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NOVEL NOOR ITERATIVE METHODS FOR MIXED TYPE ASYMPTOTICALLY NONEXPANSIVE MAPPINGS FROM THE PERSPECTIVE OF CONVEX PROGRAMMING IN HYPERBOLIC SPACES

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Abstract. *This note aims to propose a novel Noor iterative method in view of three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive nonself-mappings and prove some strong convergence theorems of the proposed iteration scheme in the setting of hyperbolic spaces. This strong convergence result is more robust than delta and weak convergence results. An illustrative example from the perspective of convex programming is also provided. The latter found that our proposed iteration performs better when compared with the JE-iteration introduced by Jayashree and Eldred.*
Keywords: Noor iteration, Asymptotically nonexpansive self-mappings, Asymptotically nonexpansive nonself-mappings, Strong convergence, Uniformly convex hyperbolic space

1. **Introduction.** Let $(\mathcal{X}, d)$ be a metric space, and let $\mathcal{K}$ be a nonempty subset of $\mathcal{X}$. We denote the fixed point set of a mapping $\mathcal{T}$ by $F(\mathcal{T}) = \{u \in \mathcal{K} : \mathcal{T}u = u\}$ and

$$d(u, F(\mathcal{T})) = \inf\{d(u, p) : p \in F(\mathcal{T})\}.$$

A self-mapping $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is said to be

(i) nonexpansive if $d(\mathcal{T}u, \mathcal{T}v) \le d(u, v)$ for all $u, v \in \mathcal{K}$,

(ii) asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $d(\mathcal{T}^n u, \mathcal{T}^n v) \le k_n d(u, v)$ for all $u, v \in \mathcal{K}$ and $n \ge 1$,

(iii) uniformly $\mathcal{L}$-Lipschitzian if there exists a constant $\mathcal{L} > 0$ such that

$$d(\mathcal{T}^n u, \mathcal{T}^n v) \le \mathcal{L} d(u, v)$$

for all $u, v \in \mathcal{K}$ and $n \ge 1$.

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From the above definitions, one clearly sees that each nonexpansive mapping is an asymptotically nonexpansive mapping with $k_n = 1, \forall n \geq 1$. Both nonexpansive mappings and asymptotically nonexpansive mappings are Lipschitzian continuous. To be more precise, each nonexpansive mapping is $\mathcal{L}$-Lipschitzian and each asymptotically nonexpansive mapping is uniformly $\mathcal{L}$-Lipschitzian mapping with $\mathcal{L} = \sup_{n\in\mathbb{N}}\{k_n\}$.

Some interesting results concerning fixed-point iteration processes for nonexpansive nonself mappings can be found in [1-4].

In 1972, Goebel and Kirk [5] introduced the class of asymptotically nonexpansive self-mappings. They proved that if $\mathcal{K}$ is nonempty closed convex subset of a real uniformly convex Banach space and $\mathcal{T}$ is an asymptotically nonexpansive self-mapping on $\mathcal{K}$, then $\mathcal{T}$ has a fixed point.

In 1991, Schu [6] introduced the following modified Mann iteration process

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n \mathcal{T}^n u_n, \quad n \geq 1, \tag{1}$$

to approximate fixed points of asymptotically nonexpansive self-mappings in a Hilbert space. Since then, Schu's iteration process (1) has been widely used to approximate fixed points of asymptotically nonexpansive self-mappings in Hilbert spaces or Banach spaces; see, e.g., [7-9] and the references therein.

For asymptotically nonexpansive nonself-mappings Chidume et al. [10] studied the following iterative sequence

$$u_{n+1} = \mathcal{P}\left((1-\alpha_n)u_n + \alpha_n \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}u_n\right) \tag{2}$$

to approximate some fixed point of $\mathcal{T}$. They obtained a convergence theorem under suitable conditions in real uniformly convex Banach spaces. If $\mathcal{T}$ is a self-mapping, then $\mathcal{P}$ becomes the identity mapping. Hence, (2) reduces to (1).

In 2006, Wang [11] considered the following iteration process which is a generalization of (2) (see also [12]),

$$\begin{aligned} v_n &= \mathcal{P}\left((1-\beta_n)u_n + \beta_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n\right), \\ u_{n+1} &= \mathcal{P}\left((1-\alpha_n)u_n + \alpha_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n\right), \ n \geq 1, \end{aligned} \tag{3}$$

where $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{E}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in [0,1). They obtain a strong convergence theorem under weak restrictions imposed on the control parameters.

In 2012, Guo et al. [13] further studied the following iteration scheme

$$\begin{aligned} v_n &= \mathcal{P}\left((1-\beta_n)\mathcal{S}_2^n u_n + \beta_n \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1}u_n\right), \\ u_{n+1} &= \mathcal{P}\left((1-\alpha_n)\mathcal{S}_1^n u_n + \alpha_n \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1}v_n\right), \ n \geq 1, \end{aligned} \tag{4}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{E}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in [0,1). Weak and strong convergence theorems of common fixed points of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{T}_1$ and $\mathcal{T}_2$ were obtained.

Another classical iteration precess was introduced by Noor [14] which is formulated as follows: $u_1 = u \in \mathcal{K}$,

$$\begin{aligned} w_n &= (1-\gamma_n)u_n + \gamma_n \mathcal{S}u_n, \\ v_n &= (1-\beta_n)u_n + \beta_n \mathcal{S}w_n, \\ u_{n+1} &= (1-\alpha_n)u_n + \alpha_n \mathcal{S}v_n, \ n \geq 1, \end{aligned} \tag{5}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$. Such iterative method is called Noor-iteration. By updating the solution and the auxiliary principle, Noor investigated the iterative method (5) for finding the approximations of solutions to variational inclusions (inequalities) in a Hilbert space.

Because of its simplicity, the method (5) has been widely utilized to solve the fixed point problem, and as a result, it has been enhanced by many works, as seen in [15-18].

Elastoviscoplasticity, liquid crystal, and eigenvalue problems were all solved by Glowinski and Le Tallec [19] using a three-step iterative method. They demonstrated that compared to the two-step and one-step iterative techniques, the three-step approximation method performs better.

Haubruge et al. [20] investigated the convergence analysis of the three-step iterative schemes of Glowinski and Le Tallec [19]. They applied these three-step iterations to obtaining new splitting-type algorithms for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions. Under certain conditions, they also demonstrated that three-step iterations result in highly parallelized algorithms. Thus, it is evident that three-step schemes play an important role in solving numerous problems in the pure and applied sciences.

Several fixed point results and iterative algorithms for approximating the fixed points of nonlinear mappings in Hilbert and Banach spaces have been obtained in literature, for example, see [21-35]. Some scholars studied the fixed point theory and cone b-metric space in recent years, see [36]. Besides the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. It is easier working with Banach space due to its convex structures. However, metric space does not naturally enjoy this structure. Therefore, the need to introduce convex structures to it arises. The concept of convex metric space was first introduced by Takahashi [37] who studied the fixed points for nonexpansive mappings in the setting of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced on hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [38-40]). Although the class of hyperbolic spaces defined by Kohlenbach [39] is slightly restrictive compared with the class of hyperbolic spaces introduced in [38], it is, however, more general than the class of hyperbolic spaces introduced in [40]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in [39]. Some other examples of this class of hyperbolic spaces include Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and R-trees (see [38-43]).

