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COMBINING DIRECT DATA DRIVEN AND MODEL PREDICTIVE CONTROL WITH SET MEMBERSHIP UNCERTAINTY

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Abstract. *To extend the always studied system with statistical uncertainty, this paper considers the control problem for the system with set membership uncertainty, thus being more suited in practice. As the most important element in applying model predictive control strategy is the unknown output predictor, the new idea of direct data driven is used to identify the unknown state in the framework of set membership uncertainty, such as process noise and measurement process. After our recursive state estimation is substituted in our own mathematical derivations, the output predictor is yielded sequently. Then one quadratic programming problem is constructed to satisfy the goal of model predictive control. An ellipsoid optimization algorithm is proposed to solve this quadratic programming problem, whose optimal control input corresponds to the center of the final ellipsoid. Finally, one simulation example is used to prove the efficiency of our proposed theories.*
Keywords: Model predictive control, Direct data driven, Set membership uncertainty, Ellipsoid optimization algorithm

1. **Introduction.** Due to no need for mathematical equation for the unknown system, direct data driven control is widely studied in theory research and practice, such as paper machine, chemical process and water networks. Generally, direct data driven control includes some kinds of forms, for example, virtual reference feedback tuning and iterative correlation tuning. As system identification is the original idea for direct data driven control, lots of new concepts, coming from system identification, can be also applied to direct data driven control, for example, persistent excitation, model validation and optimal input signal. Consider one special case, where constraint conditions are imposed in unknown parameters in system identification, then lots of existing numerical optimization methods are introduced to identify the unknown parameters, and Lyapunov theory is used to analyze the parameter convergence and algorithm stability. Similarly, if constraint conditions are also considered in controller design, then one novel control strategy is yielded to be model predictive control.

In case of the unknown but bounded noise, one bounded error identification is proposed to identify the unknown systems with time varying parameters. Then one feasible parameter set is constructed to include the unknown parameter with a given probability level. In [1], the feasible parameter set is replaced by one confidence interval, as this confidence interval can accurately describe the actual probability that the future predictor will fall into the constructed confidence interval. The problem about how to construct this confidence interval is solved by a linear approximation/programming approach [2],

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which can identify the unknown parameter only for linear regression model. According to the obtained feasible parameter set or confidence interval, the midpoint or center can be deemed as the final parameter estimation. This robustness corresponds to other external noises, such as outlier, and unmeasured disturbance [3]. The above mentioned identification strategy, used to construct one set or interval for unknown parameter, is called as set membership identification, dealing with the unknown but bounded noise. There are two kinds of descriptions on external noise, one is probabilistic description, and the other is deterministic description, corresponding to the unknown but bounded noise here [4]. For the probabilistic description on external noise, the noise is always assumed to be one white noise, and its probabilistic density function is known in advance. In reality or practice, bounded noise is more common than white noise. Within the deterministic description on external noise, set membership identification is adjusted to design controllers with two degrees of freedom [5]; it corresponds to data driven control or set membership control. Set membership control is applied to designing feedback control in a closed loop system with nonlinear system in [6], where the considered system is identified by set membership identification, and the obtained system parameter will be benefit for the prediction output. After substituting the obtained system parameter into the prediction output to construct one cost function, [7] takes the derivative of the above cost function with respect to control input to achieve one optimal input. Set membership identification can be not only applied in MC, but also in stochastic adaptive control [8]. Based on the bounded noise, many parameters are also included in known intervals in prior, and then robust optimal control with adjustable uncertainty sets is studied in [9], where robust optimization is introduced to consider uncertain noise and uncertain parameter simultaneously. To solve the expectation operation with dependence on the uncertainty, sample size of random convex programs is considered to replace the expectation by finite sum [10]. Generally, many practical problems in systems and control, such as controller synthesis and state estimation, are often formulated as optimization problems [11].

Our combination with these two control strategies can remedy their own shortcomings, and extend to other forms, such as set membership uncertainty. More specifically, consider one simple linear system with set membership uncertainty on process noise and measurement noise. The final goal is to apply model predictive control to guaranteeing the output predictive to track one given desired output value at each time instant, so the most important factor is to obtain or construct the output prediction at each time instant. However, when to construct these output predictions, no any information on state is given in case of set membership uncertainty. To get information about state estimations at all time instants, classical Kalman filtering is useless, due to this set membership uncertainty, not the probabilistic uncertainty. Then direct data driven idea is to solve this state estimation problem with set membership uncertainty, i.e., recursive state estimation is given. Based on our recursive state estimation, the output predictive at each time instant is easily obtained only through some simple mathematical operations. After comparing the output predictive with its own desired output value at each time instant, one cost function for model predictive control is yielded, while also considering some constraint conditions on input signal, such as persistent excitation or limited amplitude. Then one novel ellipsoid optimization algorithm is proposed to solve this above constraint optimization problem, whose decision variables are a sequence of ellipsoids with decreasing volumes. Then the central point of that final ellipsoid is deemed as the acceptable decision variable, i.e., the optimal input signal.

Generally, the main contributions are formulated as follows. Direct data driven is used to achieve the goal of state estimation in case of set membership uncertainty, and then the state estimation is extended to yield the output prediction at each time instant. Combining the output prediction and constraint condition to form one constraint optimization problem for model predictive control, then one novel ellipsoid optimization algorithm is proposed to solve this constraint optimization problem.

2. **Problem Statement.** Suppose that we are giving a linear dynamic system with control input

$$x(k+1) = Ax(k) + Bu(k) + w(k), \quad k = 0, 1, \ldots, N-1 \tag{1}$$

where $x(k) \in R^n$ and $u(k) \in R^m$ denote the state and control input vectors, respectively, and $w(k) \in R^n$ is the disturbance vector. Suppose also that at each time instant k, we receive a measurement $y(k) \in R^l$ of the form

$$y(k) = Cx(k) + v(k) \tag{2}$$

where $v(k) \in R^l$ is an unknown observation noise vector, and matrices A, B, C are given.

