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MODELLING FOR MECHANICAL ELEMENTS OF T-SHAPED CROSS-SECTIONAL BEAMS WITH STRAIGHT HAUNCHES: PART I

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Abstract. *This paper shows a model for T-shaped cross-sectional beams with straight haunches under uniformly distributed load that considers bending and shear deformations to find the fixed-end moments, carry-over and stiffness factors, which is the main contribution of this investigation. The methodology is developed by the conjugate beam method to find the rotations in the supports and by the superposition method this type of problems is solved. The traditional model takes account of only the bending deformations, and other authors consider bending and shear deformations, but the proportions shown in the tables are restricted. A numerical example has been developed to observe the application of the proposed model, and a comparison between the proposed approaches that considers bending and shear deformations against the traditional model that takes account of only the bending deformations presented in the tables and graphics. A significant advantage of the model proposed in this paper on any other document is that the fixed-end moments, carry-over, and stiffness factors can be obtained for any T-shaped beams with straight haunches.*

Keywords: T-shaped beams, Fixed-end moments factors, Carry-over factors, Stiffness factors, Uniformly distributed load, Bending and shear deformations

1. **Introduction.** The beams of reinforced concrete with haunches in its ends are differed from prismatic beams because the beam height has a gradual variation in all or in part of its length. The main application is in buildings of medium height, and highway and railway bridges.

The main problems in the analysis of structures with moment of inertia variable along of its length are to obtain the fixed-end moments, carry-over and stiffness factors.

In the last century, Guldan published several aids for the design of non-prismatic beams [1], and the tables presented by the PCA (Portland Cement Association) that presents the stiffness factors and the fixed-end moments for the nonprismatic beams [2]. In these tables the bending deformations are considered only and the shear deformations are neglected, and also the length-height relationship of the beam is not considered in definition of the various stiffness factors, simplifications that can lead to significant errors in the determination of the stiffness factors.

There are important contributions in the non-prismatic beams analysis that are based on the theory of Euler-Bernoulli beams [3-6].

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The new design aids are found in Appendix B of the book of "Analysis of Structures with Matrix Methods" that replace the old PCA tables, and these tables provide the fixed-end moments, carry-over, and stiffness factors for beams of sections in shape of "I" and "T" [7]. However, these are restricted to some relations, and the height of the haunches is the same in the two ends.

The topic of non-prismatic beams in the last 15 years has aroused great interest among researchers, and the main contributions are the following. Yuksel investigated the modeling, analysis and behavior of the non-prismatic beams under temperature changes [8]. Yuksel presented the behavior of the NBSPH (Non-prismatic Beams with Symmetrical Parabolic Haunches) that has a constant haunch length relationship of 0.5 by means of FEA (Finite Element Analyses) due to vertical loads to obtain the stiffness coefficients and the carry-over factors [9]. Other researchers have presented mathematical models for beams of variable rectangular section of linear or parabolic and symmetric shape, and under uniformly distributed load and/or concentrated load, but shear deformations are not considered [10-13]. Luévanos Rojas presented a model for the I-shaped beams under a uniformly distributed load with straight haunches considering the bending and shear deformations to obtain the fixed-end moments, carry-over and stiffness factors, and the methodology is presented by the method of consistent deformation and the rotations are obtained by the conjugate beam method [14]. Several researchers have developed mathematical models for beams of rectangular cross section with straight and parabolic haunches under uniformly distributed load and concentrated load, these models take account of the bending and shear deformations to obtain the fixed-end moments, carry-over and stiffness factors, this methodology is presented by the method of consistent deformation and the rotations are obtained by the conjugate beam method [15-18]. Balduzzi et al. analyzed the simple compatibility, equilibrium, and constitutive equations for a non-prismatic planar beam [19]. Luévanos Soto and Luévanos Rojas proposed a model for the I-shaped beams under a concentrated load localized anywhere of the beam with straight haunches considering the bending and shear deformations to obtain the fixed-end moments factors, and the methodology is presented using the equations of compatibility and equilibrium, and the rotations are obtained by the principle of virtual work [20]. Balduzzi et al. presented the combination of the effective beam theory with a simple and accurate numerical technique opening the door to the prediction of the structural behavior of planar beams characterized by a continuous variation of the cross-section geometry that in general deeply influences the stress distribution and, therefore, leads to non-trivial constitutive relations [21]. Balduzzi et al. investigated a Timoshenko-like model for planar multilayer (i.e., non-homogeneous) non-prismatic beams, and the main peculiarity of multilayer non-prismatic beams is a non-trivial stress distribution within the cross-section [22].

Also, several design aids have been published for beams subject to uniformly distributed loading or a concentrated load of rectangular cross section with straight haunches [23,24] and with parabolic haunches [25,26] to obtain the fixed-end moments, carry-over and stiffness factors.

The bibliographic review shows that the closest studies for the T-shaped beams with straight haunches are the tables provided the fixed-end moments, carry-over and stiffness factors for beams of sections in shape of "I" and "T" [7]. These tables for the T-shaped beams consider the flange width "b_f" is equal to the web thickness "b_w" more 16 times flange thickness "t_f", the relationship between the beam length and the effective depth is equal to 10, the relationship between the flange thickness "t_f" and the effective depth is equal to 1/4, the relationship between the web thickness "b_w" and the effective depth is equal to 3/4, and the height of the straight haunches is the same in the two ends. Therefore, there are no papers on the subject with the level of current knowledge that provide a way to obtain the fixed-end moments, carry-over and stiffness factors for T-shaped beams with straight haunches for the general case.

This paper shows a mathematical model for T-shaped cross-sectional beams with straight haunches under uniformly distributed load that includes the shear and bending deformations to find the fixed-end moments, carry-over and stiffness factors. The conjugate beam method is used to obtain the rotations in the supports and by the superposition method this type of problem is solved. The traditional model only considers the bending deformations, and other authors consider bending and shear deformations for proportions restricted that are shown in tables. A numerical example has been developed to observe the application of the proposed model, and a comparison between the proposed approaches that considers bending and shear deformations against the traditional model that considers bending deformations only are presented in the tables and graphics to observe differences.

The remainder of the paper is organized as follows. Section 2 shows the formulation of the mathematical model. Section 2.1 shows the formulation of the equations for fixed-end moments factors. Section 2.2 presents the formulation of the equations for carry-over factors and stiffness factors. Section 3 describes the application of the proposed model. Section 4 is dedicated to the results through the comparison of the two models, the proposed model (PM), and the traditional model (TM). Section 5 presents the conclusions.