Recently, Jayashree and Eldred [44] introduced and studied the following mixed type iteration scheme (JE-iteration for short) in a uniformly convex hyperbolic space and proved some strong convergence theorems for mixed type asymptotically nonexpansive mappings:

$$\begin{aligned} v_n &= \mathcal{P}\left(\mathcal{H}\left(\mathcal{S}_2^n u_n, \mathcal{T}_2(\mathcal{P}\mathcal{T}_2)^{n-1} u_n, \alpha_n\right)\right), \\ u_{n+1} &= \mathcal{P}\left(\mathcal{H}\left(\mathcal{S}_1^n u_n, \mathcal{T}_1(\mathcal{P}\mathcal{T}_1)^{n-1} v_n, \beta_n\right)\right),\ n \geq 1, \end{aligned} \tag{6}$$

where $\mathcal{S}_1, \mathcal{S}_2 : \mathcal{K} \to \mathcal{K}$ are asymptotically nonexpansive self-mappings, $\mathcal{T}_1, \mathcal{T}_2 : \mathcal{K} \to \mathcal{X}$ are asymptotically nonexpansive nonself-mappings and $\{\alpha_n\}$, $\{\beta_n\}$ are two sequences in $[0, 1)$.

Let $\mathcal{K}$ be a nonempty closed convex subset of a real uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three asymptotically nonexpansive nonself-mappings. Thianwan [45] presented the new two-step iteration process for mixed type asypmtotically nonexpansive mappings $\mathcal{S}_1$, $\mathcal{S}_2$

and $\mathcal{T}_1$, $\mathcal{T}_2$ as follows: $u_1 \in \mathcal{K}$,

$$\begin{aligned} v_n &= \mathcal{H}\left(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n u_n, \zeta_n\right), \\ u_{n+1} &= \mathcal{H}\left(\mathcal{S}_1^n v_n, (\mathcal{P}\mathcal{T}_1)^n v_n, \vartheta_n\right), \end{aligned} \tag{7}$$

where $\{\vartheta_n\}$ and $\{\zeta_n\}$ are two sequences in $[0, 1)$. More precisely, he established the strong convergence theorem in a uniformly convex hyperbolic space.

Motivated by these recent works, incorporating the ideas of Noor [14], Jayashree and Eldred [44], and Thianwan [45], for an arbitrary $u_1 \in \mathcal{K}$, we suggest the following novel Noor iterative scheme which is the modification of (5) and (6) for mixed type asypmtotically nonexpansive mappings

$$\begin{aligned} w_n &= \mathcal{H}\left(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n, \alpha_n\right), \\ v_n &= \mathcal{H}\left(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n w_n, \beta_n\right), \\ u_{n+1} &= \mathcal{H}\left(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_3)^n v_n, \gamma_n\right), \end{aligned} \tag{8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1)$.

Note that the novel Noor process (8) and Thianwan process (7) are independently neither reduced to the other.

In this paper, we study the strong convergence of the novel Noor iteration scheme (8) for three asymptotically nonexpansive self-mappings $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and three asymptotically nonexpansive nonself-mappings $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ in the setting of uniformly convex hyperbolic spaces. To support our main results, we perform a numerical example and convergence comparisons of the proposed iteration with the JE-iteration process.

2. **Preliminaries.** A subset $\mathcal{K}$ of $\mathcal{X}$ is said to be retract if there exists a continuous mapping $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ such that $\mathcal{P}u = u$, $\forall u \in \mathcal{K}$. $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ is said to be a retraction if $\mathcal{P}^2 = \mathcal{P}$. If $\mathcal{P}$ is a retraction, then $u = \mathcal{P}u$ for all u in the range of $\mathcal{P}$. We refer to [41, 46, 47] for more details.

For any nonempty subset $\mathcal{K}$ of a real metric space $(\mathcal{X}, d)$, let $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Then, $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is said to be an asymptotically nonexpansive nonself-mapping (see [10]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d\left(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}u, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}v\right) \le k_n d(u, v) \tag{9}$$

for all $u, v \in \mathcal{K}$ and $n \ge 1$. We denote by $(\mathcal{P}\mathcal{T})^0$ the identity map from $\mathcal{K}$ onto itself. We see that if $\mathcal{T}$ is a self-mapping.

In addition, if $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is asymtotically nonexpansive in light of (9) and $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ is a nonexpansive retraction, then $\mathcal{P}\mathcal{T} : \mathcal{K} \to \mathcal{K}$ is asymtotically nonexpansive mapping (see (10)). Indeed, for all $u, v \in \mathcal{K}$ and $n \ge 1$, by (9), it follows that

$$\begin{aligned} d\left((\mathcal{P}\mathcal{T})^n u, (\mathcal{P}\mathcal{T})^n v\right) &= d\left(\mathcal{P}\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}u, \mathcal{P}\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}v\right) \\ &\le d\left(\mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}u, \mathcal{T}(\mathcal{P}\mathcal{T})^{n-1}v\right) \\ &\le k_n d(u, v). \end{aligned}$$

Therefore, we now introduce the following definition.

Definition 2.1. *For any nonempty subset $\mathcal{K}$ of a real metric space $(\mathcal{X}, d)$, let $\mathcal{P} : \mathcal{X} \to \mathcal{K}$ be a nonexpansive retraction of $\mathcal{X}$ onto $\mathcal{K}$. Then, $\mathcal{T} : \mathcal{K} \to \mathcal{X}$ is said to be an asymptotically nonexpansive nonself-mapping if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$d\left((\mathcal{P}\mathcal{T})^n u, (\mathcal{P}\mathcal{T})^n v\right) \le k_n d(u, v), \tag{10}$$

for all $u, v \in \mathcal{K}$ and $n \ge 1$.

Throughout this paper, we work in the setting of hyperbolic spaces introduced by Kohlenbach [39]. Recall that a hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is a metric space $(\mathcal{X}, d)$ together with a mapping $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0, 1] \to \mathcal{X}$ satisfying

($\mathcal{H}1$): $d(w, \mathcal{H}(u, v, \beta)) \leq (1 - \beta)\, d(w, u) + \beta d(w, v)$,

($\mathcal{H}2$): $d(\mathcal{H}(u, v, \beta)), \mathcal{H}(u, v, \gamma) = |\beta - \gamma|\, d(u, v)$,

($\mathcal{H}3$): $\mathcal{H}(u, v, \beta) = \mathcal{H}(v, u, (1 - \beta))$,

($\mathcal{H}4$): $d(\mathcal{H}(u, w, \beta), \mathcal{H}(v, s, \beta)) \leq (1 - \beta)\, d(u, v) + \beta d(w, s)$

for all $u, v, s, w \in \mathcal{X}$ and $\beta, \gamma \in [0, 1]$.

