International Journal of Innovative Computing, Information and Control
Volume 19, Number 5, October 2023
 ISSN 1349-4198
pp. 1501–1516

A NEW APPROACH TO SOLVING TOPSIS RANK REVERSAL BASED ON S-TYPE UTILITY FUNCTION

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Received November 2022; revised February 2023

Abstract. *In recent decades, scholars have used different multi-attribute decision-making methods to help decision makers choose better alternatives for various decision-making problems or make comprehensive evaluation of projects. However, these methods have been criticized in the literature because they present a problem called rank reversal. In particular, analyzing this problem in relation to the TOPSIS method is still limited. Thus, we introduce the generation mechanism of rank reversal in the comprehensive evaluation of multi-attribute evaluation TOPSIS method, and the problems of the current solutions to rank reversal, such as redundant work, anti-objectivity and use restrictions. It is pointed out that the S-shaped curve can effectively avoid the occurrence of rank reversal. Next, we present a new method of utility construction by fitting the index utility function with the normal distribution function, which improves the existing S-type utility construction method to make its slope more scientific and more in line with the expectations and preferences of decision makers. Finally, by comparing and analyzing the ranking results of the traditional TOPSIS method and the new method, it is demonstrated that the new method improves the consistency of the results on the basis of the original TOPSIS method, and has good stability.*
Keywords: Multi-attribute decision-making, TOPSIS, Rank reversal, S-type utility function

1. **Introduction.** In engineering practice, we have to face various complex decision-making problems. When making decisions, we must comprehensively and systematically consider the areas to be evaluated from the perspective of system engineering, and establish a systematic and scientific decision-making process. The multi-attribute evaluation has attracted extensive attention in its theory and application research because of its scientificity and objectivity. At present, TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) has become one of the most widely used multi-attribute comprehensive evaluation methods in practical problem solving and scientific research due to its long-term theoretical development, simple and convenient application and wide adaptability. In the process of research and use, we found that the traditional decision-making

DOI: 10.24507/ijicic.19.05.1501

methods assume that the decision-makers have determined the indexes and alternatives before ranking, so it is difficult for them to complete the decision-making under the dynamic situation [1]. When the alternative scheme is changed, the rank reversal problem may occur. That means the sorting result after adding a new sample is inconsistent with the original sample, resulting in inconsistent sorting results. By comparing the ranking results of ten multi-attribute decision-making problems, Mousavi-Nasab and Sotoudeh-Anvari found that the rank reversal problem appeared in five cases of TOPSIS method, and the worst performance was found in the three decision-making methods of TOPSIS, COPRAS, and SAW [2]. However, at present, most of the studies related to TOPSIS are discussing and promoting its application scenarios [3], and there are few studies on the improvement of the method itself, especially on the solution of rank reversal problems, and some of them have a long history, while most of the recent improvement forms are similar. Although some researchers believe that the rank reversal problem does not often occur, in some cases some types of rank reversal may not mean wrong decisions [4,5]. However, it is undeniable that the rank reversal problem will more or less destroy the independence and scientific of the evaluation, make the evaluation results difficult to explain, and may make some decision makers no longer favor this decision-making method [6]. Especially with the progress and development of science and technology, decision makers will face more complex decision-making problems, such as traffic road selection [7]. At this time, decision makers are likely to prefer decision-making methods that have less or no rank reversal problems.

In order to eliminate the rank reversal problem in the application of TOPSIS method and give full play to the advantages of TOPSIS method in comprehensive evaluation, this paper improves the existing S-type utility construction method, and puts forward a new utility construction method by fitting the index utility function through the normal distribution function, so as to solve the rank reversal problem of TOPSIS method, which improves the consistency of the results on the basis of the original TOPSIS method and has good stability.

In the second part, this paper will introduce the algorithm of TOPSIS method, summarize the generation mechanism of TOPSIS rank reversal problem and the current methods to solve this problem through literature review, and introduce the S-type utility function and its application in multi-attribute decision-making. In the third part, the ideas of this new method will be presented in detail. In the fourth part, a practical case will be used to compare the traditional method with the new method to verify the effectiveness and reliability of the new method. Finally, we conclude the paper in the last section by stating the advantages and limitations of the proposed methodology.

2. **Background.**

2.1. **TOPSIS method.**

Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) is one of the most widely used multi-attribute decision analysis methods. In this method, the best alternative is the one nearest to the Positive Ideal Solution (PIS) and farthest from the Negative Ideal Solution (NIS). PIS is a hypothetical alternative that maximizes the benefit criteria and simultaneously minimizes the cost criteria. On the contrary, NIS maximizes the cost criteria and simultaneously minimizes the benefit criteria. Based on this, TOPSIS calculates the closeness degree of each alternative and PIS, and the alternatives are ranked in descending order by using the result obtained.

Algorithm 1. TOPSIS method

Suppose there are m schemes and n indexes, the original decision matrix is X_{mn}, and the weight coefficient $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$. The specific process is as follows.

① Obtain the normalized decision matrix by normalizing the index data under the original matrix. The general normalization method is $s_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}$, and then get normalized decision matrix $S_{mn} = \{s_{ij}\}$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$.