2. **Formulation of the Mathematical Model.** Figure 1 presents in detail the asymmetric T-shaped beam under uniformly distributed load where the flange width "b_f", flange thickness "t_f", web thickness "b_w" are constant, GC is center of gravity of the cross section, L_1 and L_2 are the lengths of the haunches, and the web height "d_x" that varies with the linear shape at the ends, and the beam in the center has a constant section.

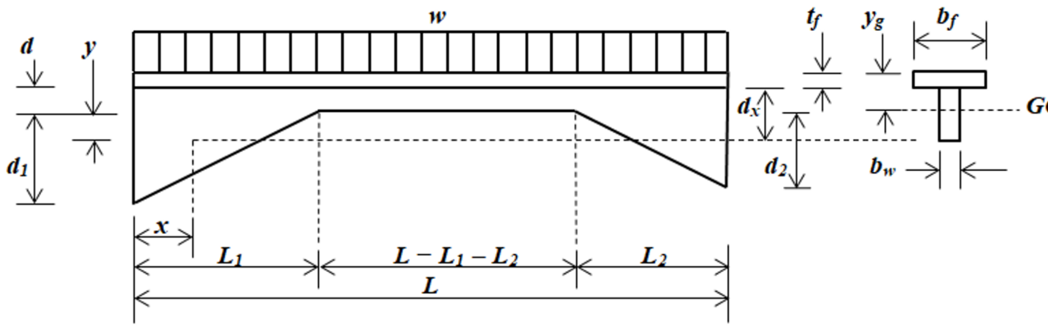

FIGURE 1. T-shaped beam with straight haunches under uniformly distributed load

The geometrical properties of T-shaped cross-sectional beams in general are

$$y_g = \frac{b_f t_f^2 + b_w d_x (d_x + 2t_f)}{2(b_f t_f + b_w d_x)} \tag{1}$$

$$I_x = \frac{b_f t_f^3 + 3b_f t_f (2y_g - t_f)^2 + b_w d_x^3 + 3b_w d_x (d_x + 2t_f - 2y_g)^2}{12} \tag{2}$$

$$A_{sx} = b_w (d_x + t_f) \tag{3}$$

where y_g is the center of gravity of the cross section, I_x is the moment of inertia at a distance "x" in function of "d_x", and A_{sx} is the shear area at a distance "x" in function of "d_x".

The geometrical properties for the first cross section of the beam are

$$d_{x1} = \frac{dL_1 + d_1(L_1 - x)}{L_1} \tag{4}$$

$$I_{x1} = \frac{b_f t_f^3 L_1^3 + b_w[dL_1 - d_1(x - L_1)]^3}{12L_1^3} + \frac{b_w b_f t_f[dL_1 - d_1(x - L_1)][t_f L_1 + dL_1 - d_1(x - L_1)]^2}{4L_1^2\{b_f t_f L_1 + b_w[dL_1 - d_1(x - L_1)]\}} \tag{5}$$

$$A_{sx1} = \frac{b_w[t_f L_1 + dL_1 - d_1(x - L_1)]}{L_1} \tag{6}$$

The geometrical properties for the second cross section of the beam are

$$d_{x2} = d \tag{7}$$

$$I_{x2} = \frac{(b_f - b_w)\left(b_f t_f^4 - b_w d^4\right) + b_f b_w (d + t_f)^4}{12(b_f t_f + b_w d)} \tag{8}$$

$$A_{sx2} = b_w(d + t_f) \tag{9}$$

The geometrical properties for the third cross section of the beam are

$$d_{x3} = \frac{dL_3 + d_2(x - L + L_2)}{L_1} \tag{10}$$

$$I_{x3} = \frac{b_f t_f^3 L_2^3 + b_w[dL_2 - d_1(L - L_2 - x)]^3}{12L_1^3} + \frac{b_w b_f t_f[dL_2 - d_1(L - L_2 - x)][t_f L_2 + dL_2 - d_1(L - L_2 - x)]^2}{4L_2^2\{b_f t_f L_2 + b_w[dL_2 - d_1(L - L_2 - x)]\}} \tag{11}$$

$$A_{sx3} = \frac{b_w[t_f L_1 + dL_1 - d_1(L - L_2 - x)]}{L_1} \tag{12}$$

2.1. **Formulation for fixed-end moments.** Figure 2(a) shows a beam with fixed ends under a uniformly distributed load. The fixed-end moments are obtained by the principle of superposition (sum of the individual rotations in each support "A" and "B"). The individual rotations are "β_{A1}" and "β_{B1}" due to the load applied with the simply supported beam, "β_{A2}" and "β_{B2}" due to a moment "M_{AB}" applied in the support "A", and "β_{A3}" and "β_{B3}" due to a moment "M_{BA}" applied in support "B". The moments are positive in counterclockwise direction, and negative in clockwise direction. Figure 2(b) presents the same simply supported beam in its two supports under the same load applied to obtaining the rotations "β_{A1}" and "β_{B1}". Figure 2(c) shows the moment "M_{AB}" applied in the support "A" that generates the rotations "β_{A2}" and "β_{B2}". Figure 2(d) presents the moment "M_{BA}" applied in the support "B" that generates the rotations "β_{A3}" and "β_{B3}" [27-32].

Now, by the principle of superposition and using the beam compatibility equations for the rotations in each support obtain the following equations:

$$\beta_{A1} - \beta_{A2} + \beta_{A3} = 0 \tag{13}$$

$$\beta_{B1} - \beta_{B2} + \beta_{B3} = 0 \tag{14}$$

Figure 2 is analyzed to obtain the rotations (β_{A1} and β_{B1}, β_{A2} and β_{B2}, β_{A3} and β_{B3}) and the flexural and shear deformations are taken into account due to the loads applied on beam.