Combining state Equation (1) and measurement Equation (2) to be one following state space form, that is also plotted in Figure 1.

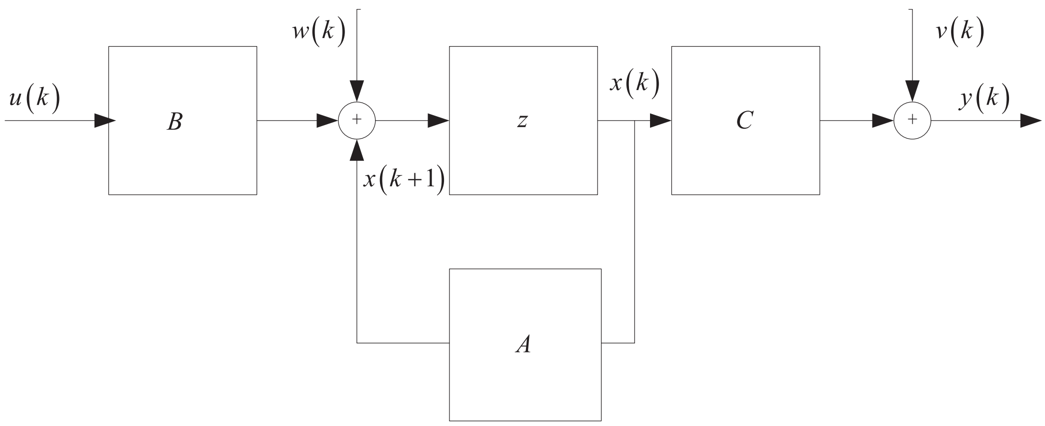


FIGURE 1. State space model

In Figure 1, z is one time delay operator, i.e., $zx(k+1) = x(k)$. For convenience, two noises $w(k)$ and $v(k)$ correspond to process noise and measurement noise, respectively.

$$x(k+1) = Ax(k) + Bu(k) + w(k); \quad y(k) = Cx(k) + v(k) \tag{3}$$

To be convenient for formulating the later cost function in model predictive control, output predictor for each output value $y(k)$ at each time instant k is needed. Observing Equation (3) again, it tells us that firstly state estimation for each state $x(k)$ is identified within the condition of noises, which are known to be in one set, instead of the known probability distributions.

An important and generic problem is to estimate the state value of each $x(k)$, given the observations $y(1), \ldots, y(k)$, accumulated up to time k. The uncertain variables here are the initial state $x(0)$, the process noises $w(0), \ldots, w(N-1)$, and the measurement noise $v(1), \ldots, v(N)$. If the joint probability distribution for these uncertain variables is given, one may calculate the conditional distribution of $x(k)$ given $y(1), \ldots, y(k)$, and obtain state estimations such as the conditional expectation $E\{x(k)|y(1), \ldots, y(k)\}$. This approach leads to the classical Kalman filtering algorithm.

3. **State Estimation with Set Membership Uncertainty.** Assume that, instead of a probability distribution, a set $\mathcal{R}$ includes the vector of unknown variables.

$$r = (x(0), w(0), \ldots, w(N-1), v(1), \ldots, v(N)) \tag{4}$$

It means $r \in \mathcal{R}$, i.e., all unknown variables, including initial state, process noises and measurement noises, are known to belong to a set $\mathcal{R}$.

Observing state equation again, then the state $x(k+1)$ can be expressed in terms of r and control input as follows.

$$x(k+1) = A^k x(0) + \begin{bmatrix} A^k & A^{k-1} & \cdots & A & 1 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(k) \end{bmatrix} + \begin{bmatrix} B & AB & \cdots & A^k B \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(0) \end{bmatrix} \tag{5}$$

Changing the time instant $k+1$ to k, then it holds that

$$x(k) = \begin{bmatrix} A^{k-1} & A^{k-1} & A^{k-2} & \cdots & A & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x(0) \\ w(0) \\ \vdots \\ w(k) \\ v(1) \\ \vdots \\ v(k) \end{bmatrix} + \begin{bmatrix} B & AB & \cdots & A^k B \end{bmatrix} \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(0) \end{bmatrix} \tag{6}$$

or more abstractly as

$$x(k) = L_k r + O_k u \tag{7}$$

where L_k and O_k are two appropriate matrices, and their explicit forms are given as

$$L_k = \begin{bmatrix} A^{k-1} & A^{k-1} & A^{k-2} & \cdots & A & 1 & 0 & \cdots & 0 \end{bmatrix}$$
$$r = [x(0), w(0), \ldots, w(k), v(1), \ldots, v(k)]; \quad u = [u(k), u(k-1), \ldots, u(0)]$$
$$O_k = \begin{bmatrix} B & AB & \cdots & A^k B \end{bmatrix}$$

Thus, knowing that $r \in \mathcal{R}$ and before any measurements are considered, the state $x(k)$ is known to belong to the set.

$$X_k = L_k \mathcal{R} + O_k u = \{L_k r + O_k u | r \in \mathcal{R}\} \tag{8}$$

Similarly, it holds that

$$X_k - O_k u = \{L_k r | r \in \mathcal{R}\} \tag{9}$$

Each measurement $y(i)$ restricts the set of possible values of r to be such that

$$y(i) = \begin{bmatrix} CL_i & 1 \end{bmatrix} \begin{bmatrix} r \\ v(i) \end{bmatrix} + CO_i u = E_i r + CO_i u \tag{10}$$

for an appropriate matrix E_i.