② Obtain the weighted normalized matrix K_{mn} by multiplying S with the weighted coefficient ω.

$$K_{mn} = S_{mn} * \omega = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mn} \end{pmatrix}.$$

③ Get the positive ideal solution under each attribute $k^* = (k_1^*, k_2^*, \ldots, k_n^*)$ and the negative ideal solution $k^0 = \left(k_1^0, k_2^0, \ldots, k_n^0\right)$, in which

$$k_j^* = \begin{cases} \max_i k_{ij}, & j \text{ index is benefit type attribute} \\ \min_i k_{ij}, & j \text{ index is cost type attribute} \end{cases},$$

$$k_j^0 = \begin{cases} \min_i k_{ij}, & j \text{ index is benefit type attribute} \\ \max_i k_{ij}, & j \text{ index is cost type attribute} \end{cases},$$

$$i = 1, 2, \ldots, m; \; j = 1, 2, \ldots, n.$$

④ Calculate the distance between each scheme and the positive ideal solution $d^* = (d_1^*, d_2^*, \ldots, d_m^*)^T$ or the negative ideal solution $d^0 = \left(d_1^0, d_2^0, \ldots, d_m^0\right)^T$, in which

$$d_i^* = \sqrt{\sum_{j=1}^{n} \left(k_{ij} - k_j^*\right)^2}, \quad d_i^0 = \sqrt{\sum_{j=1}^{n} \left(k_{ij} - k_j^0\right)^2}, \quad i = 1, 2, \ldots, m.$$

⑤ Calculate the closeness between the scheme and the positive ideal solution

$$C^* = (C_1^*, C_2^*, \ldots, C_m^*)^T, \text{ in which } C_i^* = \frac{d_i^0}{d_i^0 + d_i^*}, \; i = 1, 2, \ldots, m.$$

⑥ Sort the value of closeness degree C_i^* to get the order from best to worst of each scheme.

2.2. **Literature review.**

2.2.1. *Generation mechanism of rank reversal problem in TOPSIS method.* The traditional TOPSIS method measures the value of each scheme by calculating the distance between each scheme and the best and worst points, thus obtaining their closeness to the best scheme. When the evaluation situation and its own conditions change, the final score will change accordingly. The rank reversal problem is mainly caused by the change of index weight and the change in positive and negative ideal solutions [11]. Firstly, the change of index weight leads to the contradiction of ranking: in TOPSIS method, researchers need to determine the weight ω of each index, and multiply it by the normalized value under the corresponding index as a coefficient. If researchers apply entropy weight method to assuming the weight of TOPSIS method, that is, the degree of data dispersion under each index is used as the weighting standard, and the more data is dispersed, the greater the weight is, and the smaller the weight is concentrated, whereby the newly introduced samples may cause the rank reversal problem as a result of changes in the original weight. However, the probability of rank reversal caused by weight change alone is not high, which only appears when entropy weighting method is applied. The rank reversal problem caused by weight change can be completely avoided by applying subjective weighting or other objective weighting rules. In recent years, there are many studies on the weight selection of multi-attribute decision-making. You can refer to [12] for details. Therefore, this paper will not go into in-depth research in this field. Secondly, the change in positive and negative ideal points leads to the rank reversal of results, which is the main subject to be discussed and addressed in this paper. When the introduced new scheme changes the positive and negative ideal points, causing the range of certain indexes to expand, whereby reducing the discrimination ability of such indicators against the original scheme, it may lead to the contradiction of the evaluation results and the occurrence of the rank reversal problem. The following is a simple example to illustrate this.

TABLE 1. Samples and standardized values before adding y_4 to a certain index[1]

	y_1	y_2	y_3
Original value	1	5	10
Value after standardization of TOPSIS	0.089087081	0.445435403	0.890870806

TABLE 2. Samples and standardized values after adding y_4 to a certain index

	y_1	y_2	y_3	y_4
Original value	1	5	10	1000
Value after standardization of TOPSIS	0.000999937	0.004999685	0.00999937	0.999937006

Comparing the two statistical sheets, it can be seen intuitively that if the traditional TOPSIS normalization method is adopted, the range of the index will greatly increase after the scheme with the extreme value y_4 is introduced, and the discrimination between the index and the original sample will obviously decrease. For example, the difference between the normalized values of y_1 and y_3 under the index will decrease from 0.8 to 0.009, and the difference between the two schemes will be reduced to about 1/100 of that before the aforesaid scheme is introduced. Therefore, in the original scheme, adding a new scheme with an extreme value under a certain index is equivalent to indirectly reducing the weight of such index. If the data does not change obviously under other indicators, it is very likely that the rank reversal will occur because of the introduction of the new scheme.

2.2.2. *Current solutions of TOPSIS rank reversal.* At present, there are few studies on TOPSIS rank reversal problem, and the ideas and methods used by researchers are relatively simple. This section will summarize the previous TOPSIS rank reverse research literature.