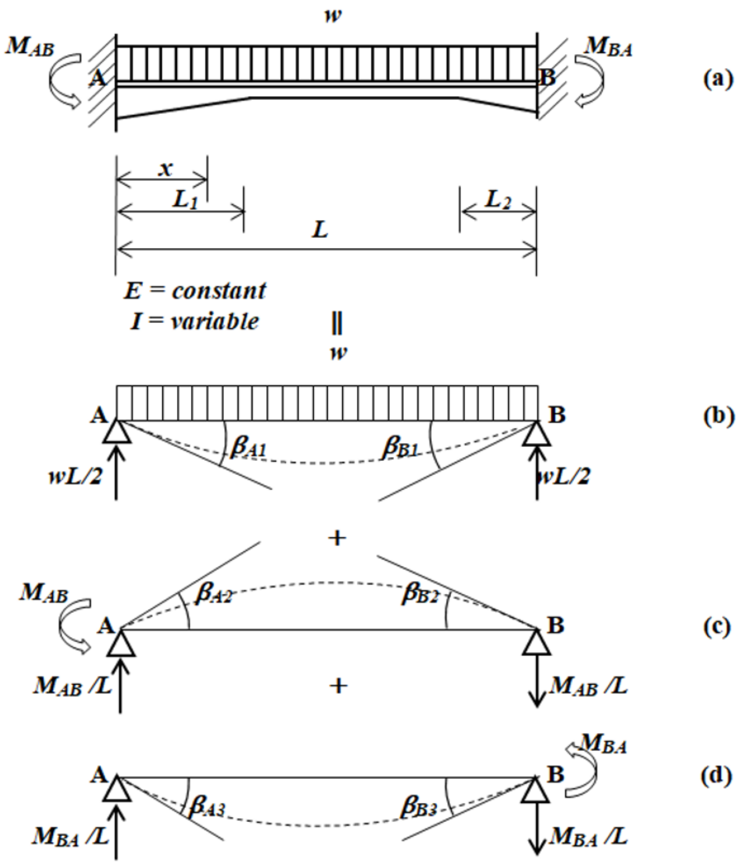

FIGURE 2. T-shaped beam with fixed supports

The rotations in non-prismatic members (β_{Ai} and β_{Bi}) for general case are obtained in the following way [7]:

$$\beta_{Bi} = \frac{1}{L}\int_0^L \frac{V_x}{GA_{sx}}dx + \frac{1}{L}\int_0^L \frac{M_x x}{EI_x}dx \tag{15}$$

$$\beta_{Ai} = \int_0^L \frac{M_x}{EI_x}dx - \beta_{Bi} \tag{16}$$

where $i = 1$ (Due to the uniformly distributed load), 2 (Due to the moment "M_{AB}" applied in the support "A"), 3 (Due to the moment "M_{BA}" applied in the support "B"), V_x is the shear force at a distance x, G is the shear modulus, M_x is the moment at a distance x and E is the modulus of elasticity. $G = E/[2(1+\nu)]$, and ν is Poisson's ratio.

The shear forces and moments located in anywhere on the beam according to Figure 2(b) are

$$V_x = \frac{w(L-2x)}{2} \tag{17}$$

$$M_x = \frac{w(L-x)x}{2} \tag{18}$$

Substituting Equations (17) and (18) into Equations (15) and (16), the rotations "β_{A1}" and "β_{B1}" are obtained:

$$\beta_{A1} = \frac{w}{EL}\left[\int_0^{L_1} \frac{(L-x)^2 x}{2I_{x1}}dx + \int_{L_1}^{L-L_2} \frac{(L-x)^2 x}{2I_{x2}}dx + \int_{L-L_2}^{L} \frac{(L-x)^2 x}{2I_{x3}}dx\right]$$

$$-\frac{w}{GL}\left[\int_0^{L_1}\frac{(L-2x)}{2A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{(L-2x)}{2A_{sx2}}dx+\int_{L-L_2}^{L}\frac{(L-2x)}{2A_{sx3}}dx\right] \quad (19)$$

$$\beta_{B1}=\frac{w}{EL}\left[\int_0^{L_1}\frac{(L-x)x^2}{2I_{x1}}dx+\int_{L_1}^{L-L_2}\frac{(L-x)x^2}{2I_{x2}}dx+\int_{L-L_2}^{L}\frac{(L-x)x^2}{2I_{x3}}dx\right]$$
$$+\frac{w}{GL}\left[\int_0^{L_1}\frac{(L-2x)}{2A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{(L-2x)}{2A_{sx2}}dx+\int_{L-L_2}^{L}\frac{(L-2x)}{2A_{sx3}}dx\right] \quad (20)$$

The shear forces and moments located in anywhere on the beam according to Figure 2(c) are

$$V_x=\frac{M_{AB}}{L} \quad (21)$$

$$M_x=-\frac{M_{AB}(L-x)}{L} \quad (22)$$

Substituting Equations (21) and (22) into Equations (15) and (16), the rotations "β_{A2}" and "β_{B2}" are obtained:

$$\beta_{A2}=\frac{M_{AB}}{EL^2}\left[\int_0^{L_1}\frac{(L-x)^2}{I_{x1}}dx+\int_{L_1}^{L-L_2}\frac{(L-x)^2}{I_{x2}}dx+\int_{L-L_2}^{L}\frac{(L-x)^2}{I_{x3}}dx\right]$$
$$+\frac{M_{AB}}{GL^2}\left[\int_0^{L_1}\frac{1}{A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{1}{A_{sx2}}dx+\int_{L-L_2}^{L}\frac{1}{A_{sx3}}dx\right] \quad (23)$$

$$\beta_{B2}=\frac{M_{AB}}{EL^2}\left[\int_0^{L_1}\frac{(L-x)x}{I_{x1}}dx+\int_{L_1}^{L-L_2}\frac{(L-x)x}{I_{x2}}dx+\int_{L-L_2}^{L}\frac{(L-x)x}{I_{x3}}dx\right]$$
$$-\frac{M_{AB}}{GL^2}\left[\int_0^{L_1}\frac{1}{A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{1}{A_{sx2}}dx+\int_{L-L_2}^{L}\frac{1}{A_{sx3}}dx\right] \quad (24)$$