A subset $\mathcal{K}$ of a hyperbolic space $\mathcal{X}$ is convex if $\mathcal{H}(u, v, \beta) \in \mathcal{K}$ for all $u, v \in \mathcal{K}$ and $\beta \in [0, 1]$.

Recall that a hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ is said to be

(i) strictly convex [37] if for any $u, v \in \mathcal{X}$ and $\beta \in [0, 1]$, there exists a unique element $z \in \mathcal{X}$ such that $d(z, u) = \beta d(u, v)$ and $d(z, v) = (1 - \beta)d(u, v)$;

(ii) uniformly convex [48] if for all $u, v, w \in \mathcal{X}$, $r > 0$ and $\epsilon \in (0, 2]$, there exists $\delta \in (0, 1]$ such that $d\left(\mathcal{H}\left(u, v, \frac{1}{2}\right), u\right) \leq (1 - \delta)r$ whenever $d(u, w) \leq r$, $d(v, w) \leq r$ and $d(u, v) \geq \epsilon r$.

Recall that a mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such $\delta = \eta(r, \epsilon)$ for given $r > 0$ and $\epsilon \in (0, 2]$ is called modulus of uniform convexity. We call η-monotone if it decreases with r (for a fixed ϵ). A uniformly convex hyperbolic space is strictly convex (see [49]).

To show our main convergence theorems, we shall need the following useful lemmas.

Lemma 2.1. [50] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of non-negative real numbers such that $a_{n+1} \leq (1 + b_n)a_n + c_n$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.2. [51] *Let u_n and v_n be two sequences of a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ such that, for $\mathcal{R} \in [0, \infty)$, $\lim_{n\to\infty} \sup d(u_n, a) \leq \mathcal{R}$, $\lim_{n\to\infty} \sup d(v_n, a) \leq \mathcal{R}$ and $\lim_{n\to\infty} d(\mathcal{H}(u_n, v_n, \alpha_n)) = \mathcal{R}$ where $\alpha_n \in [a, b]$ with $0 < a < b < 1$, then we have, $\lim_{n\to\infty} d(u_n, v_n) = 0$.*

3. **Main Results.** In this section, we consider a uniformly convex hyperbolic space $(\mathcal{X}, d, \mathcal{H})$ and prove a strong convergence theorem for $\mathcal{X}$, using the iterative scheme given in (8).

Lemma 3.1. *Let $(\mathcal{X}, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings with $\left\{k_n^{(1)}\right\}, \left\{k_n^{(2)}\right\}, \left\{k_n^{(3)}\right\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three asymptotically nonexpansive nonself-mappings with $\left\{l_n^{(1)}\right\}, \left\{l_n^{(2)}\right\}, \left\{l_n^{(3)}\right\} \subset [1, \infty)$ such that, $\sum_{n=1}^{\infty} \left(k_n^{(i)} - 1\right) < \infty$ and $\sum_{n=1}^{\infty} \left(l_n^{(i)} - 1\right) < \infty$ for $i = 1, 2, 3$, respectively and $\Omega = F(\mathcal{S}_1) \bigcap F(\mathcal{S}_2) \bigcap F(\mathcal{S}_3) \bigcap F(\mathcal{T}_1) \bigcap F(\mathcal{T}_2) \bigcap F(\mathcal{T}_3) \neq \emptyset$. Let $\{u_n\}$ be a sequence defined by (8) where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1)$. Then $\lim_{n\to\infty} d(u_n, q)$ exists for any $q \in \Omega$.*

Proof: Using (8) and setting $h_n = \max\left\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\right\}$, we have

$$\begin{aligned} d(w_n, q) &= d(\mathcal{H}(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n, \alpha_n), q) \\ &\leq (1 - \alpha_n)d(\mathcal{S}_1^n u_n, q) + \alpha_n d((\mathcal{P}\mathcal{T}_1)^n u_n, q) \\ &\leq (1 - \alpha_n)h_n d(u_n, q) + \alpha_n h_n d(u_n, q) \end{aligned}$$

$$= h_n d(u_n, q) \tag{11}$$

and

$$\begin{aligned} d(v_n, q) &= d(\mathcal{H}(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n w_n, \beta_n), q) \\ &= d(\mathcal{H}(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n (\mathcal{H}(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n, \alpha_n)), \beta_n), q) \\ &\leq (1-\beta_n) d(\mathcal{S}_2^n u_n, q) + \beta_n d((\mathcal{P}\mathcal{T}_1)^n (\mathcal{H}(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n, \alpha_n)), q) \\ &\leq (1-\beta_n) h_n^2 d(u_n, q) + \beta_n h_n^2 d(u_n, q) \\ &= h_n^2 d(u_n, q). \end{aligned} \tag{12}$$

Also,

$$\begin{aligned} d(u_{n+1}, q) &= d(\mathcal{H}(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_3)^n v_n, \gamma_n), q) \\ &= d(\mathcal{H}(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_3)^n (\mathcal{H}(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n (\mathcal{H}(\mathcal{S}_1^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n, \alpha_n)), \beta_n)), \gamma_n), q) \\ &\leq (1-\gamma_n) d(\mathcal{S}_3^n u_n, q) \\ &\quad + \gamma_n d((\mathcal{P}\mathcal{T}_3)^n (\mathcal{H}(\mathcal{S}_2^n u_n, (\mathcal{P}\mathcal{T}_2)^n (\mathcal{H}(\mathcal{S}_1^n u_n (\mathcal{P}\mathcal{T}_1)^n u_n, \alpha_n)), \beta_n)), q) \\ &\leq (1-\gamma_n) h_n^3 d(u_n, q) + \gamma_n h_n^3 d(u_n, q) \\ &= \left(1 + \left(h_n^3 - 1\right)\right) d(u_n, q). \end{aligned} \tag{13}$$

By the hypothesis, $\sum_{n=1}^{\infty} \left(k_n^{(i)} - 1\right) < \infty$ and $\sum_{n=1}^{\infty} \left(l_n^{(i)} - 1\right) < \infty$ for $i = 1, 2, 3$. Therefore, $\sum_{n=1}^{\infty} (h_n^3 - 1) < \infty$ for $i = 1, 2, 3$. Using Lemma 2.1, $\lim_{n\to\infty} d(u_n, q)$ exists.

Lemma 3.2. *Let $(\mathcal{X}, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive self-mappings with $\left\{k_n^{(1)}\right\}, \left\{k_n^{(2)}\right\}, \left\{k_n^{(3)}\right\} \subset [1, \infty)$ and $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ be three asymptotically nonexpansive nonself-mappings with $\left\{l_n^{(1)}\right\}, \left\{l_n^{(2)}\right\}, \left\{l_n^{(3)}\right\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} \left(k_n^{(i)} - 1\right) < \infty$ and $\sum_{n=1}^{\infty} \left(l_n^{(i)} - 1\right) < \infty$ for $i = 1, 2, 3$, respectively and $\Omega = F(\mathcal{S}_1) \bigcap F(\mathcal{S}_2) \bigcap F(\mathcal{S}_3) \bigcap F(\mathcal{T}_1) \bigcap F(\mathcal{T}_2) \bigcap F(\mathcal{T}_3) \neq \emptyset$. Let $\{u_n\}$ be the sequence defined by (8) and the following conditions hold:*

(i) $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0, 1)$,
(ii) $d(u, \mathcal{T}_i v) \leq d(\mathcal{S}_i u, \mathcal{T}_i v)$ for all $u, v \in \mathcal{K}$ and $i = 1, 2, 3$.