It is pointed out by Lu and Tang that the general TOPSIS method has shortcomings in the determination of positive and negative ideal points, the determination of weights and

[1]In this paper, the traditional index standardization method refers to the most commonly used standardization method in TOPSIS research: $y_i^* = \frac{y_i}{\sqrt{\sum_{j=1}^m y_j^2}}$. In addition, the min-max normalization method and z-score standardization method are used. The min-max normalization method $y_i^* = \frac{y_i - \min\{Y\}}{\max\{Y\} - \min\{Y\}}$, z-score standardization method: $y_i^* = \frac{(y_i - \bar{y})}{S_y}$. The results obtained by other traditional normalization methods are similar, not all examples here.

the calculation of closeness [13]. He improved the calculation formula of degree of closeness by projection, and demonstrated its good stability by examples. As pointed out by Chen, the positive and negative ideal solutions defined under the traditional TOPSIS method are closely connected with the decision scheme group and change is easy to occur following the change in scheme, thus causing the rank reversal [14]. Therefore, he proposed that absolute positive and negative ideal solutions can be introduced by expert method or theoretical analysis method to minimize the risk of rank reversal. Farias Aires and Ferreira proposed the R-TOPSIS method for the rank reversal problem. This method needs to determine the feasible region for each index, and complete the standardization steps through the max method or max-min method [15]. On the basis of the above, Yang et al. improved the standardization steps and named the method NR-TOPSIS method [16]. According to the empirical conclusion, the ranking order obtained by these two methods is more stable than the traditional TOPSIS method. On the basis of determining absolute maximum and minimum values and linear max-min normalization, Yang et al. also artificially determined PIS and NIS, which can not only avoid rank reversal, but also achieve the purpose of simplifying operations [17]. García-Cascales and Lamata improved the standardization method of indicators, set absolute positive and negative ideal solutions on the basis of this standardization method, and demonstrated that this method can avoid the rank reversal problem through geometric and empirical analysis [18]. Senouci et al., based on the standardization concept of max-min, put forward four improved methods to solve the rank reversal problem: virtualizing positive and negative ideal solutions, virtualizing negative ideal solutions, deleting alternatives without updating ideal solutions, and identifying & deleting outliers [19]. All four methods were proved to be able to minimize or eliminate the rank reversal problem. Mufazzal and Muzakkir pointed out that if the data are unevenly distributed, those closely arranged data will be prone to rank reversal problems [10]. In their paper, the basis of the final ranking is changed from the relative distance to the positive and negative ideal solution to the distance to the positive ideal solution, thus reducing the influence of changes in tail data on the ranking results.

According to the above literature, it can be seen that currently the solution to the rank reversal problem mainly consists of changing the normalization method and introducing absolute positive and negative ideal solutions. At present, there are some defects in the former research, such as the inability of standardized method of [18] to deal with the indicators with negative numbers. Moreover, the majority of researchers only improve the normalization process to facilitate the setting of positive and negative ideal solutions. The normalization process, when used alone, may aggravate the rank reversal problem [19]. Although the theory of the latter is more perfect, it has not been widely used for many years. We believe that the main reasons include the following. ① The process is complicated. ② Excessive subjective judgment on the objectivity of injury evaluation methods. Because not all data ranked by the traditional TOPSIS method causes the rank reversal problem, using this method in the data without the rank reversal problem not only effectively reflects the subjective preference of decision makers, but also causes unnecessary loss to the objectivity of the evaluation method. ③ It is difficult to apply to quantitative indicators. All possible values should be included between the positive and negative ideal points, but in reality, the theoretical upper and lower limits of quantitative indicators are likely to be extreme or even have no definite upper and lower limits. If the absolute ideal point is set too far away from the group, it will have a great influence on the scientific of the evaluation system. However, without doing so, the new scheme may still change the positive and negative ideal points, leading to the rank reversal problem. Therefore, we need to find a new improvement concept to avoid the rank reversal problem of TOPSIS method.

2.2.3. *S-type utility function and its application in multi-attribute decision making.* The concept of utility was first applied to the economic system. In the narrow sense, it refers to a measure of consumers' gratification of their own needs and desires through consumption, while in the broad sense, it refers to a measure of personal inner pleasure. In the decision-making problems of management, indicator non-dimensionalization through some nonlinear utility functions based on the specific evaluation environment can reflect the satisfaction of decision-makers with a certain scheme under a specific indicator better, thereby making the evaluation results more targeted and better reflecting the value judgment of decision-makers. Su summarized the research on utility function evaluation at that time according to the previous literature, and divided the common dimensionless functions into five classes, namely, convex upward function, convex downward function, sigmoid function, line function and hybrid function [20]. He pointed out that the derivative form of sigmoid function is consistent with the concept of "marginal benefit" in economics, which is more consistent with the universal value judgment of human beings than other forms such as line utility, and allows the evaluation conclusion to be more reasonable and accurate. In the past scholars often used exponential function to establish sigmoid utility function. Dai put forward a new sigmoid exponential utility function [21]. In order to reflect his idea of rewarding the good and punishing the bad, the original evaluation value was converted to interval $[-1, 1]$ by exponential transformation, and this method was applied to the multi-index comprehensive evaluation model based on artificial neural network. Li and Wei established three sigmoid utility functions in the decision-making model to address cost-type, benefit-type and interval-type indicators, respectively [22]. Similarly, in recent years, many studies have been conducted on the introduction of sigmoid utility into decision-making methods. Liu introduced hyperbolic absolute risk aversion function into the framework of sigmoid utility function, put forward sigmoid hyperbolic absolute risk aversion utility function, and applied it to intuitionistic fuzzy number multi-attribute decision making method [23]. Gao et al. proposed the interval intuitionistic fuzzy number multi-attribute decision making method of sigmoid utility function based on Bounded Rational Hypothesis [11]. The sigmoid function used by [24] is very similar to that used by [22], but the former requires researchers to specify a coefficient reflecting user sensitivity. Many scholars, combined with their respective empirical studies, have demonstrated the feasibility of sigmoid utility function in the evaluation system and its scientific advantages compared with traditional line utility by theoretical narration, questionnaire and interview.