Thus, with each new measurement, the set of possible vector r is further restricted. More specifically, given the measurements $y(1), y(2), \ldots, y(k)$, the set of possible vector r is given by

$$\begin{aligned} &\mathcal{R}_k(y(1), y(2), \ldots, y(k)) \\ &= \mathcal{R} \cap \{r|y(1) = E_1 r + CO_1 u\} \cap \cdots \cap \{r|y(k) = E_k r + CO_k u\} \end{aligned} \tag{11}$$

and by a linear transformation, yields the set of possible state estimation $x(k)$ as

$$X_k(y(1), y(2), \ldots, y(k)) = L_k(\mathcal{R}_k(y(1), y(2), \ldots, y(k))) + CO_k u \tag{12}$$

where in Equation (12), no any constraint conditions are on control input u and here the case corresponds to that the matrices A, B, and C are all known.

Consider the most favorable case, where the set of possible states $X_k(y(1), y(2), \ldots, y(k))$ turns out to be an ellipsoid, and $CO_k u$ means one translation operation in geometry. Suppose that the vector of unknown variables r is known in a set of the following form.

$$\begin{aligned} \mathcal{R} = \Bigg\{ r| \left(x(0) - \hat{x}(0)\right)^T S^{-1} \left(x(0) - \hat{x}(0)\right) + \sum_{i=0}^{N-1} \big(w^T(i) M_i^{-1} w(i) \\ + v^T(i+1) N_{i+1}^{-1} v(i+1)\big) \leq 1 \Bigg\} \end{aligned} \tag{13}$$

where S, M_i and N_i are all positive definite symmetric matrices, and $\hat{x}(0)$ is a given vector. The above first term denotes the uncertainty of the initial state. Thus, we have $\xi \in X_k(y(1), y(2), \ldots, y(k))$ if and only if $V_k(\xi)$ is the optimal cost of the problem of minimizing the quadratic cost.

$$\begin{aligned} &\sum_{i=0}^{N-1} [y(i+1) - Cx(i+1)]^T N_{i+1}^{-1} [y(i+1) - Cx(i+1)] \\ &+ \left(x(0) - \hat{x}(0)\right)^T S^{-1} \left(x(0) - \hat{x}(0)\right) + \sum_{i=0}^{N-1} w^T(i) M_i^{-1} w(i) \end{aligned} \tag{14}$$

As the cost function $V_k(\xi)$ is quadratic in ξ, and in the above problem, the system (1) and the cost function is quadratic, the forward recursion can be used to calculate the optimal cost, for example,

$$\begin{aligned} V_i(x(i)) = \min\nolimits_{x(i-1)} \big\{ &V_{i-1}(x(i-1)) + [x(i) - Ax(i-1) - Bu(i-1)]^T M_{i-1}^{-1} [*] \\ &+ [y(i) - Cx(i)]^T N_i^{-1} [y(i) - Cx(i)] \big\} \end{aligned} \tag{15}$$

Start with the initial condition

$$V_0(x(0)) = \left(x(0) - \hat{x}(0)\right)^T S^{-1} \left(x(0) - \hat{x}(0)\right) \tag{16}$$

Here recursive state estimation is applied to finding an ellipsoid, within which the possible state at each time instant k lies.

Comment: Given initial state $\hat{x}(0)$ and set $\mathcal{R}$ within which the vector of unknown quantities, a bounded set $\hat{X}_k$ corresponding to all possible states, is given by the following ellipsoid.

$$\hat{X}_k = \left\{ x(k) : (x(k) - \hat{x}(k))^T \Sigma_k^{-1} \left(x(k) - \hat{x}(k)\right) + \delta_k \leq 1 \right\} \tag{17}$$

where $\hat{x}(k)$ and Σ_k^{-1} are generated by the recursions.

$$\begin{aligned} \hat{x}(k) &= A\hat{x}(k-1) + \Sigma_k N_k^{-1} \left(y(k) - CA\hat{x}(k-1)\right) \\ \Sigma_k &= \left[A \Sigma_{k-1} A^T + C^T N_k^{-1} C + B M_{k-1} B^T\right]^{-1} \end{aligned} \tag{18}$$

with the initial condition

$$\Sigma_0 = S \tag{19}$$

and δ_k is given by

$$\delta_k = \sum_{i=1}^{k} [y(i) - CA\hat{x}(i-1)]^T \left[C\Sigma_i C^T + N_i\right]^{-1} [y(i) - CA\hat{x}(i-1)] \tag{20}$$

From ellipsoid (17), we see that at each time instant k, state estimation $x(k)$ will be in this ellipsoid, i.e., $x(k) \in \hat{X}_k$. Rough speaking, in practice recursive state estimation $\hat{x}(k)$ can be deemed as the state estimation at time instant k, i.e., $x(k) = \hat{x}(k)$. For a special case, we now consider a different type of set description of the uncertainty. In particular, assume that the initial state, the process noise and the measurement noise are independently constrained to lie in ellipsoids. In other words, we have

$$\begin{aligned} &x^T(0)S^{-1}x(0) \le 1; \quad w^T(i)M_i^{-1}w(i) \le 1 \\ &v^T(i+1)N_{i+1}^{-1}v(i+1) \le 1, \quad i = 0, 1, \ldots, N \end{aligned} \tag{21}$$

where S, M_i, N_i are all given symmetric positive definite matrices. Thus, we have

$$\begin{aligned} &[x(i) - Ax(i-1) - Bu(i-1)]^T M_i^{-1}[x(i) - Ax(i-1) - Bu(i-1)] \le 1 \\ &\left(x(i) - C^{-1}y(i)\right)^T N_i^{-1} \left(x(i) - C^{-1}y(i)\right) \le 1, \quad i = 0, 1, 2, \ldots, N \end{aligned} \tag{22}$$

Then the problem of the best inner ellipsoidal approximation of the intersection of two ellipsoids is the explicit semidefinite program.