Then $\lim_{n\to\infty} d(u_n, \mathcal{S}_i u_n) = \lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_i) u_n) = 0$ for $i = 1, 2, 3$.

Proof: For any given $q \in \Omega$, $\lim_{n\to\infty} d(u_n, q)$ exists, by Lemma 3.1. Taking $h_n = \max\left\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\right\}$. Suppose that $\lim_{n\to\infty}(u_n, q) = c$. By (13) and $\sum_{n=1}^{\infty}(h_n^3 - 1) < \infty$, we have

$$\lim_{n\to\infty} d((\mathcal{H}(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_3)^n v_n, \gamma_n)), q) = c \tag{14}$$

and

$$\limsup_{n\to\infty} d(\mathcal{S}_3^n u_n, q) \leq \limsup_{n\to\infty} h_n d(u_n, q) = c. \tag{15}$$

Taking lim sup on both sides of (12) we obtain

$$\limsup_{n\to\infty} d(v_n, q) \leq c,$$

and so we have

$$\lim_{n\to\infty} d((\mathcal{P}\mathcal{T}_3)^n v_n, q) \leq \limsup_{n\to\infty} d(v_n, q) \leq c. \tag{16}$$

Using (14), (15) and (16), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_3)^n v_n) = 0. \tag{17}$$

By the condition (ii), we have

$$d(u_n,(\mathcal{P}\mathcal{T}_3)^n v_n) \le d(\mathcal{S}_3^n u_n,(\mathcal{P}\mathcal{T}_3)^n v_n). \tag{18}$$

It follows from (17) and (18) that

$$\lim_{n\to\infty} d(u_n,(\mathcal{P}\mathcal{T}_3)^n v_n) = 0. \tag{19}$$

In additon,

$$\begin{aligned} d(u_n,q) &\le d(u_n,(\mathcal{P}\mathcal{T}_3)^n v_n) + d((\mathcal{P}\mathcal{T}_3)^n v_n,q) \\ &\le d(u_n,(\mathcal{P}\mathcal{T}_3)^n v_n) + h_n d(v_n,q). \end{aligned} \tag{20}$$

In Inequality (20), taking infimum on both sides and applying (19), we obtain

$$\lim_{n\to\infty}\inf d(v_n,q) \ge c,$$

and since $\lim_{n\to\infty}\sup d(v_n,q) \le c$, $\lim_{n\to\infty} d(v_n,q) = c$. Using the arguments in (12) and by $\sum_{n=1}^{\infty}\left(h_n^{(2)}-1\right) < \infty$, we have

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_2^n u_n,(\mathcal{P}\mathcal{T}_2)^n w_n,\beta_n),q) = c. \tag{21}$$

In additon,

$$\lim_{n\to\infty}\sup d(\mathcal{S}_2^n u_n,q) \le \lim_{n\to\infty}\sup h_n d(u_n,q) = c. \tag{22}$$

Taking lim sup on both sides of (11), we have

$$\lim_{n\to\infty}\sup d(w_n,q) \le c. \tag{23}$$

Using (23), we have

$$\lim_{n\to\infty}\sup d((\mathcal{P}\mathcal{T}_2)^n w_n,q) \le \lim_{n\to\infty}\sup h_n d(w_n,q) \le c. \tag{24}$$

Applying by Lemma 2.2, using (21), (22) and (24), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n u_n,(\mathcal{P}\mathcal{T}_2)^n w_n) = 0. \tag{25}$$

From condition (ii), we get

$$d(u_n,(\mathcal{P}\mathcal{T}_2)^n w_n) \le d(\mathcal{S}_2^n u_n,(\mathcal{P}\mathcal{T}_2)^n w_n). \tag{26}$$

It follows from (25) and (26) that

$$\lim_{n\to\infty} d(u_n,(\mathcal{P}\mathcal{T}_2)^n w_n) = 0. \tag{27}$$

In addition,

$$\begin{aligned} d(u_n,q) &\le d(u_n,(\mathcal{P}\mathcal{T}_2)^n w_n) + d((\mathcal{P}\mathcal{T}_2)^n w_n,q) \\ &\le d(u_n,(\mathcal{P}\mathcal{T}_2)^n w_n) + h_n d(w_n,q). \end{aligned} \tag{28}$$

In Inequality (28), taking infimum on both sides and applying (27), we obtain

$$\lim_{n\to\infty}\inf d(w_n,q) \ge c,$$

and since $\lim_{n\to\infty}\sup d(w_n,q) \le c$, $\lim_{n\to\infty} d(w_n,q) = c$. Using the arguments in (11) and by $\sum_{n=1}^{\infty}(h_n-1) < \infty$ we have

$$\lim_{n\to\infty} d(\mathcal{H}(\mathcal{S}_1^n u_n,(\mathcal{P}\mathcal{T}_1)^n u_n,\alpha_n),q) = c. \tag{29}$$

In addition,

$$\lim_{n\to\infty}\sup d(\mathcal{S}_1^n u_n,q) \le \lim_{n\to\infty}\sup h_n d(u_n,q) = c \tag{30}$$

and

$$\lim_{n\to\infty}\sup d((\mathcal{PT}_1)^n u_n, q) \leq \lim_{n\to\infty}\sup h_n d(u_n, q) = c. \tag{31}$$

Applying by Lemma 2.2, using (29), (30) and (31), again we have

$$\lim_{n\to\infty} d(\mathcal{S}_1^n u_n, (\mathcal{PT}_1)^n u_n) = 0. \tag{32}$$

From condition (ii), we get

$$d(u_n, (\mathcal{PT}_1)^n u_n) \leq d(\mathcal{S}_1^n u_n, (\mathcal{PT}_1)^n u_n). \tag{33}$$

It follows from (32) and (33) that

$$\lim_{n\to\infty} d(u_n, (\mathcal{PT}_1)^n u_n) = 0. \tag{34}$$

Using (8), we have

$$\begin{aligned} d(w_n, \mathcal{S}_1^n u_n) &\leq (1-\alpha_n) d(\mathcal{S}_1^n u_n, \mathcal{S}_1^n u_n) + \alpha_n d(\mathcal{S}_1^n u_n, (\mathcal{PT}_1)^n u_n) \\ &= \alpha_n d(\mathcal{S}_1^n u_n, (\mathcal{PT}_1)^n u_n). \end{aligned}$$