3. Framework for Evaluating Rank Reversal in TOPSIS.

3.1. **Proposed method.** S-type utility function has been widely used in the study of multi-attribute decision making, but researchers have not yet explored its role in solving rank reversal problems. Similar to the principle of describing the subjective risk preference of decision makers in finance, when the subject performs poorly compared with other subjects under a certain index, it is shown as risk preference; otherwise it is shown as risk aversion [25]. It should also be noted that in the competition and comparison relationship, the marginal utility of indexes should be related to the distribution of data under indexes. The marginal utility of a subject's score under a certain index tends to increase before decreasing, because the same added value can surpass more "rivals" in the data set, which allows the subject to have more advantages in the competition. In real life, the data are mostly concentrated near the mean value, and the farther away from the mean value, the more discrete the data. Combined with the causes of the rank reversal problem mentioned above, in the traditional TOPSIS method, the introduction of extreme values will greatly change the position of positive and negative ideal solutions, resulting in the change of evaluation criteria. However, in fact, for decision makers, their like or dislike toward extreme values is often not as intense as the original data showing. Therefore, we can introduce the sigmoid utility function to process the data, emphasize the middle gap, weaken the tail gap, so as to reduce the influence of the introduced extreme value on the decision-making of the original sample, thus reducing the possibility of the rank reversal problem while allowing the utility added value to reflect the subjective judgment of decision makers correctly. Meanwhile, the introduction of exponential function to construct sigmoid utility function causes a new problem: there is no unified construction method for exponential utility functions applied in previous literature, where the application of exponential functions is the only common point among them. Moreover, the parameters related to marginal utility in the function are often not fully and reasonably explained. For example, if [22] directly uses $U(x) = \begin{cases} \frac{1}{1+e^{-x}}, & \text{benefit indicator} \\ \frac{1}{1+e^{x}}, & \text{cost indicator} \end{cases}$ as a utility function, an obvious problem would arise: which standard does the researcher use to define the relevant parameters in the function? Why cannot the utility function be $U(x) = \begin{cases} \frac{1}{1+2e^{-x}}, & \text{benefit indicator} \\ \frac{1}{1+2e^{x}}, & \text{cost indicator} \end{cases}$ or $U(x) = \begin{cases} \frac{1}{1+e^{-2x}}, & \text{benefit indicator} \\ \frac{1}{1+e^{2x}}, & \text{cost indicator} \end{cases}$? This problem has not been defined in the current research. Although the authors of [24] considered this problem and used sigmoid function with unknown parameters that had not been defined in advance, they did not explain the principle or method of setting the final parameters in empirical cases. The realistic basis of the function is difficult to explain, which reduces the scientificity of the method to some extent and increases the difficulty of application. Therefore, this paper proposes to fit the utility function with the normal distribution function, establish a new sigmoid utility function and test its feasibility.

As mentioned above, the more competitors the entity surpasses under a certain index, the greater the competitive advantage of the entity under this index. For any index, utility is closely related to data distribution: the more centralized the data, the greater the partial marginal utility, and the more discrete the data, the smaller the partial marginal utility. In practice, the data often obey a normal or nearly normal distribution, so we can use normal approximation instead of exponential function to process the data. Assume data under any index $X_j \sim N\left(\bar{X}, s^2\right)$, the density function $f_j(x)$ corresponding to X_j is the marginal utility function of X_j, and the distribution function corresponding to X_j is the utility function of X_j, that is, the utility function is fitted with the normal distribution function. For benefit indexes, the closer to 1 after standardization means the better performance of the scheme, and the closer to 0 means the worse performance of the scheme; For cost indexes, the opposite is true. At this time, the utility can be interpreted as the dissatisfaction of decision makers. The closer to 1, the worse the performance of the scheme. The closer to 0, the better. The results are shown in Figure 1.