The shear forces and moments located in anywhere on the beam according to Figure 2(d) are

$$V_x=\frac{M_{BA}}{L} \quad (25)$$

$$M_x=\frac{M_{BA}x}{L} \quad (26)$$

Substituting Equations (25) and (26) into Equations (15) and (16), the rotations "β_{A3}" and "β_{B3}" are obtained:

$$\beta_{A3}=\frac{M_{BA}}{EL^2}\left[\int_0^{L_1}\frac{(L-x)x}{I_{x1}}dx+\int_{L_1}^{L-L_2}\frac{(L-x)x}{I_{x2}}dx+\int_{L-L_2}^{L}\frac{(L-x)x}{I_{x3}}dx\right]$$
$$-\frac{M_{BA}}{GL^2}\left[\int_0^{L_1}\frac{1}{A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{1}{A_{sx2}}dx+\int_{L-L_2}^{L}\frac{1}{A_{sx3}}dx\right] \quad (27)$$

$$\beta_{B3}=\frac{M_{BA}}{EL^2}\left[\int_0^{L_1}\frac{x^2}{I_{x1}}dx+\int_{L_1}^{L-L_2}\frac{x^2}{I_{x2}}dx+\int_{L-L_2}^{L}\frac{x^2}{I_{x3}}dx\right]$$
$$+\frac{M_{BA}}{GL^2}\left[\int_0^{L_1}\frac{1}{A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{1}{A_{sx2}}dx+\int_{L-L_2}^{L}\frac{1}{A_{sx3}}dx\right] \quad (28)$$

Now, Equations (19), (23) and (27) that correspond to support "A" are substituted into Equation (13), and Equations (20), (24) and (28) that correspond to support "B" are substituted into Equation (14). Subsequently, the two generated equations are solved to obtain the values of “M_{AB}” and “M_{BA}”. The equations for moments are presented as follows:

$$M_{AB} = \frac{wL^2}{m_{AB}} \tag{29}$$

$$M_{BA} = \frac{wL^2}{m_{BA}} \tag{30}$$

where m_{AB} and m_{BA} are the fixed-end moment factors.

The equations for fixed-end moments “M_{AB}” and “M_{BA}” are shown in Appendix A.

2.2. **Formulation for the carry-over factors and stiffness factors.** The principle of superposition and the beam compatibility equations are used to obtain the carry-over and stiffness factors. The procedure is the following: when the moment “M_{AB}” is applied clockwise at support “A” of a simply supported straight member of non-constant cross-section at one end (support “A”) and fixed at the other end (support “B”) to obtain the rotation “β_{A1}” in the support “A” and the fixed-end moment “M_{BA}” in the support “B” (see Figure 3(a)). Rotations “β_{A1}” and “$\beta_{B1} = 0$” in the supports are caused by the moments “M_{AB}” and “M_{BA}”, respectively. Figure 3(b) shows that the rotations “β_{A2}” and “β_{B2}” are caused by “M_{AB}”, and Figure 3(c) presents the rotations “β_{A3}” and “β_{B3}” caused by “M_{BA}”.

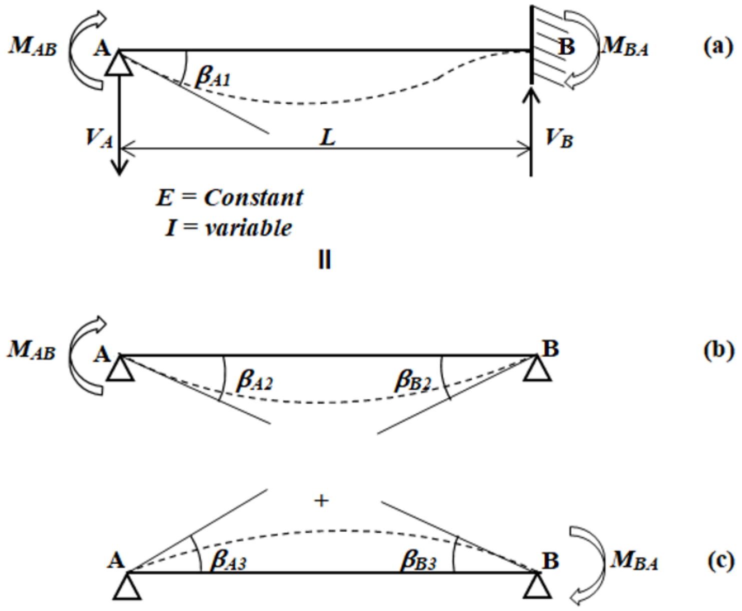

FIGURE 3. A simply supported beam at one end and fixed at the other

Now, by the principle of superposition and using the beam compatibility equations for the rotations in each support obtain the following equations:

$$\beta_{A1} = \beta_{A2} - \beta_{A3} \tag{31}$$

$$0 = \beta_{B2} - \beta_{B3} \tag{32}$$

The beam is analyzed to obtain “β_{A2}” and “β_{B2}” in the function of “M_{AB}” (see Figure 3(b)); these appear in Equations (23) and (24). The beam is analyzed to obtain “β_{A3}” and “β_{B3}” in the function of “M_{BA}” (see Figure 3(c)); these appear in Equations (27) and (28).

Now, substituting Equations (24) and (28) into Equation (32), “M_{BA}” is obtained in the function of “M_{AB}”. Therefore, the carry-over factor of “A” to “B” (C_{AB}) is the percentage of the moment produced in the support "B" due to a moment applied in the support "A"; the procedure to find the carry-over factor of "B" to "A" (C_{BA}) is the same used above. Therefore, equations are shown as follows:

$$M_{BA} = C_{AB} M_{AB} \tag{33}$$

$$M_{AB} = C_{BA} M_{BA} \tag{34}$$

The equations for carry-over factors "C_{AB}" and "C_{BA}" are shown in Appendix B.

Now, substituting Equations (23) and (27) into Equation (31) is presented in the function of "M_{AB}", "M_{BA}" and "β_{A1}", and subsequently Equation (33) is substituted in this same equation to find "M_{AB}" in function of "β_{A1}". Therefore, the stiffness "K_{AB}" (Absolute stiffness from point A to B) is the moment applied in the support "A" to produce a rotation of 1 radian in the support "A"; the procedure to find the stiffness "K_{BA}" (Absolute stiffness from point B to A) is the same used above. Therefore, equations are presented as follows:

$$M_{AB} = K_{AB}\beta_{A1} = \frac{k_{AB} E I_{x2}}{L}\beta_{A1} \tag{35}$$

$$M_{BA} = K_{BA}\beta_{B1} = \frac{k_{BA} E I_{x2}}{L}\beta_{B1} \tag{36}$$

The equations for stiffness factors "k_{AB}" (Stiffness factor from point A to B) and "k_{BA}" (Stiffness factor from point B to A) are shown in Appendix C.