It follows from (32) that

$$\lim_{n\to\infty} d(w_n, \mathcal{S}_1^n u_n) = 0. \tag{35}$$

Since

$$d(w_n, u_n) \leq d(w_n, \mathcal{S}_1^n u_n) + d(\mathcal{S}_1^n u_n, (\mathcal{PT}_1)^n u_n) + d((\mathcal{PT}_1)^n u_n, u_n),$$

it follows from (32), (34) and (35) that

$$\lim_{n\to\infty} d(w_n, u_n) = 0. \tag{36}$$

In addition,

$$d(u_n, \mathcal{S}_1^n u_n) \leq d(u_n, w_n) + d(w_n, \mathcal{S}_1^n u_n).$$

Following from (35) and (36), we have

$$\lim_{n\to\infty} d(u_n, \mathcal{S}_1^n u_n) = 0. \tag{37}$$

From (8), we have

$$d(v_n, \mathcal{S}_2^n u_n) \leq \beta_n d(\mathcal{S}_2^n u_n, (\mathcal{PT}_2)^n w_n). \tag{38}$$

Following from (25) and (38), we have

$$\lim_{n\to\infty} d(v_n, \mathcal{S}_2^n u_n) = 0. \tag{39}$$

Furthermore,

$$d(v_n, u_n) \leq d(v_n, \mathcal{S}_2^n u_n) + d(\mathcal{S}_2^n u_n, (\mathcal{PT}_2)^n w_n) + d((\mathcal{PT}_2)^n w_n, u_n),$$

by using (25), (27) and (39), we have

$$\lim_{n\to\infty} d(v_n, u_n) = 0. \tag{40}$$

Since

$$d(u_n, \mathcal{S}_2^n u_n) \leq d(u_n, v_n) + d(v_n, \mathcal{S}_2^n u_n),$$

using (39) and (40), we have

$$\lim_{n\to\infty} d(u_n, \mathcal{S}_2^n u_n) = 0. \tag{41}$$

Since

$$\begin{aligned} d(u_{n+1}, \mathcal{S}_3^n u_n) &= d((\mathcal{H}(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n v_n, \gamma_n)), \mathcal{S}_3^n u_n) \\ &\leq (1-\gamma_n) d(\mathcal{S}_3^n u_n, \mathcal{S}_3^n u_n) + \gamma_n d((\mathcal{PT}_3)^n v_n, \mathcal{S}_3^n u_n) \\ &= \gamma_n d((\mathcal{PT}_3)^n v_n, \mathcal{S}_3^n u_n), \end{aligned}$$

using (17), we have

$$\lim_{n\to\infty} d(u_{n+1}, \mathcal{S}_3^n u_n) = 0. \tag{42}$$

In addition,

$$\begin{aligned} d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n u_n) &\le d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n v_n) + d((\mathcal{PT}_3)^n v_n, (\mathcal{PT}_3)^n u_n) \\ &\le d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n v_n) + h_n d(v_n, u_n). \end{aligned}$$

It follows from (17) and (40) that

$$\lim_{n\to\infty} d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n u_n) = 0. \tag{43}$$

By condition (ii), we know that

$$d(u_n, (\mathcal{PT}_3)^n u_n) \le d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n u_n).$$

Using (43), we have

$$\lim_{n\to\infty} d(u_n, (\mathcal{PT}_3)^n u_n) = 0. \tag{44}$$

In addition,

$$\begin{aligned} d(\mathcal{S}_2^n u_n, (\mathcal{PT}_2)^n u_n) &\le d(\mathcal{S}_2^n u_n, (\mathcal{PT}_2)^n w_n) + d((\mathcal{PT}_2)^n w_n, (\mathcal{PT}_2)^n u_n) \\ &\le d(\mathcal{S}_2^n u_n, (\mathcal{PT}_2)^n w_n) + h_n d(w_n, u_n). \end{aligned}$$

Using (25) and (36), we have

$$\lim_{n\to\infty} d(\mathcal{S}_2^n u_n, (\mathcal{PT}_2)^n u_n) = 0. \tag{45}$$

Again by condition (ii), using (45), we also have

$$\begin{aligned} d(u_n, (\mathcal{PT}_2)^n u_n) &\le d(\mathcal{S}_2^n u_n, (\mathcal{PT}_2)^n u_n) \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{46}$$

Using (36), (42) and (43), we have

$$\begin{aligned} d(u_{n+1}, (\mathcal{PT}_3)^n w_n) &\le d(u_{n+1}, \mathcal{S}_3^n u_n) + d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n u_n) + d((\mathcal{PT}_3)^n u_n, (\mathcal{PT}_3)^n w_n) \\ &\le d(u_{n+1}, \mathcal{S}_3^n u_n) + d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n u_n) + h_n d(u_n, w_n) \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{47}$$

Since

$$d(\mathcal{S}_3^n u_n, u_n) \le d(\mathcal{S}_3^n u_n, (\mathcal{PT}_3)^n u_n) + d(u_n, (\mathcal{PT}_3)^n u_n),$$

using (43) and (44), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n u_n, u_n) = 0. \tag{48}$$

Since

$$d(\mathcal{S}_3^n u_n, (\mathcal{PT}_2)^n u_n) \le d(\mathcal{S}_3^n u_n, u_n) + d(u_n, (\mathcal{PT}_2)^n u_n),$$

it follows from (46) and (48) that

$$\lim_{n\to\infty} d(\mathcal{S}_3^n u_n, (\mathcal{PT}_2)^n u_n) = 0. \tag{49}$$

In addition,

$$\begin{aligned} d(u_{n+1}, (\mathcal{PT}_2)^n w_n) &\le d(u_{n+1}, \mathcal{S}_3^n u_n) + d(\mathcal{S}_3^n u_n, (\mathcal{PT}_2)^n u_n) + d((\mathcal{PT}_2)^n u_n, (\mathcal{PT}_2)^n w_n) \\ &\le d(u_{n+1}, \mathcal{S}_3^n u_n) + d(\mathcal{S}_3^n u_n, (\mathcal{PT}_2)^n u_n) + h_n d(u_n, w_n). \end{aligned}$$

Using (36), (42) and (49), we have

$$\lim_{n\to\infty} d(u_{n+1}, (\mathcal{PT}_2)^n w_n) = 0. \tag{50}$$

Since

$$d(\mathcal{S}_3^n u_n, (\mathcal{PT}_1)^n u_n) \le d(\mathcal{S}_3^n u_n, u_n) + d(u_n, (\mathcal{PT}_1)^n u_n),$$

using (34) and (48), we have

$$\lim_{n\to\infty} d(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n) = 0. \tag{51}$$

Moreover, we have

$$\begin{aligned} d(u_{n+1}, (\mathcal{P}\mathcal{T}_1)^n w_n) &\leq d(u_{n+1}, \mathcal{S}_3^n u_n) + d(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n) + d((\mathcal{P}\mathcal{T}_1)^n u_n, (\mathcal{P}\mathcal{T}_1)^n w_n) \\ &\leq d(u_{n+1}, \mathcal{S}_3^n u_n) + d(\mathcal{S}_3^n u_n, (\mathcal{P}\mathcal{T}_1)^n u_n) + h_n d(u_n, w_n). \end{aligned}$$