According to the definition of distribution function and density function of normal distribution, it can be seen that the marginal utility of X is large when being close to the mean value and small when falling at both ends of the data, which conforms to a symmetrical bell-shaped distribution with non-negativity. With normative characteristics, the utility function $0 \leq \mu(X) \leq 1$ of X is subject to a sigmoid distribution. This method inherits the advantages of sigmoid utility function, wherein the parameters are estimated by samples, which need not be specified by researchers. Besides, the explanation of marginal utility is more realistic than the ordinary exponential utility function. The parameters $\bar{X}$ and s used in our fitted normal distribution are easily affected by the extreme values of the data. If the distribution is fitted directly with the original sequence, to the extent

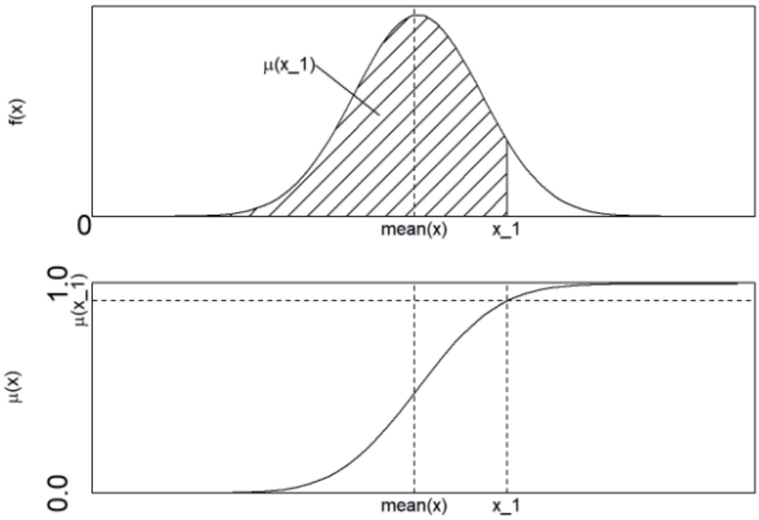

FIGURE 1. Marginal utility and utility of x fitted by normal density and distribution function

where the samples with abnormal values are introduced, the skewness of the data under certain indicators may be too large, causing $\bar{X}$ to deviate greatly and making it difficult to represent the concentration trend of the data. As a result, the fitting result of utility function will be greatly changed, leaving the rank reversal problem partially addressed still. Therefore, it is necessary to eliminate outliers in the sample before fitting the utility function. According to the rule of experience, if the data obey the normal distribution, about 68% of the data are between $\left(\bar{X}-\sigma, \bar{X}+\sigma\right)$, about 95% of the data are between $\left(\bar{X}-2\sigma, \bar{X}+2\sigma\right)$, and about 99.7% of the data are between $\left(\bar{X}-3\sigma, \bar{X}+3\sigma\right)$. We can use the sample variance approximation to replace the population variance, and select an appropriate interval according to the size of the sample range. The values outside the interval are defined as extreme values, which should be eliminated when fitting the normal distribution. Although the specific rules can be set by the decision-maker, we set one condition that the expected value of data outside the interval less than 1 should be met at least.

In short, researchers in the past mainly used the more direct "truncation method" to avoid the rank reversal problem. That is artificially set a limit that includes all possible values, and do not allow any extreme data beyond the specified limit. In fact, this is difficult to implement in most evaluation scenarios. This paper will use "weakening method" to improve TOPSIS method, and introduce S-type utility function to improve the dimensionless step of traditional TOPSIS, so as to reduce the change of extreme value to the upper and lower limits of index data. While allowing the "abnormal scheme" to have the evaluation system, the consistency and scientificity of the evaluation results shall be ensured.

3.2. **Specific algorithm.** Suppose there are m schemes and n indexes, the original decision matrix is $X_{mn} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$, the weight coefficient $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$.

① Specify a coefficient z according to the rule of experience, so that the expected number of normal samples falling outside the limit $\left(\bar{X} - z\cdot\sigma, \bar{X} + z\cdot\sigma\right)$ is less than 1. Eliminate the data which is outside $\left(\bar{X} - z \cdot \sigma, \bar{X} + z \cdot \sigma\right)$ when fitting the normal *utility function*. This step needs to be repeated until all the obtained data fall into the theoretical range. Set the data under the jth index as $Y_j = (y_1, y_2, \ldots, y_p)^T$, $j = 1, 2, \ldots, n$; $0 < p \leq i$ after completely eliminating the abnormal value.

② Calculate the mean value $\left(\overline{Y_1}, \overline{Y_2}, \ldots, \overline{Y_n}\right)$ of each index after excluding extreme data and standard deviation $(s_{Y_1}, s_{Y_2}, \ldots, s_{Y_n})$, get distribution $x_{ij} \sim N\left(\overline{Y_j}, {s_{Y_j}}^2\right)$, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. And based on this to calculate the utility function matrix

$$\mu_{mn} = \begin{pmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mn} \end{pmatrix} = \left(\varphi\left(\frac{x_{ij} - \overline{Y_j}}{s_{Y_j}}\right)\right)$$

$$= \begin{pmatrix} \varphi\left(\frac{x_{11} - \overline{Y_1}}{s_{Y_1}}\right) & \varphi\left(\frac{x_{12} - \overline{Y_2}}{s_{Y_2}}\right) & \cdots & \varphi\left(\frac{x_{1n} - \overline{Y_n}}{s_{Y_n}}\right) \\ \varphi\left(\frac{x_{21} - \overline{Y_1}}{s_{Y_1}}\right) & \varphi\left(\frac{x_{22} - \overline{Y_2}}{s_{Y_2}}\right) & \cdots & \varphi\left(\frac{x_{2n} - \overline{Y_n}}{s_{Y_n}}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi\left(\frac{x_{m1} - \overline{Y_1}}{s_{Y_1}}\right) & \varphi\left(\frac{x_{m2} - \overline{Y_2}}{s_{Y_2}}\right) & \cdots & \varphi\left(\frac{x_{mn} - \overline{Y_n}}{s_{Y_n}}\right) \end{pmatrix},$$

$$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.$$

In this way, we can use μ_{mn} instead of $s_{ij} = \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} x_{ij}^2}}$ as the normalization result of the original data to reduce the impact of extreme values on data standardization.