Maximize t

$$\begin{aligned} \text{subject to } &\begin{bmatrix} I & \sqrt{M_i}(z - Ax(i-1) - Bu(i-1)) & \sqrt{M_i}Z \\ \sqrt{M_i}(z - Ax(i-1) - Bu(i-1)) & 1 - \lambda_1 & 0 \\ \sqrt{M_i}Z & 0 & \lambda_1 I \end{bmatrix} \\ &\ge 0 \\ &t \le (Det Z)^{\frac{1}{n}}, \quad Z \ge 0 \\ &\begin{bmatrix} I & \sqrt{N_i}\,(z - C^{-1}y(i)) & \sqrt{N_i}Z \\ \sqrt{N_i}\,(z - C^{-1}y(i)) & 1 - \lambda_2 & 0 \\ \sqrt{N_i}Z & 0 & \lambda_2 I \end{bmatrix} \ge 0 \end{aligned} \tag{23}$$

with the design variables $Z, z, \lambda_1, \lambda_2, t \in R$. The largest ellipsoid contained in the intersection is given by an optimal solution $Z_*, z_*, \lambda_1^*, \lambda_2^*, t_* \in R$ via the relation.

$$X = \left\{x(k) = Z_* a + z_* | a^T a \le 1\right\} \tag{24}$$

Comment: The above Equation (24) is also an ellipsoid. Set $x(k) = Z_* a + z_*$ and $a^T a \le 1$, then $Z_* a = x(k) - z_*$ or $a = Z_*^{-1}[x(k) - z_*]$. After applying the inequality condition $a^T a \le 1$, then it holds that $[x(k) - z_*]^{-1} Z_*^{-1} Z_*^{-T} \le 1$. It means Equation (24) is equivalent to

$$X = \left\{x(k) | [x(k) - z_*]^{-1} Z_*^{-1} Z_*^{-T} [x(k) - z_*] \le 1\right\} \tag{25}$$

which is also an ellipsoid.

Observing that linear matrix inequality in Equation (23), it shows the fact that the ellipsoid $\left\{x(k) = Z_* a + z_* | a^T a \le 1\right\}$ with $Z_* \ge 0$ is contained in every one of the ellipsoids (22), i.e., it is contained in the intersection of these ellipsoids.

4. **Constraint Model Predictive Control.** As the goal of identification or estimation is for control, i.e., identification for control, in this section we start to apply the above derived state estimation to yielding the output prediction, which is the most element in model predictive control, and then the optimal control input is generated through solving one optimization problem in model predictive control.

For notational clarity, rewrite that state space form (3) as follows again.

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + w(k) \\ y(k) &= Cx(k) + w(k) \end{aligned} \tag{26}$$

where in Equation (26), for convenience set $w(k) = v(k)$; otherwise the latter derivations are similar.

Assume $\phi = A - C$ is stable and the input-output sequence $z(k) = [u(k), y(k)]^T$ at any time instant k has one bounded power spectral density. After iterating recursively Equation (26) again, the following input-output relation is obtained.

$$Y_t = C\phi^s X_{t-s} + E_0 Z_{[t-s,t)} + W_t \tag{27}$$

where Equation (27) shows the relations among state $x(k)$, control input $u(k)$ and disturbance $w(k)$, and E_0 includes Markov parameters. Y_t is the future output sequence, W_t is the innovation sequence, and X_{t-s} is the unknown initial state sequence. Matrix $Z_{[t-s,t)}$ is constituted by past input-output data. The above variables are given, respectively as follows.

$$\begin{aligned}
&E_0 = \begin{bmatrix} C\phi^s B & C\phi^s & \cdots & C\phi & C \end{bmatrix}; \quad Y_t = \begin{bmatrix} y(t) & y(t+1) & \cdots & y(t+N-1) \end{bmatrix} \\
&W_t = \begin{bmatrix} w(t) & w(t+1) & \cdots & w(t+N-1) \end{bmatrix} \\
&X_{t-s} = \begin{bmatrix} x(t-s) & x(t-s+1) & \cdots & x(t-s+N-1) \end{bmatrix} \\
&Z_{[t-s,t)} = \begin{bmatrix} u(t-s) & u(t-s+1) & \cdots & u(t-s+N-1) \\ y(t-s) & y(t-s+1) & \cdots & y(t-s+N-1) \\ \vdots & \vdots & \vdots & \vdots \\ u(t-1) & u(t) & \cdots & u(t+N-2) \\ y(t-1) & y(t) & \cdots & y(t+N-2) \end{bmatrix}
\end{aligned}$$

In Equation (16), the estimation corresponding to E_0 is given as

$$\hat{E}_0 = Y_t Z^*_{[t-s,t)} \tag{28}$$

where $Z^*_{[t-s,t)}$ is the pseudo inverse matrix of data matrix $Z_{[t-s,t)}$. The uniqueness of pseudo inverse matrix $Z^*_{[t-s,t)}$ can be guaranteed due to the fact that the input-output sequence has one bounded power spectral density.

Set the prediction horizon as f, then the past input-output data matrix is expressed as follows in our control problem.

$$\bar{Z}_{[t-s,t)} = \begin{bmatrix} u(k-s) & y(k-s) & \cdots & u(k-1) & y(k-1) \end{bmatrix}^T \tag{29}$$