It follows from (36), (42) and (51) that

$$\lim_{n\to\infty} d(u_{n+1}, (\mathcal{P}\mathcal{T}_1)^n w_n) = 0. \tag{52}$$

Again, since $(\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-1} w_{n-1}, u_n \in \mathcal{K}$ for $i = 1, 2, 3$, and $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ are three asymptotically nonexpansive nonselt-mappings, we have

$$\begin{aligned} d((\mathcal{P}\mathcal{T}_i)^n w_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n) &= d((\mathcal{P}\mathcal{T}_i)(\mathcal{P}\mathcal{T}_i)^{n-1} w_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n) \\ &\leq \max\left\{ l_1^{(1)}, l_1^{(2)}, l_1^{(3)} \right\} d((\mathcal{P}\mathcal{T}_i)^{n-1} w_{n-1}, u_n). \end{aligned} \tag{53}$$

For $i = 1, 2, 3$, using (47), (50), (52) and (53), we have

$$\lim_{n\to\infty} d((\mathcal{P}\mathcal{T}_i)^n w_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n) = 0. \tag{54}$$

Since

$$d(u_{n+1}, w_n) \leq d(u_{n+1}, (\mathcal{P}\mathcal{T}_2)^{n-1} w_n) + d((\mathcal{P}\mathcal{T}_2)^{n-1} w_n, u_n) + d(u_n, w_n),$$

from (27), (36) and (50), we have

$$\lim_{n\to\infty} d(u_{n+1}, w_n) = 0. \tag{55}$$

In addition, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(u_n, (\mathcal{P}\mathcal{T}_i) u_n) &\leq d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + d((\mathcal{P}\mathcal{T}_i)^n u_n, (\mathcal{P}\mathcal{T}_i)^n w_{n-1}) \\ &\quad + d((\mathcal{P}\mathcal{T}_i)^n w_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n) \\ &\leq d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + \max\left\{ \sup_{n\geq 1} l_1^{(1)}, \sup_{n\geq 1} l_2^{(2)}, \sup_{n\geq 1} l_3^{(3)} \right\} d(w_{n-1}, u_n) \\ &\quad + d((\mathcal{P}\mathcal{T}_i)^n w_{n-1}, (\mathcal{P}\mathcal{T}_i) u_n). \end{aligned}$$

Thus, it follows from (34), (44), (46), (54) and (55), we have

$$\lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_1) u_n) = \lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_2) u_n) = \lim_{n\to\infty} d(u_n, (\mathcal{P}\mathcal{T}_3) u_n) = 0.$$

The first part of the theorem is hence proved. We prove the next part of the theorem, i.e.,

$$\lim_{n\to\infty} d(u_n, \mathcal{S}_1 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{S}_2 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{S}_3 u_n) = 0.$$

In fact, for $i = 1, 2, 3$, we have

$$\begin{aligned} d(u_n, \mathcal{S}_i u_n) &\leq d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + d((\mathcal{P}\mathcal{T}_i)^n u_n, \mathcal{S}_i u_n) \\ &\leq d(u_n, (\mathcal{P}\mathcal{T}_i)^n u_n) + d((\mathcal{P}\mathcal{T}_i)^n u_n, \mathcal{S}_i^n u_n). \end{aligned}$$

Thus, it follows from (32), (34), (43), (44), (45) and (46) that

$$\lim_{n\to\infty} d(u_n, \mathcal{S}_1 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{S}_2 u_n) = \lim_{n\to\infty} d(u_n, \mathcal{S}_3 u_n) = 0.$$

The proof is completed.

Next, we can prove a strong convergence theorem.

Theorem 3.1. *Let $\mathcal{K}$, $\mathcal{X}$, $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ satisfy the hypotheses of Lemma 3.2. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[\varepsilon, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$ and $\mathcal{S}_i$, $\mathcal{T}_i$ for all $i = 1, 2, 3$ satisfy the condition (ii) in Lemma 3.2. If there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0,\infty)$ such that*

$$f(d(u,\Omega)) \le d(u,\mathcal{S}_1u)+d(u,\mathcal{S}_2u)+d(u,\mathcal{S}_3u)+d(u,(\mathcal{P}\mathcal{T}_1)u)+d(u,(\mathcal{P}\mathcal{T}_2)u)+d(u,(\mathcal{P}\mathcal{T}_3)u)$$

for all $u \in \mathcal{K}$, where $d(u,\Omega) = \inf\{d(u,q) : q \in \Omega\}$. Then the sequence $\{u_n\}$ defined by algorithm (8) *converges strongly to a common fixed point of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.*

Proof: From Lemma 3.2, we have $\lim_{n\to\infty} d(u_n,\mathcal{S}_iu_n) = 0 = \lim_{n\to\infty} d(u_n,\mathcal{T}_iu_n)$ for $i = 1, 2, 3$. It follows from the hypothesis that

$$\begin{aligned}\lim_{n\to\infty} f(d(u_n,\Omega)) &\le \lim_{n\to\infty} (d(u_n,\mathcal{S}_1u_n) + d(u_n,\mathcal{S}_2u_n) + d(u_n,\mathcal{S}_3u_n) \\ &\quad + d(u_n,(\mathcal{P}\mathcal{T}_1)u_n) + d(u_n,(\mathcal{P}\mathcal{T}_2)u_n) + d(u_n,(\mathcal{P}\mathcal{T}_3)u_n)) \\ &= 0.\end{aligned}$$

Thus, $\lim_{n\to\infty} f(d(u_n,\Omega)) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$, by Lemma 3.1, we obtain that $\lim_{n\to\infty} d(u_n,\Omega)$ exists. This implies that $\lim_{n\to\infty} d(u_n,\Omega) = 0$. Next, we show that $\{u_n\}$ is a Cauchy sequence in $\mathcal{K}$. Using (13), we have

$$d(u_{n+1},q) \le \left(1 + \left(h_n^3 - 1\right)\right) d(u_n,q)$$

for each $n \ge 1$, where $h_n = \max\left\{k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, l_n^{(1)}, l_n^{(2)}, l_n^{(3)}\right\}$ and $q \in \Omega$. For any m, n, $m > n \ge 1$, we have

$$\begin{aligned}d(u_m,q) &\le \left(1 + \left(h_{m-1}^3 - 1\right)\right) d(u_{m-1},q) \\ &\le e^{h_{m-1}^3-1} d(u_{m-1},q) \\ &\le e^{h_{m-1}^3-1} e^{h_{m-2}^3-1} d(u_{m-2},q) \\ &\;\;\vdots \\ &\le e^{\sum_{i=n}^{m-1}\left(h_i^3-1\right)} d(u_n,q) \\ &\le M d(u_n,q),\end{aligned}$$

where $M = e^{\sum_{i=1}^{\infty}\left(h_i^3-1\right)}$. So, for any $q \in \Omega$, we have

$$d(u_n,u_m) \le d(u_n,q) + d(u_m,q) \le (1+M)d(u_n,q).$$

Taking the infimum over all $q \in \Omega$, we have

$$d(u_n,u_m) \le (1+M)d(u_n,\Omega).$$

Thus, it follows from $\lim_{n\to\infty} d(u_n,\Omega) = 0$ that $\{u_n\}$ is a Cauchy sequence. Since $\mathcal{K}$ is a closed subset in a complete hyperbolic space $\mathcal{X}$, the sequence $\{u_n\}$ converges strongly to some $q^* \in \mathcal{K}$. It is easy to prove that $F(\mathcal{S}_1)$, $F(\mathcal{S}_2)$, $F(\mathcal{S}_3)$, $F(\mathcal{T}_1)$, $F(\mathcal{T}_2)$ and $F(\mathcal{T}_3)$ are all closed, that is, Ω is closed subset of $\mathcal{K}$. Since $\lim_{n\to\infty} d(u_n,\Omega) = 0$ gives that $d(q^*,\Omega) = 0$, we have $q^* \in \Omega$. The proof is completed.

