} \subseteq A \hat{\cdot}_r C = B$ and $B = B \hat{\cdot}_r \{r\} \subseteq B \hat{\cdot}_r D = A$. Therefore, $A = B$. The converse is clear. □

From (i) and (iii), we can conclude that $\mathcal{R} = \Delta_{\mathcal{P}(W_\tau^r(X_n))}$. Next, we consider other three remained relations, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$.

Definition 5.2. *Let $A, B \in \mathcal{P}(W_\tau^r(X_n))$. Green's relations $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ are defined as follows:*

(i) $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$;

(ii) $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$;

(iii) $A\mathcal{J}B$ if and only if there are $C, D, E, F \in \mathcal{P}(W_\tau^r(X_n))$ such that $C \hat{\cdot}_r A \hat{\cdot}_r D = B$ and $E \hat{\cdot}_r B \hat{\cdot}_r F = A$.

We remark that $\mathcal{D}$ is the smallest equivalent relation containing both $\mathcal{L}$ and $\mathcal{R}$. Since $\mathcal{R} = \Delta_{\mathcal{P}(W_\tau^r(X_n))} \subseteq \Delta_{\mathcal{P}(W_\tau^r(X_n))} \cup \{(A, B) | r \notin \mathrm{Sub}(A) \text{ and } r \notin \mathrm{Sub}(B)\} = \mathcal{L}$, we directly obtain the characterizations of $\mathcal{H}$ and $\mathcal{D}$ as follows.

Theorem 5.3. *For $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$, the characterizations of $\mathcal{H}$ and $\mathcal{D}$ are*

(i) $\mathcal{H} = \Delta_{\mathcal{P}(W_\tau^r(X_n))}$;

(ii) $\mathcal{D} = \Delta_{\mathcal{P}(W_\tau^r(X_n))} \cup \{(A, B) | r \notin \mathrm{Sub}(A) \text{ and } r \notin \mathrm{Sub}(B)\}$.

For Green's relation $\mathcal{J}$, it requires the following lemma.

Lemma 5.1. *Let $A, B, C \in \mathcal{P}(W_\tau^r(X_n))$ and $r \in \mathrm{Sub}(A)$. If $A = B \hat{\cdot}_r A \hat{\cdot}_r C$ for some $B, C \in \mathcal{P}(W_\tau^r(X_n))$, then $r \in B$ and $r \in C$.*

Proof: Assume that $A = B \hat{\cdot}_r A \hat{\cdot}_r C$ for some $B, C \in \mathcal{P}(W_\tau^r(X_n))$. Since $r \in \mathrm{Sub}(A) = \mathrm{Sub}(B \hat{\cdot}_r A \hat{\cdot}_r C)$. By Lemma 3.2 (ii), $r \in \mathrm{Sub}(B)$ and $r \in \mathrm{Sub}(C)$. Hence, $\bar{B}, \bar{C} \neq \emptyset$. Suppose that $r \notin B$ or $r \notin C$. Since $r \in \mathrm{Sub}(A)$, we have $\bar{A} \neq \emptyset$. Let m be the least natural number such that $A_m \neq \emptyset$. Let $h \in \bar{A}$. Then $h \in A$ and $r \in \mathrm{Sub}(\{h\}) = \mathrm{sub}(h)$. Since $A = B \hat{\cdot}_r A \hat{\cdot}_r C$ and $B = \bar{B} \cup B^0$, we have $A = \left(\bar{B} \hat{\cdot}_r A \hat{\cdot}_r C\right) \cup \left(B^0 \hat{\cdot}_r A \hat{\cdot}_r C\right)$. Since $r \notin \mathrm{Sub}\left(B^0\right)$, Lemma 3.2 (ii) implies that $r \notin \mathrm{Sub}\left(B^0 \hat{\cdot}_r A \hat{\cdot}_r C\right)$. Then $h \in \bar{B} \hat{\cdot}_r A \hat{\cdot}_r C$. For $r \notin B$, we have $\mathrm{mindepth}_r(b) \geq 1$ for each $b \in \bar{B}$. Then

$$\begin{aligned}\mathrm{mindepth}_r(h) &\geq \mathrm{mindepth}_r(d) + \mathrm{mindepth}_r(c) \text{ for some } d \in \bar{B} \hat{\cdot}_r A \text{ and } c \in \bar{C} \\ &\geq \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a) + \mathrm{mindepth}_r(c) \text{ for some } b \in \bar{B}, a \in \bar{A} \\ &\quad \text{and } c \in \bar{C} \\ &\geq 1 + m + 0 > m.\end{aligned}$$