3. **Numerical Example of the Proposed Model.** A non-prismatic reinforced concrete T-shaped beam with straight haunches supported on four supports is analyzed (see Figure 4). The first beam (A-B) has a length of 16.00 m, and does not have straight haunch in support A, only in support B. The second beam (B-C) has a length of 16.00 m, and has two straight haunches, and the haunches are perfectly symmetrical. The third beam (C-D) has a length of 16.00 m, and does not have straight haunch in support D, only in support C. Figure 5 shows the three beams separately and their fixed moments (Fixed-end moments) in each support. Constant data over all the cross section are $\nu = 0.20$ for concrete, w (Uniformly distributed load) $= 10$ kN/m, $b_f = 1.50$ m, $t_f = 0.30$ m, $b_w = 0.50$ m. The final moments are obtained using the proposed model by the matrix methods.

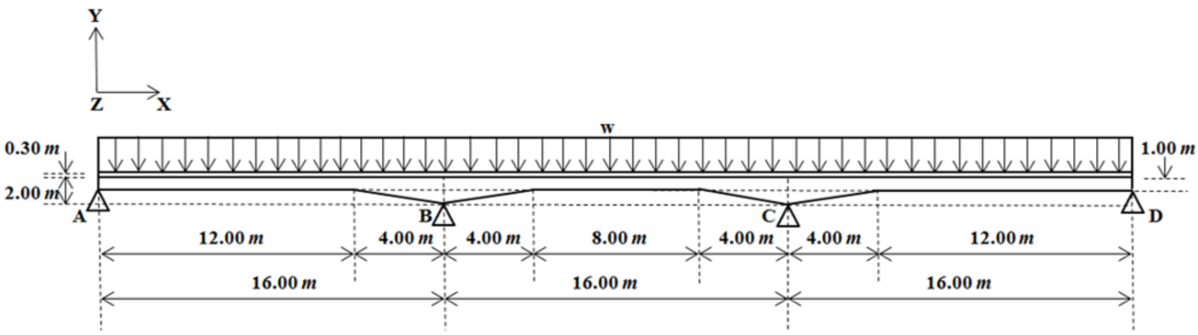

FIGURE 4. Non-prismatic reinforced concrete T-shaped beam

The data for beam A-B are $L_1 = 0.00$ m; $L_2 = 4.00$ m; $d = 1.00$ m; $d_1 = 0.00$ m; $d_2 = 1.00$ m; $L = 16.00$ m. The load $w = 10$ kN/m. The fixed-end moments are $M_{FAB} = 175.2594$ kN-m and $M_{FBA} = 300.9206$ kN-m. The carry-over factors are $C_{AB} = 0.6857$ and $C_{BA} = 0.4596$. The stiffness factors are $k_{AB} = 4.3789$ and $k_{BA} = 6.5337$. The absolute stiffnesses are $K_{AB} = 4.3789 E I_{x2}/L$ and $K_{BA} = 6.5337 E I_{x2}/L$.

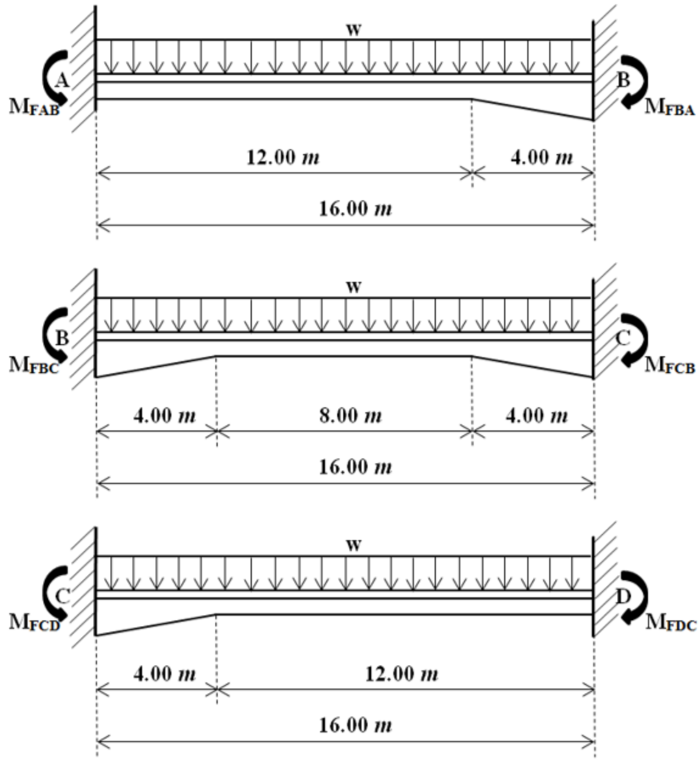

FIGURE 5. Three separate beams

The data for beam B-C are $L_1 = 4.00$ m; $L_2 = 4.00$ m; $d = 1.00$ m; $d_1 = 1.00$ m; $d_2 = 1.00$ m; $L = 16.00$ m. The load $w = 10$ kN/m. The fixed-end moments are $M_{FBC} = 253.1357$ kN-m and $M_{FCB} = 253.1357$ kN-m. The carry-over factors are $C_{BC} = 0.6407$ and $C_{CB} = 0.6407$. The stiffness factors are $k_{BC} = 7.6580$ and $k_{CB} = 7.6580$. The absolute stiffnesses are $K_{BC} = 7.6580EI_{x2}/L$ and $K_{CB} = 7.6580EI_{x2}/L$.

The data for beam C-D are $L_1 = 4.00$ m; $L_2 = 0.00$ m; $d = 1.00$ m; $d_1 = 1.00$ m; $d_2 = 0.00$ m; $L = 16.00$ m. The load $w = 10$ kN/m. The fixed-end moments are $M_{FCD} = 300.9206$ kN-m and $M_{FDC} = 175.2594$ kN-m. The carry-over factors are $C_{CD} = 0.4596$ and $C_{DC} = 0.6857$. The stiffness factors are $k_{CD} = 6.5337$ and $k_{DC} = 4.3789$. The absolute stiffnesses are $K_{CD} = 6.5337EI_{x2}/L$ and $K_{DC} = 4.3789EI_{x2}/L$.