If $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ are self-mappings, then $\mathcal{P}$ becomes the identity mapping. By using the same ideas and techniques as in Lemma 3.1, Lemma 3.2 and Theorem 3.1, we can obtain a strong convergence theorem for asymptotically nonexpansive mappings in a uniformly convex hyperbolic space. Therefore, we can state the following result without proofs.

Theorem 3.2. *Let $(\mathcal{X}, d, \mathcal{H})$ be a uniformly convex hyperbolic space and $\mathcal{K}$ a nonempty closed convex subset of $\mathcal{X}$. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3 : \mathcal{K} \to \mathcal{K}$ be three asymptotically nonexpansive* *self-mappings with* $\left\{k_n^{(1)}\right\}, \left\{k_n^{(2)}\right\}, \left\{k_n^{(3)}\right\} \subset [1,\infty)$ *and* $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 : \mathcal{K} \to \mathcal{X}$ *be three asymptotically nonexpansive nonself-mappings with* $\left\{l_n^{(1)}\right\}, \left\{l_n^{(2)}\right\}, \left\{l_n^{(3)}\right\} \subset [1,\infty)$ *such that* $\sum_{n=1}^{\infty}\left(k_n^{(i)}-1\right) < \infty$ *and* $\sum_{n=1}^{\infty}\left(l_n^{(i)}-1\right) < \infty$ *for* $i = 1,2,3,$ *respectively and* $\Omega \neq \emptyset$. *Assume* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are real sequences in* $[\varepsilon, 1-\varepsilon]$ *for some* $\varepsilon \in (0,1)$ *and* $\mathcal{S}_i$, $\mathcal{T}_i$ *for all* $i = 1,2,3$ *satisfy the condition (ii) in Lemma 3.2. If there is a nondecreasing function* $f : [0,\infty) \to [0,\infty)$ *with* $f(0) = 0$ *and* $f(r) > 0$ *for all* $r \in (0,\infty)$ *such that*

$$f(d(u,\Omega)) \leq d(u,\mathcal{S}_1u) + d(u,\mathcal{S}_2u) + d(u,\mathcal{S}_3u) + d(u,\mathcal{T}_1u) + d(u,\mathcal{T}_2u) + d(u,\mathcal{T}_3u)$$

for all $u \in \mathcal{K}$, *where* $d(u,\Omega) = \inf\{d(u,q) : q \in \Omega\}$. *Then the sequence* $\{u_n\}$ *defined by*

$$\begin{aligned} w_n &= \mathcal{H}(\mathcal{S}_3^n u_n, \mathcal{T}_3^n u_n, \alpha_n),\\ v_n &= \mathcal{H}(\mathcal{S}_2^n u_n, \mathcal{T}_2^n w_n, \beta_n),\\ u_{n+1} &= \mathcal{H}(\mathcal{S}_1^n u_n, \mathcal{T}_1^n v_n, \gamma_n) \end{aligned}$$

converges strongly to a common fixed point of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{T}_1$, $\mathcal{T}_2$ *and* $\mathcal{T}_3$.

4. **Numerical Example.** Now, we present a numerical example to illustrate the convergence and efficiency of the proposed algorithms.

Example 4.1. [8] *Let* $\mathcal{X}$ *be a real line with metric* $d(u,v) = |u-v|$ *and* $\mathcal{K} = [-1,1]$. *Define* $\mathcal{H} : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ *by* $\mathcal{H}(u,v,\alpha) := \alpha u + (1-\alpha)v$ *for all* $u, v \in \mathcal{X}$ *and* $\alpha \in [0,1]$. *Then* $(\mathcal{X}, d, \mathcal{H})$ *is complete uniformly hyperbolic space with a monotone modulus of uniform convexity and* $\mathcal{K}$ *is a nonempty closed convex subset of* $\mathcal{X}$. *Define two mappings* $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ *by*

$$\mathcal{T}u = \begin{cases} -2\sin\dfrac{u}{2}, & \text{if } u \in [0,1],\\ 2\sin\dfrac{u}{2}, & \text{if } u \in [-1,0) \end{cases}$$

and

$$\mathcal{S}u = \begin{cases} u, & \text{if } u \in [0,1],\\ -u, & \text{if } u \in [-1,0). \end{cases}$$

Clearly, $F(\mathcal{T}) = \{0\}$ *and* $F(\mathcal{S}) = \{u \in \mathcal{K}; 0 \leq u \leq 1\}$. *Now, we show that* $\mathcal{T}$ *is nonexpansive. In fact, if* $u, v \in [0,1]$ *or* $u, v \in [-1,0)$, *then*

$$d(\mathcal{T}u,\mathcal{T}v) = |\mathcal{T}u - \mathcal{T}v| = 2\left|\sin\frac{u}{2} - \sin\frac{v}{2}\right| \leq |u-v| = d(u,v).$$

If $u \in [0,1]$ *and* $v \in [-1,0)$ *or* $u \in [-1,0)$ *and* $v \in [0,1]$, *then* $d(\mathcal{T}u,\mathcal{T}v) = |\mathcal{T}u - \mathcal{T}v|$ $= 2\left|\sin\frac{u}{2} + \sin\frac{v}{2}\right| = 4\left|\sin\frac{u+v}{4}\cos\frac{u-v}{4}\right| \leq |u+v| \leq |u-v| = d(u,v).$

That is, $\mathcal{T}$ *is nonexpansive. It follows that* $\mathcal{T}$ *is an asymptotically nonexpansive mapping with* $k_n = 1$ *for each* $n \geq 1$. *Similarly, we can show that* $\mathcal{S}$ *is an asymptotically nonexpansive mapping with* $l_n = 1$ *for each* $n \geq 1$. *Next, to show that* $\mathcal{S}$ *and* $\mathcal{T}$ *satisfy the condition (ii) in Lemma 3.2, we have to consider the following cases:*

Case 1. Let $u, v \in [0,1]$. *It follows that*

$$d(u,\mathcal{T}v) = |u - \mathcal{T}v| = \left|u + 2\sin\frac{v}{2}\right| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u,\mathcal{T}v).$$

Case 2. Let $u, v \in [-1,0)$. *It follows that*

$$d(u,\mathcal{T}v) = |u - \mathcal{T}v| = \left|u - 2\sin\frac{v}{2}\right| \leq \left|-u - 2\sin\frac{v}{2}\right| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u,\mathcal{T}v).$$

Case 3. Let $u \in [-1, 0)$ and $v \in [0, 1]$. It follows that

$$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = \left|u + 2\sin\frac{v}{2}\right| \leq \left|-u + 2\sin\frac{v}{2}\right| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$$

Case 4. Let $u \in [0, 1]$ and $y \in [-1, 0]$. It follows that

$$d(u, \mathcal{T}v) = |u - \mathcal{T}v| = \left|u - 2\sin\frac{v}{2}\right| = |\mathcal{S}u - \mathcal{T}v| = d(\mathcal{S}u, \mathcal{T}v).$$

Hence, the condition (ii) in Lemma 3.2 is satisfied. In addition, let $\alpha_n = \frac{n}{2n+1}$, $\beta_n = \frac{n}{3n+1}$ and $\gamma_n = \frac{n}{4n+1}$, $\forall n \geq 1$. Consequently, the conditions of Lemma 3.2 are fulfilled. Thus, the convergence of the sequence $\{u_n\}$ generated by (8) to a point $0 \in F(\mathcal{T}) \cap F(\mathcal{S})$ can be received.