Then the future f step output predictor is obtained as

$$\hat{Y}_{[k,k+f)} \simeq \begin{bmatrix} E_0 \\ E_1 \\ \vdots \\ E_{f-1} \end{bmatrix} \bar{Z}_{[k-s,k)} + \begin{bmatrix} C\phi^s x(k-s) \\ C\phi^{s+1} x(k-s) \\ \vdots \\ C\phi^{s+f-1} x(k-s) \end{bmatrix}$$

$$+\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \varphi_1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi_{f-1} & \varphi_{f-2} & \cdots & \varphi_1 & 0 \end{bmatrix} \begin{bmatrix} u(k) \\ y(k) \\ \vdots \\ u(k-f-1) \\ y(k-f-1) \end{bmatrix} \tag{30}$$

where

$$\begin{aligned} \varphi_i &= C\phi_{i-1} \begin{bmatrix} B & 1 \end{bmatrix}, \quad i = 1, 2, \ldots, f-1 \\ E_i &= \begin{bmatrix} 0 & C\phi^{s-1}B & C\phi^{s-1} & \cdots & C\phi^i B & C\phi^i \end{bmatrix} \end{aligned} \tag{31}$$

and $U_{[k,k+f-1)}$ is the future control input, i.e.,

$$\begin{aligned} \hat{Y}_{[k,k+f)} &= \begin{bmatrix} \hat{Y}(k) & \hat{Y}(k+1) & \cdots & \hat{Y}(k+f-1) \end{bmatrix}^T \\ U_{[k,k+f)} &= \begin{bmatrix} \hat{u}(k) & \hat{u}(k+1) & \cdots & \hat{u}(k+f-2) \end{bmatrix}^T \end{aligned} \tag{32}$$

Based on identified Markov parameter $\hat{E}_0$, substitute and formulate above terms to yield

$$\hat{Y}_{[k,k+f)} = \begin{bmatrix} \Gamma_0 \\ \Gamma_1 \\ \vdots \\ \Gamma_{f-1} \end{bmatrix} \bar{Z}_{[k-s,k)} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \Lambda_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \Lambda_{f-1} & \Lambda_{f-2} & \cdots & \Lambda_1 \end{bmatrix} U_{[k,k+f)} \tag{33}$$

where the recursive forms for parameters $\{\Gamma_i, \Lambda_j / i, j = 1, 2, \ldots, f-1\}$ are given as

$$\begin{aligned} &\Gamma_0 = \hat{E}_0; \quad \Gamma_i = \bar{E}_i + \sum_{\tau=0}^{i-1} C\phi^{i-\tau-1}\Gamma_\tau; \quad \Lambda_j = C\phi^{j-1}B + \sum_{\tau=1}^{j-1} C\phi^{j-\tau-1}\Lambda_\tau; \quad \Lambda_1 = CB \\ &\hat{E}_i = \begin{bmatrix} 0 & C\phi^{s-1}B & C\phi^{s-1} & \cdots & C\phi^i B & C\phi^i \end{bmatrix} \end{aligned} \tag{34}$$

where the condition about $\phi = A - C$ being stable, is used in deriving Equation (34), when s is chosen as large enough, then we guarantee that

$$\|\phi^s\| \ll 1$$

From Equation (34), we see that the first row is a zero row in the second matrix, i.e., output predictor $\hat{y}(k)$ is independent of control input, so we delete the first row to yield.

$$\hat{Y}_{[k,k+f)} = \Gamma \bar{Z}_{[k-s,k)} + \Lambda U_{[k,k+f-1)} \tag{35}$$

As the goal of predictive control is to control the system in order to track a desired output reference and reject disturbance during one period of time, assume the expected trajectory is priori known as

$$\Upsilon_{[k+1,k+f)} = \begin{bmatrix} r(k+1) & r(k+2) & \cdots & r(k+f-1) \end{bmatrix}^T \tag{36}$$

One commonly used quadratic objective function is chosen in predictive control field.

$$\begin{aligned} J(U) &= \left(\hat{Y}_{[k,k+f)} - \Upsilon_{[k+1,k+f)}\right)^T \left(\hat{Y}_{[k,k+f)} - \Upsilon_{[k+1,k+f)}\right) + U^T_{[k,k+f-1)} R U_{[k,k+f-1)} \\ &= \left\|\hat{Y} - \Upsilon\right\|_Q^2 + \|U\|_R^2 \end{aligned} \tag{37}$$

where two matrices Q and R are positive definite weight matrices, and the decision variable is $U_{[k,k+f-1)}$, $\|v\|_Q^2 = v^T Q v$ is defined as one L2 norm.

Similarly, output predictor $\hat{y}(t+k/t)$ for a station normal process can also be substituted into Equation (37), and then a same quadratic objective function is derived to solve that quadratic programming problem (37). One ellipsoid optimization algorithm is proposed to solve the future control input $U_{[k,k+f-1)}$.

5. **Ellipsoid Optimization Algorithm.** If no any limitations are imposed on control input $U_{[k,k+f-1)}$, we substitute Equation (36) into (37) and take the partial derivation with respect to U. By setting the partial derivative equal to zero, the optimal control input is obtained. However, there are many limitations on control input or output, for example, input and output constraint.

$$u_{\min} \leq u(k+i) \leq u_{\max}; \quad y_{\min} \leq y(k+i) \leq y_{\max}$$

Rewrite the quadratic programming problem in model predictive control field as follows.

$$\begin{aligned} &\min_U J(U) = \left\| \hat{Y} - \Upsilon \right\|_Q^2 + \|U\|_R^2 \\ &\text{subject to } \Omega = \{U | q_j(U) \leq 0, j = 1, 2, \ldots, M\} \end{aligned} \tag{38}$$

where Ω is a convex compact set with a nonempty interior, and Ω is expressed as some linear matrix inequalities, which are used to replace these constraints on control input and output. Define

$$q(U) = \max_j q_j(U); \quad g(U) = \begin{cases} \nabla q(U) & \text{if } q(U) < 0 \\ \nabla J(U) & \text{if } q(U) \geq 0 \end{cases} \tag{39}$$

Set the initial ellipsoid as that

$$\varepsilon^{(0)} = \left\{ U : \left(U - U^{(0)}\right)^T P_0^{-1} \left(U - U^{(0)}\right) \leq 1 \right\} \supseteq \Omega \tag{40}$$

where $U^{(0)}$ is the center of the initial ellipsoid.