The stiffness matrix of the beam "A-B" is

$$K_{AB} = \begin{bmatrix} k_{11}^{AB} & k_{12}^{AB} \\ k_{21}^{AB} & k_{22}^{AB} \end{bmatrix} = \begin{bmatrix} 4.3789 & 3.0026 \\ 3.0026 & 6.5337 \end{bmatrix} \frac{EI_{x2}}{L}$$

where $k_{11}^{AB} = K_{AB}$; $k_{22}^{AB} = K_{BA}$; $k_{12}^{AB} = C_{AB}K_{AB}$; $k_{21}^{AB} = C_{BA}K_{BA}$; $k_{12}^{AB} = k_{21}^{AB}$.

The stiffness matrix of the beam "B-C" is

$$K_{BC} = \begin{bmatrix} k_{11}^{BC} & k_{12}^{BC} \\ k_{21}^{BC} & k_{22}^{BC} \end{bmatrix} = \begin{bmatrix} 7.6580 & 4.9065 \\ 4.9065 & 7.6580 \end{bmatrix} \frac{EI_{x2}}{L}$$

where $k_{11}^{BC} = K_{BC}$; $k_{22}^{BC} = K_{CB}$; $k_{12}^{BC} = C_{BC}K_{BC}$; $k_{21}^{BC} = C_{CB}K_{CB}$; $k_{12}^{BC} = k_{21}^{BC}$.

The stiffness matrix of the beam "C-D" is

$$K_{CD} = \begin{bmatrix} k_{11}^{CD} & k_{12}^{CD} \\ k_{21}^{CD} & k_{22}^{CD} \end{bmatrix} = \begin{bmatrix} 6.5337 & 3.0026 \\ 3.0026 & 4.3789 \end{bmatrix} \frac{EI_{x2}}{L}$$

where $k_{11}^{CD} = K_{CD}$; $k_{22}^{CD} = K_{DC}$; $k_{12}^{CD} = C_{CD}K_{CD}$; $k_{21}^{CD} = C_{DC}K_{DC}$; $k_{12}^{CD} = k_{21}^{CD}$.

General stiffness matrix "K_G" of the continuous beam is

$$K_G = \begin{bmatrix} k_{11}^{AB} & k_{12}^{AB} & 0 & 0 \\ k_{21}^{AB} & k_{22}^{AB} + k_{11}^{BC} & k_{12}^{BC} & 0 \\ 0 & k_{21}^{BC} & k_{22}^{BC} + k_{11}^{CD} & k_{12}^{CD} \\ 0 & 0 & k_{21}^{CD} & k_{22}^{CD} \end{bmatrix} = \begin{bmatrix} 4.3789 & 3.0026 & 0 & 0 \\ 3.0026 & 14.1917 & 4.9065 & 0 \\ 0 & 4.9065 & 14.1917 & 3.0026 \\ 0 & 0 & 3.0026 & 4.3789 \end{bmatrix} \frac{EI_{x2}}{L}$$

Fixed-end moments of the beams (phase 1) are

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} +175.2594 \\ -300.9206 \end{bmatrix}; \quad \begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} +253.1357 \\ -253.1357 \end{bmatrix}; \quad \begin{bmatrix} M_{CD} \\ M_{DC} \end{bmatrix} = \begin{bmatrix} +300.9206 \\ -175.2594 \end{bmatrix}$$

The vector of effective moments that act on the continuous beam is

$$\begin{bmatrix} M_A \\ M_B \\ M_C \\ M_D \end{bmatrix} = \begin{bmatrix} -175.2594 \\ +300.9206 - 253.1357 \\ +253.1357 - 300.9206 \\ +175.2594 \end{bmatrix} = \begin{bmatrix} -175.2594 \\ +47.7849 \\ -47.7849 \\ +175.2594 \end{bmatrix}$$

Force-displacement relationship is

$$[P] = [K][d]$$

where $[P]$ is the vector of effective moments that acts on the continuous beam, $[K]$ is the general stiffness matrix, and $[d]$ is the vector of displacements.

$$\begin{bmatrix} -175.2594 \\ +47.7849 \\ -47.7849 \\ +175.2594 \end{bmatrix} = \begin{bmatrix} 4.3789 & 3.0026 & 0 & 0 \\ 3.0026 & 14.1917 & 4.9065 & 0 \\ 0 & 4.9065 & 14.1917 & 3.0026 \\ 0 & 0 & 3.0026 & 4.3789 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} \beta_A \\ \beta_B \\ \beta_C \\ \beta_D \end{bmatrix}$$

The solution of the system is

$$\begin{bmatrix} \beta_A \\ \beta_B \\ \beta_C \\ \beta_D \end{bmatrix} = \begin{bmatrix} -55.9611 \\ +23.2428 \\ -23.2428 \\ +55.9611 \end{bmatrix} \frac{L}{EI_{x2}}$$

The mechanical elements associated to the analysis moments (phase 2) are

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} k_{11}^{AB} & k_{12}^{AB} \\ k_{21}^{AB} & k_{22}^{AB} \end{bmatrix} \begin{bmatrix} \beta_A \\ \beta_B \end{bmatrix} = \begin{bmatrix} 4.3789 & 3.0026 \\ 3.0026 & 6.5337 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} -55.9611 \\ +23.2428 \end{bmatrix} \frac{L}{EI_{x2}} = \begin{bmatrix} -175.2594 \\ -16.1676 \end{bmatrix}$$

$$\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} k_{11}^{BC} & k_{12}^{BC} \\ k_{21}^{BC} & k_{22}^{BC} \end{bmatrix} \begin{bmatrix} \beta_B \\ \beta_C \end{bmatrix} = \begin{bmatrix} 7.6580 & 4.9065 \\ 4.9065 & 7.6580 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} +23.2428 \\ -23.2428 \end{bmatrix} \frac{L}{EI_{x2}} = \begin{bmatrix} +63.9525 \\ -63.9525 \end{bmatrix}$$

$$\begin{bmatrix} M_{CD} \\ M_{DC} \end{bmatrix} = \begin{bmatrix} k_{11}^{CD} & k_{12}^{CD} \\ k_{21}^{CD} & k_{22}^{CD} \end{bmatrix} \begin{bmatrix} \beta_C \\ \beta_D \end{bmatrix} = \begin{bmatrix} 6.5337 & 3.0026 \\ 3.0026 & 4.3789 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} -23.2428 \\ +55.9611 \end{bmatrix} \frac{L}{EI_{x2}}$$
$$= \begin{bmatrix} +16.1676 \\ +175.2594 \end{bmatrix}$$

The final moments are (sum of phases 1 and 2)

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} +175.2594 \\ -300.9206 \end{bmatrix} + \begin{bmatrix} -175.2594 \\ -16.1676 \end{bmatrix} = \begin{bmatrix} 0 \\ -317.0882 \end{bmatrix}$$
$$\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} +253.1357 \\ -253.1357 \end{bmatrix} + \begin{bmatrix} +63.9525 \\ -63.9525 \end{bmatrix} = \begin{bmatrix} +317.0882 \\ -317.0882 \end{bmatrix}$$
$$\begin{bmatrix} M_{CD} \\ M_{DC} \end{bmatrix} = \begin{bmatrix} +300.9206 \\ -175.2594 \end{bmatrix} + \begin{bmatrix} +16.1676 \\ +175.2594 \end{bmatrix} = \begin{bmatrix} +317.0882 \\ 0 \end{bmatrix}$$

Therefore, these final moments are the moments for the design of the beams.