We choose $u_1 = 1$ and run our process within 100 iterations. All codes were written in Matlab 2022a. We obtain the iteration steps and its amplification factor of the proposed algorithms as shown in Table 1. For convenience, we call the iteration (8) the proposed iteration process.

TABLE 1. Numerical experiment of the proposed method for Example 4.1.

The proposed iteration process		
Iteration number (n)	$\lvert u_n\rvert$	$\frac{\lvert u_{n+1}\rvert}{\lvert u_n\rvert}$
1	1.0000e+00	1.8283e-01
2	1.8283e-01	1.1064e-01
3	2.0229e-02	7.6918e-02
4	1.5559e-03	5.8824e-02
5	9.1526e-05	4.7619e-02
⋮	⋮	⋮
10	2.6686e-15	2.4390e-02
⋮	⋮	⋮
20	1.7026e-33	1.2346e-02
⋮	⋮	⋮
40	2.0079e-75	6.2112e-03
⋮	⋮	⋮
60	1.0911e-121	4.1494e-03
⋮	⋮	⋮
80	8.0992e-171	3.1153e-03
⋮	⋮	⋮
100	4.2888e-222	2.4938e-03

Table 1 shows that the proposed method converges to zero. It can be concluded that the proposed method is linearly convergent and its amplification factor less than 0.003.

Let $\mathcal{C}$ be a nonempty closed convex subset of a Banach space $\mathcal{X}$ and $\mathcal{T} : \mathcal{C} \to \mathcal{C}$ be a mapping. Rhoades [52] gave the idea of how to compare the rate of convergence between two iterative methods as follows.

Let $\{u_n\}$ and $\{v_n\}$ be two sequences which converge to a fixed point q of T. Then $\{u_n\}$ is said to converge faster than $\{v_n\}$ if

$$\|u_n - q\| \leq \|v_n - q\|$$

for all $n \geq 1$.

We choose $u_0 = 1$ and the stop criterion is defined by $||u_n - 0|| < 10^{-15}$. Figure 1 shows the behavior of the comparative method of the JE-iteration (6) and the proposed method (8) in converging to the solutions of the numerical experiment. Table 2 also shows the numerical experiment for supporting our main results and comparing rate of convergence of the proposed method with JE-iteration. We found that the proposed method still converged faster than JE-iteration since the ratio of $|u_n - 0|$ and $|v_n - 0|$ in each iteration step is always less than one (see Rhoades [52]).

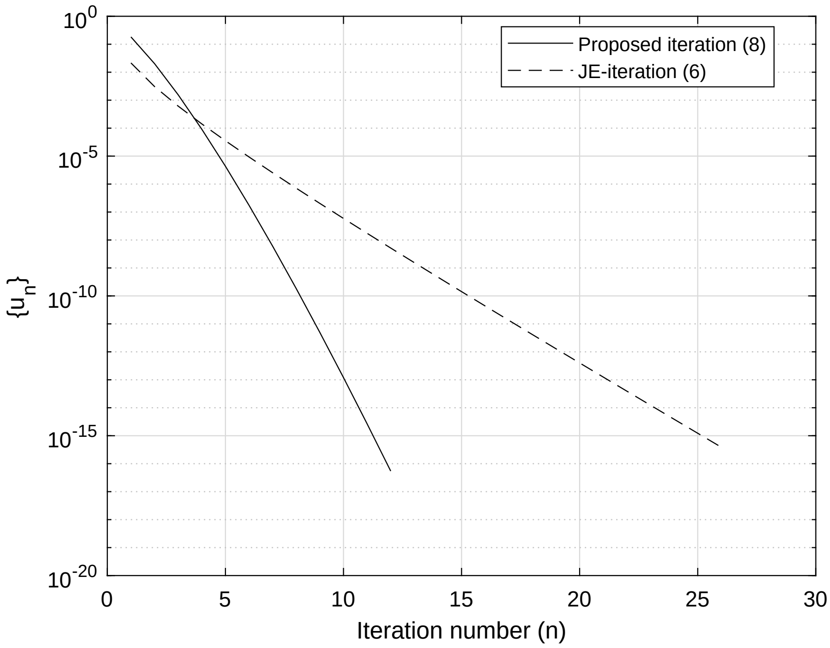

FIGURE 1. The value of $\{u_n\}$ generated by JE-iteration (6) and the proposed iteration (8)

TABLE 2. Comparative sequences generated by JE-iteration (6) and the proposed iteration (8) for numerical experiment of Example 4.1

	Comparative sequences		Rate of convergence between
	Proposed iteration (8)	JE-iteration (6)	two generated sequences
n	u_n	v_n	$\frac{\|u_n - 0\|}{\|v_n - 0\|}$
1	1.8283e-01	2.1468e-02	8.5162e+00
2	2.0229e-02	3.0670e-03	6.5955e+00
3	1.5559e-03	6.1341e-04	2.5366e+00
4	9.1526e-05	1.4156e-04	6.4657e-01
5	4.3584e-06	3.5389e-05	1.2316e-01
⋮	⋮	⋮	⋮
10	1.2008e-13	5.8990e-08	2.0357e-06
⋮	⋮	⋮	⋮
20	1.4472e-31	4.0071e-13	3.6117e-19
⋮	⋮	⋮	⋮
30	1.1468e-51	3.9664e-18	2.8913e-34

5. **Conclusions.** Authors constructed a novel Noor iterative method to approximate a common fixed point for three asymptotically nonexpansive self-mappings and three asymptotically nonexpansive nonself-mappings in the setting of uniformly convex hyperbolic spaces. The novel Noor iteration process (8) is a translation of the Noor-iteration scheme from Banach spaces to hyperbolic spaces. An illustrative example from the perspective of convex programming is also provided as Example 4.1. Our proposed iteration (8) has a better performance when compared with JE-iteration (6). The authors proved a strong convergence result which is stronger than that of delta and weak convergence results. Finally, we propose open questions for the possibility of extending the results of this work to more general classes of asymptotically nonexpansive mappings. Another, it will be interesting to find a similar iterative approximation scheme for a common fixed point of the finite family of nonself and asymptotically nonexpansive mappings in a uniformly convex hyperbolic space.

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