Then $h \notin A_m$, which contradicts $A_m \neq \emptyset$. For $h \in B \hat{\cdot}_r A \hat{\cdot}_r C$ and $r \notin C$, we have $\mathrm{mindepth}_r(c) \geq 1$ for each $c \in \bar{C}$. Therefore,

$$\begin{aligned}\mathrm{mindepth}_r(h) &\geq \mathrm{mindepth}_r(f) + \mathrm{mindepth}_r(c) \text{ for some } f \in B \hat{\cdot}_r A \text{ and } c \in \bar{C} \\ &\geq \mathrm{mindepth}_r(b) + \mathrm{mindepth}_r(a) + \mathrm{mindepth}_r(c) \text{ for some } b \in B, a \in \bar{A} \\ &\quad \text{and } c \in \bar{C} \\ &\geq 0 + m + 1 > m.\end{aligned}$$

Then $h \notin A_m$, which contradicts $A_m \neq \emptyset$. Therefore, $r \in B$ and $r \in C$. □

Finally, we can obtain the characterization of $\mathcal{J}$ as follows.

Theorem 5.4. *For the semigroup* $(\mathcal{P}(W_\tau^r(X_n)), \hat{\cdot}_r)$, $\mathcal{J} = \mathcal{L}$.

Proof: Let $A, B \in \mathcal{P}(W_\tau^r(X_n))$ such that $A\mathcal{J}B$. Then $C \hat{\cdot}_r B \hat{\cdot}_r D = A$ and $E \hat{\cdot}_r A \hat{\cdot}_r F = B$ for some $C, D, E, F \in \mathcal{P}(W_\tau^r(X_n))$. It follows that $A = C \hat{\cdot}_r E \hat{\cdot}_r A \hat{\cdot}_r F \hat{\cdot}_r D$. If $r \notin \mathrm{Sub}(A)$, then Lemma 3.2 (ii) implies that $r \notin \mathrm{Sub}(E \hat{\cdot}_r A \hat{\cdot}_r F) = \mathrm{Sub}(B)$. Hence, $A\mathcal{L}B$. For $r \in \mathrm{Sub}(A)$, we consider $A = C \hat{\cdot}_r E \hat{\cdot}_r A \hat{\cdot}_r F \hat{\cdot}_r D$. By Lemma 5.1, we have $r \in C \hat{\cdot}_r E$ and $r \in F \hat{\cdot}_r D$. Then Lemma 3.2 (i) implies that $r \in C$, $r \in E$, $r \in F$ and $r \in D$. Thus, $A = \{r\} \hat{\cdot}_r A \hat{\cdot}_r \{r\} \subseteq E \hat{\cdot}_r A \hat{\cdot}_r F = B$ and $B = \{r\} \hat{\cdot}_r B \hat{\cdot}_r \{r\} \subseteq C \hat{\cdot}_r B \hat{\cdot}_r D = A$. Thus, $A = B$ implies $A\mathcal{L}B$. Therefore, $\mathcal{J} = \mathcal{L}$. □

6. **Conclusions.** We generalized a binary operation, called an inductive composition of tree languages, from the inductive composition on terms and defined a new binary operation, an *r-inductive product* of tree languages, which is closely related to the inductive composition of tree languages. We observed that the power set $\mathcal{P}(W_\tau(X_n))$ of $W_\tau(X_n)$ along with our particular product is not actually a semigroup. However, we were able to restrict some essential conditions to establish a new base set which can form a semigroup of an inductive product of tree languages. Furthermore, we investigated the algebraic structures of such semigroup such as idempotent elements, regular elements, and Green's relations. In this work, we have not characterized special subsemigroups of our semigroup and semigroup factorizations yet. These will leave as open problems.

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