③ Calculate weighted utility matrix $K_{mn} = \begin{pmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{m1} & k_{m2} & \cdots & k_{mn} \end{pmatrix} = \mu_{mn} * \omega$.

④ Get the positive ideal solution under each attribute $k^* = (k_1^*, k_2^*, \ldots, k_n^*)$ and the negative ideal solution $k^0 = \left(k_1^0, k_2^0, \ldots, k_n^0\right)$, in which

$$k_j^* = \begin{cases} \max_i k_{ij}, & j \text{ index is benefit type attribute} \\ \min_i k_{ij}, & j \text{ index is cost type attribute} \end{cases},$$

$$k_j^0 = \begin{cases} \min_i k_{ij}, & j \text{ index is benefit type attribute} \\ \max_i k_{ij}, & j \text{ index is cost type attribute} \end{cases},$$

$$i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n.$$

⑤ Calculate the distance between each scheme and the positive ideal solution $d^* = (d_1^*, d_2^*, \ldots, d_m^*)^T$ or the negative ideal solution $d^0 = (d_1^0, d_2^0, \ldots, d_m^0)^T$, in which

$$d_i^* = \sqrt{\sum_{j=1}^{n} \left(k_{ij} - k_j^*\right)^2}, \quad d_i^0 = \sqrt{\sum_{j=1}^{n} \left(k_{ij} - k_j^0\right)^2}, \quad i = 1, 2, \ldots, m.$$

⑥ Calculate the closeness between the scheme and the positive ideal solution $C^* = (C_1^*, C_2^*, \ldots, C_m^*)^T$, in which $C_i^* = \frac{d_i^0}{d_i^0 + d_i^*}$, $i = 1, 2, \ldots, m$.

⑦ Sort the value of closeness degree C_i^* to get the order from best to worst of each scheme.

4. **Experimental Analysis.** This paper selects part index data of 8 high schools in a city of China in 2020 (Table 3), and uses the traditional TOPSIS method and the improved TOPSIS method to sort the overall performance of the middle school in four aspects: the undergraduate enrollment rate, the excellent teachers' number of city level, the students' number of school, and the average tuition and fee. Among them, the average tuition and fee is the cost type index, and the others are the benefit type index. Suppose the weight of each index is 0.25. X_8 is as the new scheme of this experiment. By comparing the decision-making results obtained by the traditional TOPSIS method and the improved TOPSIS method before and after adding the new scheme, it is judged whether the improved TOPSIS method can solve the rank reversal problem of decision-making, and whether there is too much difference between the decision-making results of the improved TOPSIS method and the traditional method.

Table 3. Statistical table of actual values of each index

	Undergraduate enrollment rate (%)	Excellent teachers' number of city level	Students' number of school	Average tuition and fee (RMB/year)
X_1	96	23	2821	4797
X_2	92	27	2648	5499
X_3	94	25	2816	4800
X_4	95	22	2632	5850
X_5	97	25	2848	7839
X_6	96	23	2656	4797
X_7	98	25	3019	4680
X_8	98	27	3024	35100

4.1. **Traditional TOPSIS method.** First, sort the original scheme group (X_1-X_7). The data are normalized and the results are shown in Table 4.

Table 4. Standardization matrix of original scheme group under traditional TOPSIS method

0.3802	0.3572	0.3835	0.3257
0.3643	0.4193	0.3600	0.3734
0.3722	0.3883	0.3828	0.3259
0.3762	0.3417	0.3578	0.3972
0.3841	0.3883	0.3872	0.5323
0.3802	0.3572	0.3611	0.3257
0.3881	0.3883	0.4104	0.3178

Weighting the standardized results, get Table 5.

According to Table 5, get the following results: the positive ideal solution $k^* = (0.0970, 0.1048, 0.1026, 0.0794)$; the negative ideal solution $k^0 = (0.0911, 0.0854, 0.0895, 0.1331)$; the distance between each scheme and the positive ideal solution $d^* = (0.2119, 0.3266, 0.2016, 0.3674, 0.2219, 0.2723, 0.0705)$; the distance between each scheme and the negative ideal solution $d^0 = (0.2593, 0.2188, 0.2600, 0.0940, 0.2707, 0.2314, 0.3824)$; closeness of each scheme before adding new samples C^*: $(0.7531, 0.6910, 0.8250, 0.5233, 0.2116, 0.7217, 0.8796)$; rank the scheme according to the closeness C_i^* (from the best to the worst): $X_7 > X_3 > X_1 > X_6 > X_2 > X_4 > X_5$.