From the convex optimization theory, half space $H^{(0)} = \left\{U : g\left(U^{(0)}\right)\left(U - U^{(0)}\right) \leq 0\right\}$ includes Ω, i.e., $\Omega \subseteq \varepsilon^{(0)} \cap H^{(0)}$. Then one new ellipsoid $\varepsilon^{(1)}$ is constructed to satisfy $\varepsilon^{(1)} \supseteq \varepsilon^{(0)} \cap H^{(0)} \supseteq \Omega$, and its volume is decreased. Through applying this algorithm to continuously performing such iterations, our proposed ellipsoid optimization algorithm is obtained. More specifically, after the ith iterations, we have $U^{(i)}$ and $P_i = P_i^T > 0$, and they satisfy

$$\varepsilon^{(i)} = \left\{ U : \left(U - U^{(i)}\right)^T P_i^{-1} \left(U - U^{(i)}\right) \leq 1 \right\} \supseteq \Omega \tag{41}$$

For convenience, the ellipsoid optimization algorithm is formulated as follows.

Step 1: Given the center of the initial ellipsoid as $U^{(0)}$ and $P_0 = P_0^T > 0$;

Step 2: Compute $g\left(U^{(0)}\right)$, $q_j\left(U^{(0)}\right)$, $q\left(U^{(0)}\right)$;

Step 3: Compute the correction term as

$$\begin{aligned} U^{(i+1)} &= U^{(i)} - \frac{1}{M+1} \frac{P_i g\left(U^{(i)}\right)}{\sqrt{g^T\left(U^{(i)}\right) P_i g\left(U^{(i)}\right)}} \\ P_{i+1} &= \frac{M^2}{M^2+1} \left[P_i - \frac{2}{M+1} \frac{P_i g\left(U^{(i)}\right) g\left(U^{(i)}\right) P_i}{g^T\left(U^{(i)}\right) P_i g\left(U^{(i)}\right)} \right] \end{aligned} \tag{42}$$

where M is the upper bound for power spectral density with respect to the input-output sequence.

Step 4: If $\left\|U^{(i+1)} - U^{(i)}\right\| + \|P_{i+1} - P_i\| \leq \delta$, ($\delta$ is one small value, for example, $\delta = 0.05$), then terminate the algorithm; otherwise, go back to Step 2.

When applying above ellipsoid optimization algorithm to solving the quadratic programming problem (39), then volume for the ith ellipsoid is

$$vol\left(\varepsilon^{(i)}\right) \leq e^{-\frac{i}{2M}} vol\left(\varepsilon^{(0)}\right) \tag{43}$$

Taking limit operation on above Inequality (43) with $i \to \infty$, then it holds that

$$\lim_{i\to\infty} vol\left(\varepsilon^{(i)}\right) \leq \lim_{i\to\infty} e^{-\frac{i}{2M}} vol\left(\varepsilon^{(0)}\right) = 0 \tag{44}$$

Equation (44) shows the volume of the final ellipsoid, obtained by the above ellipsoid optimization algorithm, will tend to zero, and it means the algorithm will converge to one point in Ω with probability 1. This point is our required minimum point or optimal control input, so if $vol(\Omega) > 0$, then there always exists one iterative step I such that

$$e^{-\frac{i}{2M}} vol\left(\varepsilon^{(0)}\right) \leq vol(\Omega), \quad \forall i \geq I \tag{45}$$

Continue to do operation to get

$$\frac{vol(\Omega)}{vol\left(\varepsilon^{(0)}\right)} \geq e^{-\frac{i}{2M}}, \quad \forall i \geq I \tag{46}$$

Taking the logarithm on both sides of the above inequality, it holds that

$$\ln\left[\frac{vol(\Omega)}{vol\left(\varepsilon^{(0)}\right)}\right] \geq -\frac{i}{2M}, \quad \forall i \geq I \tag{47}$$

i.e.,

$$i \geq 2M \ln\left[\frac{vol(\Omega)}{vol\left(\varepsilon^{(0)}\right)}\right], \quad \forall i \geq I \tag{48}$$

The algorithm is terminated, on the basis of the following upper bound on the number of iterative steps.

$$I = 2M \ln\left[\frac{vol(\Omega)}{vol\left(\varepsilon^{(0)}\right)}\right] \tag{49}$$

Combining the direct data driven and model predictive control in Figure 1, the extended structure of our considered system is given in Figure 2, where state estimation provides the output predictor for model predictive control, and ellipsoid optimization algorithm is used to solve the optimization problem for model predictive control. Furthermore, the detailed principle for the ellipsoid optimization algorithm is shown in Figure 3, where only the relation between two continuous $(i+1)$th and ith iteration is illustrated.