4. **Results.** Tables 1 and 2 present the comparison of the proposed model against the traditional model. The proposed model (PM) considers the bending and shear deformations, and the traditional model (TM) only takes account of the bending deformations and without considering the shear deformations. Tables show the fixed-end moments factors (m_{AB} and m_{BA}), the carry-over factors (C_{AB} and C_{BA}), and the stiffness factors (k_{AB} and k_{BA}) of a beam subjected to a uniformly distributed load. Table 1 for $L = 10d \to d = 0.10L$, $b_f = b_w + 16t_f = 0.475L$, $b_w = 0.75d = 0.075L$, $t_f = 0.25d = 0.025L$, $d_1 = d_2$, $\nu = 0.20$ of concrete. Table 2 for $L = 5d \to d = 0.20L$, $b_f = b_w + 16t_f = 0.95L$, $b_w = 0.75d = 0.15L$, $t_f = 0.25d = 0.05L$, $d_1 = d_2$, $\nu = 0.20$ of concrete. The values shown in Table 1 of the proposed model are identical to those in the tables presented in Appendix B by Tena-Colunga [7].

Another way to verify the proposed model is as follows.

1) Substituting "$L_1 = L_2 = 0L$" or "$d_1 = d_2 = 0$" into Equations (37) and (38) of Appendix A, the fixed-end moments factors are obtained "$m_{AB} = m_{BA} = 12$". Thus, fixed-end moments are "$M_{AB} = M_{BA} = wL^2/12$".

2) Substituting "$L_1 = L_2 = 0L$" or "$d_1 = d_2 = 0$" into Equations (39) and (40) of Appendix B and the shear deformations are neglected, the carry-over factors are obtained "$C_{AB} = C_{BA} = 0.5$".

3) Substituting "$L_1 = L_2 = 0L$" or "$d_1 = d_2 = 0$" into Equations (41) and (42) of Appendix C and the shear deformations are neglected, the stiffness factors are obtained "$k_{AB} = k_{BA} = 4$". Therefore, the absolute stiffnesses are "$K_{AB} = K_{BA} = 4EI_{x2}/L$".

Thus, the mechanical elements are verified for constant or uniform cross sections.

Tables 1 and 2 show that when the beam is symmetric, the fixed-end moments are the same in the traditional model and the proposed model, i.e., the shear deformations do not affect the fixed-end moments for symmetrical beams.

Regarding the fixed-end moment factors, if the moment is greater in one support than the opposite end, the traditional model is greater in one support, and the proposed model is greater in the opposite end, with the difference being up to 2.77% in Table 1 (m_{BA}: $L_1 = 0.2L$, $L_2 = 0.8L$, $d_2 = 2d$) and 6.58% in Table 2 (m_{BA}: $L_1 = 0.2L$, $L_2 = 0.8L$, $d_2 = 2d$). Regarding the carry-over factors, the traditional model is greater in all cases, and the difference reaches up to 15.87% in Table 1 (C_{AB}: $L_1 = 0.2L$, $L_2 = 0.8L$, $d_2 = 2d$) and 83.15% in Table 2 (C_{AB}: $L_1 = 0.2L$, $L_2 = 0.8L$, $d_2 = 2d$). Regarding the stiffness factors, also the traditional model is greater in all cases, with a difference of 30.97% in