Table 5. Weighted standardization matrix of original scheme group under traditional TOPSIS method

0.0950	0.0893	0.0959	0.0814
0.0911	0.1048	0.0900	0.0934
0.0931	0.0971	0.0957	0.0815
0.0940	0.0854	0.0895	0.0993
0.0960	0.0971	0.0968	0.1331
0.0950	0.0893	0.0903	0.0814
0.0970	0.0971	0.1026	0.0794

Table 6. Standardization matrix of new scheme group under traditional TOPSIS method

0.3544	0.3294	0.3547	0.1260
0.3396	0.3867	0.3330	0.1445
0.3470	0.3581	0.3541	0.1261
0.3507	0.3151	0.3309	0.1537
0.3581	0.3581	0.3581	0.2059
0.3544	0.3294	0.3340	0.1260
0.3618	0.3581	0.3796	0.1230
0.3618	0.3867	0.3802	0.9221

Table 7. Weighted standardization matrix of new scheme group under traditional TOPSIS method

0.0886	0.0824	0.0887	0.0315
0.0849	0.0967	0.0832	0.0361
0.0868	0.0895	0.0885	0.0315
0.0877	0.0788	0.0827	0.0384
0.0895	0.0895	0.0895	0.0515
0.0886	0.0824	0.0835	0.0315
0.0904	0.0895	0.0949	0.0307
0.0904	0.0967	0.0951	0.2305

Next, sort the scheme groups after adding X_8. The standardized results are shown in Table 6 and the weighted results are shown in Table 7.

According to Table 7, we can get the positive ideal solution $k^* = (0.0904, 0.0967, 0.0951, 0.0307)$; the negative ideal solution $k^0 = (0.0849, 0.0788, 0.0827, 0.2305)$; the distance between each scheme and the positive ideal solution $d^* = (0.0632, 0.0565, 0.0416, 0.0929, 0.0906, 0.0741, 0.0287, 0.7992)$; the distance between each scheme and the negative ideal solution $d^0 = (0.7967, 0.7810, 0.7976, 0.7685, 0.7182, 0.7964, 0.8021, 0.0897)$; closeness of each scheme after adding new samples C^*: $(0.9265, 0.9326, 0.9504, 0.8922, 0.8880, 0.9149, 0.9655, 0.1009)$; rank the scheme according to the closeness C_i^* (from the best to the worst): $X_7 > X_3 > X_2 > X_1 > X_6 > X_4 > X_5 > X_8$.

Comparing the two decision results, if the traditional TOPSIS comprehensive evaluation decision method is used, the ranking of X_1, X_2 and X_6 changed after adding the new scheme, resulting in the rank reversal problem.

4.2. **Improved TOPSIS method.** Since there are only 7-8 decision-making schemes, we specify $\left(\bar{X} - 2\sigma, \bar{X} + 2\sigma\right)$ as the reasonable interval under the index, and the data falling outside the interval need to be eliminated when fitting the normal function of the corresponding index.

First, rank the original scheme.

After inspection, the extreme value of 7839 under the index of average tuition and fee in the original scheme group needs to be eliminated. The data left for fitting the normal distribution are shown in Table 8.

Table 8. Data used to fit normal distribution of the original scheme group under the improved TOPSIS method

96	23	2821	4797
92	27	2648	5499
94	25	2816	4800
95	22	2632	5850
97	25	2848	–
96	23	2656	4797
98	25	3019	4680

Fit the normal distribution $N\left(\overline{X_j}, {s_{X_j}}^2\right)$ with the data in Table 8, and get the utility function matrix shown in Table 9.

Table 9. Utility matrix of the original scheme group under the improved TOPSIS method

0.6131	0.2253	0.6221	0.2856
0.0423	0.9444	0.1799	0.8125
0.2362	0.6624	0.6085	0.2877
0.4147	0.0899	0.1517	0.9467
0.7854	0.6624	0.6923	1.0000
0.6131	0.2253	0.1952	0.2856
0.9021	0.6624	0.9568	0.2094

Multiply the utility matrix by the weight matrix to obtain the weighted utility matrix, as shown in Table 10.

Table 10. Weighted utility matrix of the original scheme group under the improved TOPSIS method

0.1533	0.0563	0.1555	0.0714
0.0106	0.2361	0.0450	0.2031
0.0591	0.1656	0.1521	0.0719
0.1037	0.0225	0.0379	0.2367
0.1963	0.1656	0.1731	0.2500
0.1533	0.0563	0.0488	0.0714
0.2255	0.1656	0.2392	0.0524

According to Table 10, we can get the positive ideal solution $k^* = (0.2255, 0.2361, 0.2392, 0.0524)$; the negative ideal solution $k^0 = (0.0106, 0.0225, 0.0379, 0.2500)$; the distance between each scheme and the positive ideal solution $d^* = (0.2119, 0.3266, 0.2016, 0.3674, 0.2219, 0.2723, 0.0705)$; the distance between each scheme and the negative ideal solution $d^0 = (0.2593, 0.2188, 0.2600, 0.0940, 0.2707, 0.2314, 0.3824)$; closeness of each scheme after adding new samples C^*: $(0.5503, 0.4012, 0.5632, 0.2038, 0.5495, 0.4593, 0.8444)$; rank the scheme according to the closeness C_i^* (from the best to the worst): $X_7 > X_3 > X_1 > X_5 > X_6 > X_2 > X_4$.

Next, rank the scheme groups after adding X_8 data.