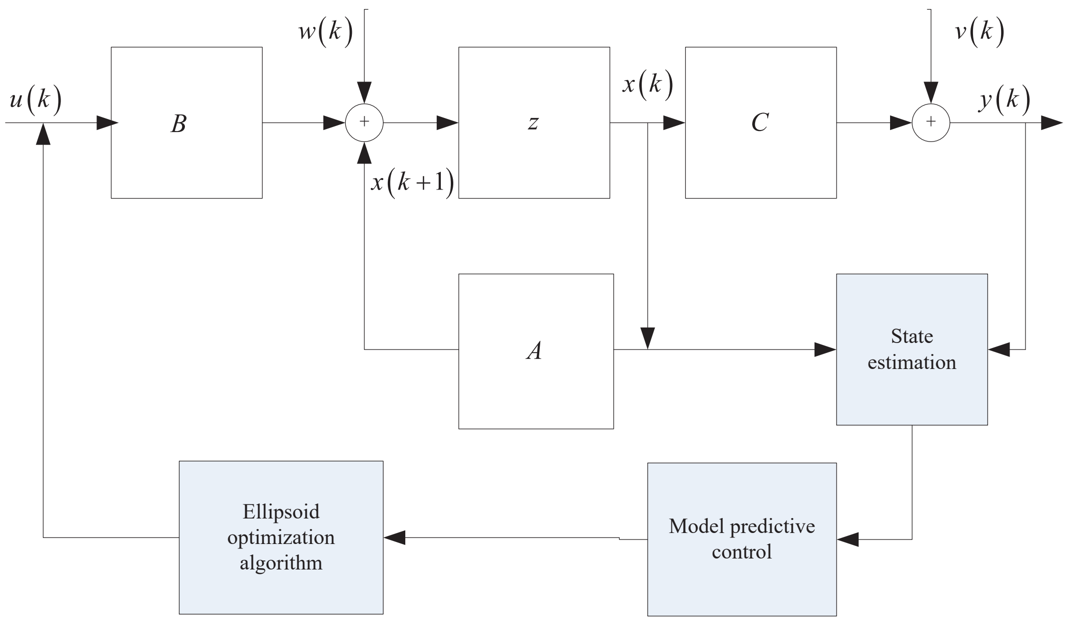


FIGURE 2. The extended structure

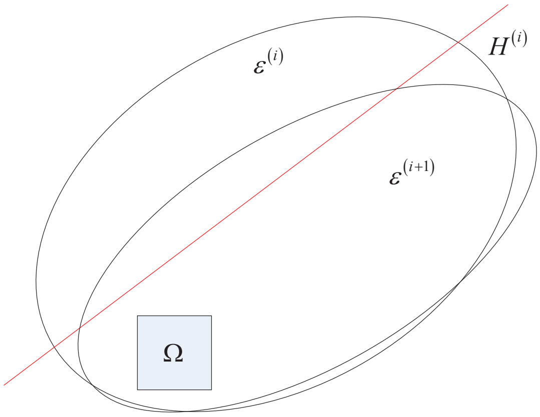


FIGURE 3. The principle for ellipsoid optimization algorithm

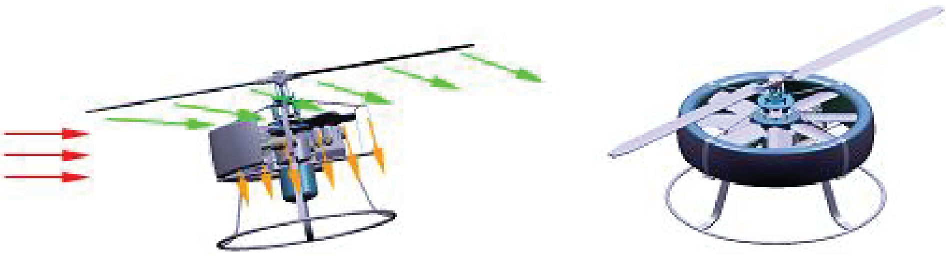


FIGURE 4. The structure of helicopter

6. **Simulation.** The structure of our considered helicopter is seen in Figure 4. Full state variable model of helicopter with rudder loop and cleaning network is chosen as follows.

$$A = \begin{bmatrix} -0.322 & 0.064 & 0.0364 & -0.9917 & 0.0003 & 0.0008 & 0 \\ 0 & 0 & 1 & 0.0037 & 0 & 0 & 0 \\ -30.64 & 0 & -3.678 & 0.0046 & -0.7333 & 0.1315 & 0 \\ 8.5396 & 0 & -0.025 & -0.476 & -0.0319 & -0.062 & 0 \\ 0 & 0 & 0 & 0 & -20.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20.2 & 0 \\ 0 & 0 & 0 & 57.29 & 0 & 0 & 0 \end{bmatrix}; \quad D = 0$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 57.2958 & 0 & 0 & 0 \\ 0 & 0 & 57.2958 & 0 & 0 & 0 & 0 \\ 57.2958 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 57.2958 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 20.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20.2 & 0 \end{bmatrix} \tag{50}$$

The control input is the servo input signal of the rudder and the aileron, and 57.2958 is the conversion value of angle and radian in output matrix C. According to the state matrix of the system, the mode of the open loop Holland roll can be solved as $-0.4425 \pm i3.063$. It can be seen that the damping of the open loop Holland rolling is insufficient, but open loop helix mode with pole -0.01631. Therefore, the output feedback regulator is needed to design the hovering control of the helicopter. The process noise $e(k)$ in original state space system is white noise with zero mean, and its variance matrix is $10^{-4}I_5$. Prediction and control horizon level are $p = L = 100$.

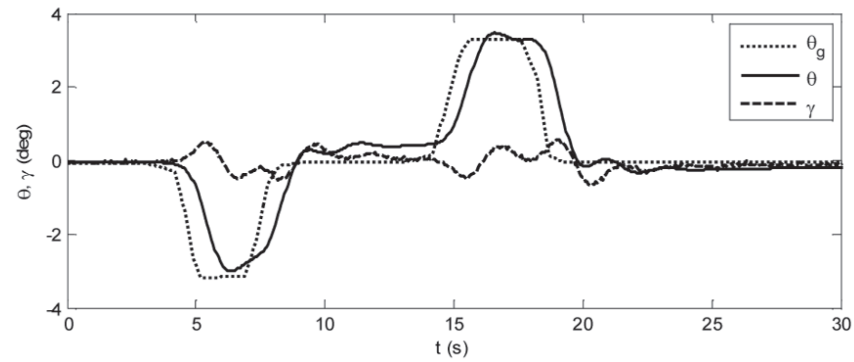
Choose 3000 input-output sample data in designing predictive controller for our considered closed loop experiment, and sample period is 0.2 s, the initial output feedback controller is chosen as

$$u(k) = -\begin{bmatrix} 0 & -0.5 & 0 \\ 0 & -0.1 & -0.1 \\ -0.2 & -0.2 & -0.2 \end{bmatrix} y(k) \tag{51}$$

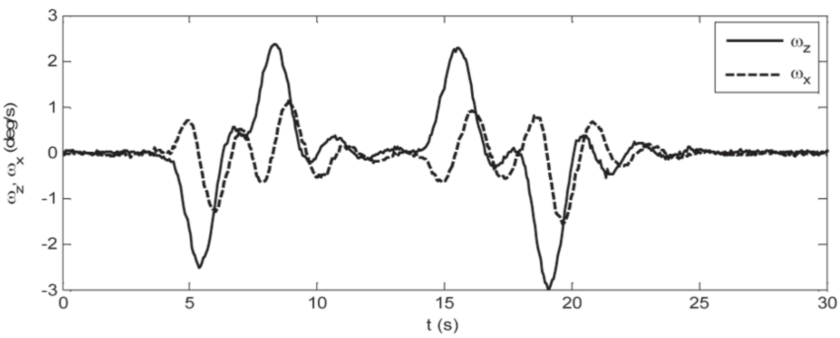
Reference signal Υ is a white noise with zero mean, and its variance is $4 \times 10^{-4} I$. Past and future horizon levels are all $p = 60$, so that $\|\phi^p\|_2 \approx 0$, $Q = R = 0.01I$. Equality and inequality constraints are imposed as

$$\begin{aligned} &-0.56u_1 - 0.44u_2 = 3.31; \quad 0 \leq \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \leq \begin{bmatrix} 1 & 0.2 & 1 \end{bmatrix} \\ &0 \leq \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} \leq \begin{bmatrix} 10 & 60 & 100 & 10 \end{bmatrix} \end{aligned} \tag{52}$$

For the optimization problem (38) of subspace predictive controller, the improved ellipsoid algorithm is used to solve the optimal predictive controller. In order to verify the performance of the closed loop system, the initial state of the system is selected as $x(0) = 10I$. The specific process using a joystick control of helicopter attitude is as follows: first before the helicopter fly rod, bow down to about 3 degrees, putting action for 3 seconds back to zero; loose rod stop 5 seconds, then pull back before the helicopter to fly rod, up to about 3 degrees, rod for 3 seconds after the return to zero. The simulation response curve of the closed loop system under above operation is shown in Figure 5.



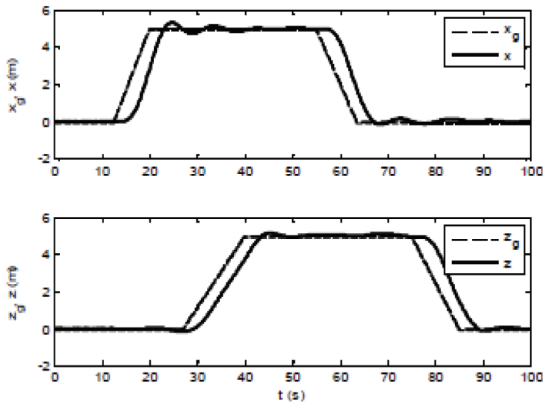
(a) Attitude angle response



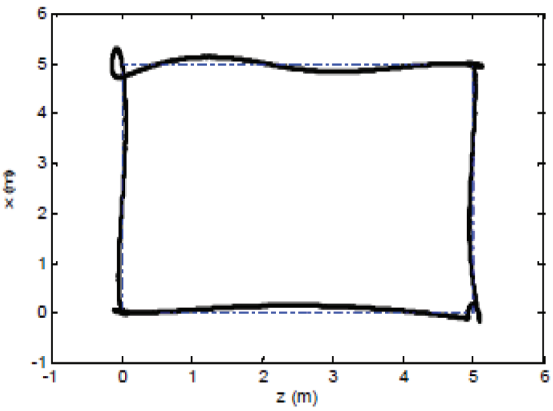
(b) Attitude angular rate response

FIGURE 5. Pitch attitude control response curve

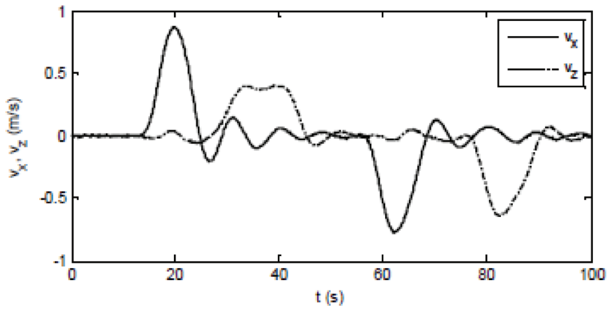
When the simulation is carried out in the hover state, the high altitude off ground coordinates of the given hover point are (−19 m, −83 m). At this time the course of helicopter is 197.8 degrees. When the unmanned helicopter enters the hover state, the instruction disk is used in turn to send the following instruction, move forward 5 meters, move right 5 meters, move 5 meters and left 5 meters, the interval of each instruction is taken to 15 seconds. The displacement of helicopter can be obtained by the output feedback control of the subspace predictive controller to the closed loop system, see Figure 6. From Figure 6, we see that the fixed point hovering closed loop control has good control response characteristics under the action of the subspace predictive controller, and the steady-state positioning error is less than 0.5 meters, which is consistent with the control accuracy requirement.



(a) Response curve under given position



(b) Flight path curve



(c) Flight velocity response curve

FIGURE 6. Displacement curve in hover state

7. **Conclusion.** In this paper, direct data driven and model predictive control are combined to study the system with set membership uncertainty. Firstly, direct data driven is proposed to obtain the state estimation in case of set membership uncertainty, as state estimation is benefit for the output predictor in model predictive control. Secondly, one quadratic programming problem is constructed to satisfy the tracking property for model predictive control through our own simple but tedious calculations. Thirdly, an ellipsoid optimization algorithm is yielded to solve the optimal control input, and the computational complexity of this algorithm is given. Due to the fact that the fault is not considered here, our future work will concern on applying our proposed theories for system with unknown fault.

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