After inspection, the two extreme value of the scheme group needs to be eliminated. The data left for fitting the normal distribution are shown in Table 11.

Table 11. Data used to fit normal distribution of the new scheme group under the improved TOPSIS method

96	23	2821	4797
92	27	2648	5499
94	25	2816	4800
95	22	2632	5850
97	25	2848	–
96	23	2656	4797
98	25	3019	4680
98	27	3024	–

Fit the normal distribution $N\left(\overline{Y_j}, {s_{Y_j}}^2\right)$ with the data in Table 11, and get the utility function matrix shown in Table 12.

Table 12. Utility matrix of the new scheme group under the improved TOPSIS method

0.5485	0.1895	0.5330	0.2856
0.0339	0.9008	0.1542	0.8125
0.1970	0.5805	0.5203	0.2877
0.3574	0.0776	0.1312	0.9467
0.7287	0.5805	0.6005	1.0000
0.5485	0.1895	0.1666	0.2856
0.8635	0.5805	0.9104	0.2094
0.8635	0.9008	0.9155	1.0000

Multiply the utility matrix by the weight matrix to obtain the weighted utility matrix, as shown in Table 13.

Table 13. Weighted utility matrix of the new scheme group under the improved TOPSIS method

0.1371	0.0474	0.1332	0.0714
0.0085	0.2252	0.0385	0.2031
0.0492	0.1451	0.1301	0.0719
0.0894	0.0194	0.0328	0.2367
0.1822	0.1451	0.1501	0.2500
0.1371	0.0474	0.0416	0.0714
0.2159	0.1451	0.2276	0.0524
0.2159	0.2252	0.2289	0.2500

According to Table 13, we can get the positive ideal solution $k^* = (0.2159, 0.2252, 0.2289, 0.0714)$; the negative ideal solution $k^0 = (0.0085, 0.0194, 0.0385, 0.2500)$; the distance between each scheme and the positive ideal solution $d^* = (0.2176, 0.3193, 0.2105, 0.3616, 0.2298, 0.2706, 0.0801, 0.1976)$; the distance between each scheme and the negative ideal solution $d^0 = (0.2435, 0.2111, 0.2421, 0.0820, 0.2444, 0.2221, 0.3685, 0.3519)$; closeness of each scheme after adding new samples C^*: $(0.5282, 0.3980, 0.5349, 0.1848, 0.5154, 0.4507, 0.8215, 0.6403)$; rank the scheme according to the closeness C_i^* (from the best to the worst): $X_7 > X_8 > X_3 > X_1 > X_5 > X_6 > X_2 > X_4$.

Comparing the results of the two experiments, we found that the order of the original scheme group did not change under the improved method. This shows that the improved method can help solve the rank reversal problem in the decision-making process and make the decision results more consistent. At the same time, according to the empirical results, we can see that the ranking results obtained by the improved method and the traditional TOPSIS method are similar for most schemes, and this method pays more attention to the performance of the evaluated individual under most indexes, and the evaluation results are less affected by the extreme performance under a few indexes, ensuring the scientificity of the evaluation results.

Compared with the other improved methods, the proposed method process is simpler and easy to apply in some comprehensive decision situation. In this way, the result can get a better explanation, and can be closer to their actual competitive ranking.

5. **Conclusion and Prospect.** This paper points out the rank reversal problem of TOPSIS method and its causes, summarizes the shortcomings of the existing improved methods, and points out the scientificity of S-type utility function and its advantages in solving the rank reversal problem. After reading and summarizing the previous literature, put forward a new construction method of S-type utility function, which uses the normal distribution function to fit the utility function of the index. This makes the new method more scientific, interpretable and universal than the original TOPSIS method, and demonstrates the above ideas with a case. Compared with the traditional TOPSIS method, the method proposed in this paper can effectively avoid the occurrence of rank reversal problem. At the same time, the results obtained by the two methods are very similar, and there is no contradiction that is difficult to explain. Compared with other S-type utility functions, the parameters in the improved method have stronger realistic basis. It can make most of the actual data obey normal distribution or approximate normal distribution. Fitting the utility function with the distribution function of normal distribution can reflect the competitive advantage of the data for the scheme under a certain index well, which is in line with the value judgment of decision makers in most cases. Compared with the previous improved methods for rank reversal problem, it avoids excessive subjective judgment and greatly reduces the workload and difficulty of decision makers. At the same time, the new method is also generalizable. Even if the data does not obey the normal distribution, the corresponding distribution function can be used to fit the utility function, and the data can be similarly standardized. Therefore, this approach is suitable for generalization in decision situations where various alternative sets are possible to change, especially in the large and comprehensive sample decision case.

Of course, the new method of fitting utility function with normal distribution function also has its limitations. When the number of schemes is quite small and the number of indexes is quite large, due to the insufficient sample size under each index, the possibility of deviation between the fitting result and the actual utility is also large. Therefore, it is necessary to be more cautious to use this method when the number of schemes is quite small.

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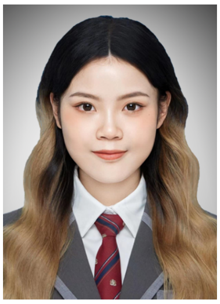

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