TABLE 1. Mechanical elements factors for $d = 0.1L$

L_2	d_2/d	Fixed-end moments						Carry-over factors						Stiffness					
		wL^2/m_{AB}			wL^2/m_{BA}			C_{AB}			C_{BA}			$k_{AB}\ EI_{x2}/L$			$k_{BA}\ EI_{x2}/L$		
		m_{AB}			m_{BA}									k_{AB}			k_{BA}		
		PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM
$L_1 = 0.2L$																			
$0.2L$	0.5	10.8253	10.8253	1.0000	10.8253	10.8253	1.0000	0.5515	0.5861	1.0627	0.5515	0.5861	1.0627	5.2594	5.6983	1.0835	5.2594	5.6983	1.0835
	1.0	10.2804	10.2804	1.0000	10.2804	10.2804	1.0000	0.5986	0.6369	1.0640	0.5986	0.6369	1.0640	6.4137	7.0896	1.1054	6.4137	7.0896	1.1054
	1.5	9.9766	9.9766	1.0000	9.9766	9.9766	1.0000	0.6288	0.6693	1.0644	0.6288	0.6693	1.0644	7.3144	8.2096	1.1224	7.3144	8.2096	1.1224
	2.0	9.7870	9.7870	1.0000	9.7870	9.7870	1.0000	0.6494	0.6912	1.0644	0.6494	0.6912	1.0644	8.0244	9.1117	1.1355	8.0244	9.1117	1.1355
$0.4L$	0.5	11.1404	11.1217	0.9983	10.2851	10.3029	1.0017	0.6106	0.6525	1.0686	0.5142	0.5473	1.0644	5.4214	5.9118	1.0905	6.4372	7.0484	1.0949
	1.0	10.8983	10.8822	0.9985	9.2917	9.3058	1.0015	0.7152	0.7687	1.0748	0.5357	0.5713	1.0665	6.8716	7.7330	1.1254	9.1735	10.4042	1.1342
	1.5	10.8805	10.8821	1.0001	8.6362	8.6349	0.9998	0.7944	0.8570	1.0788	0.5479	0.5847	1.0672	8.1532	9.4490	1.1099	11.8223	13.8495	1.1715
	2.0	10.9544	10.9844	1.0027	8.1710	8.1495	0.9974	0.8553	0.9250	1.0815	0.5553	0.5927	1.0674	9.2833	11.0486	1.1902	14.2985	17.2416	1.2058
$0.6L$	0.5	11.3466	11.2856	0.9946	10.4702	10.5331	1.0060	0.6248	0.6724	1.0762	0.4732	0.5056	1.0685	5.5164	6.0168	1.0907	7.2839	8.0010	1.0985
	1.0	11.2015	11.0920	0.9902	9.4967	9.6057	1.0115	0.7583	0.8266	1.0901	0.4668	0.5014	1.0741	7.1245	8.0462	1.1294	11.5723	13.2649	1.1463
	1.5	11.2552	11.1072	0.9869	8.7974	8.9353	1.0157	0.8746	0.9636	1.1018	0.4593	0.4949	1.0775	8.6317	10.1135	1.1717	16.4351	19.6908	1.1981
	2.0	11.4003	11.2237	0.9845	8.2614	8.4135	1.0184	0.9758	1.0845	1.1114	0.4523	0.4883	1.0796	10.0536	12.2358	1.2171	21.6882	27.1754	1.2530
$0.8L$	0.5	11.7340	11.6673	0.9943	10.7211	10.7895	1.0064	0.5986	0.6490	1.0842	0.4457	0.4794	1.0756	5.8015	6.3180	1.0890	7.7915	8.5535	1.0978
	1.0	11.7835	11.6514	0.9888	9.9019	10.0353	1.0135	0.7049	0.7817	1.1090	0.4213	0.4589	1.0892	7.8467	8.8293	1.1252	13.1286	15.0403	1.1456
	1.5	11.9450	11.7479	0.9830	9.3105	9.5030	1.0207	0.7956	0.9021	1.1339	0.4005	0.4411	1.1014	9.9017	11.5299	1.1644	19.6724	23.5818	1.1987
	2.0	12.1507	11.8895	0.9785	8.8534	9.0988	1.0277	0.8741	1.0128	1.1587	0.3829	0.4259	1.1123	11.9548	14.4233	1.2065	27.2914	34.2962	1.2567
$L_1 = 0.4L$																			
$0.2L$	0.5	10.2851	10.3029	1.0017	11.1404	11.1217	0.9983	0.5142	0.5473	1.0644	0.6106	0.6525	1.0686	6.4372	7.0484	1.0949	5.4214	5.9118	1.0905
	1.0	9.2917	9.3058	1.0015	10.8983	10.8822	0.9985	0.5357	0.5713	1.0665	0.7152	0.7687	1.0748	9.1735	10.4042	1.1342	6.8716	7.7330	1.1254
	1.5	8.6362	8.6349	0.9998	10.8805	10.8821	1.0001	0.5479	0.5847	1.0672	0.7944	0.8570	1.0788	11.8223	13.8495	1.1715	8.1532	9.4490	1.1099
	2.0	8.1710	8.1495	0.9974	10.9544	10.9844	1.0027	0.5553	0.5927	1.0674	0.8553	0.9250	1.0815	14.2985	17.2416	1.2058	9.2833	11.0486	1.1902
$0.4L$	0.5	10.5927	10.5927	1.0000	10.5927	10.5927	1.0000	0.5666	0.6067	1.0708	0.5666	0.6067	1.0708	6.6322	7.3091	1.1021	6.6322	7.3091	1.1021
	1.0	9.8633	9.8633	1.0000	9.8633	9.8633	1.0000	0.6326	0.6822	1.0784	0.6326	0.6822	1.0784	9.8271	11.3623	1.1562	9.8271	11.3623	1.1562
	1.5	9.4272	9.4272	1.0000	9.4272	9.4272	1.0000	0.6801	0.7370	1.0837	0.6801	0.7370	1.0837	13.2136	16.0717	1.2163	13.2136	16.0717	1.2163
	2.0	9.1421	9.1421	1.0000	9.1421	9.1421	1.0000	0.7156	0.7779	1.0871	0.7156	0.7779	1.0871	16.6764	21.3549	1.2805	16.6764	21.3549	1.2805
$0.6L$	0.5	10.8325	10.7934	0.9964	10.7831	10.8247	1.0039	0.5738	0.6192	1.0791	0.5193	0.5588	1.0761	6.7653	7.4522	1.1015	7.4760	8.2576	1.1045
	1.0	10.2446	10.1638	0.9921	10.0829	10.1717	1.0088	0.6500	0.7132	1.0972	0.5454	0.5947	1.0904	10.2604	11.8676	1.1566	12.2294	14.2322	1.1638
	1.5	9.9087	9.7849	0.9875	9.6134	9.7517	1.0144	0.7088	0.7893	1.1136	0.5605	0.6181	1.1028	14.1457	17.2636	1.2204	17.8881	22.0461	1.2324
	2.0	9.7035	9.5364	0.9828	9.2682	9.4558	1.0202	0.7552	0.8525	1.1288	0.5695	0.6342	1.1136	18.3203	23.6720	1.2921	24.2944	31.8193	1.3097

TABLE 2. Mechanical elements factors for $d = 0.2L$

L_2	d_2/d	Fixed-end moments						Carry-over factors						Stiffness					
		wL^2/m_{AB}			wL^2/m_{BA}			C_{AB}			C_{BA}			$k_{AB}\ EI_{x2}/L$			$k_{BA}\ EI_{x2}/L$		
		m_{AB}			m_{BA}									k_{AB}			k_{BA}		
		PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM
$L_1 = 0.2L$																			
$0.2L$	0.5	10.8253	10.8253	1.0000	10.8253	10.8253	1.0000	0.4465	0.5861	1.3127	0.4465	0.5861	1.3127	4.3388	5.6983	1.3133	4.3388	5.6983	1.3133
	1.0	10.2804	10.2804	1.0000	10.2804	10.2804	1.0000	0.4939	0.6369	1.2895	0.4939	0.6369	1.2895	5.0860	7.0896	1.3939	5.0860	7.0896	1.3939
	1.5	9.9766	9.9766	1.0000	9.9766	9.9766	1.0000	0.5183	0.6693	1.2913	0.5183	0.6693	1.2913	5.6370	8.2096	1.4564	5.6370	8.2096	1.4564
	2.0	9.7870	9.7870	1.0000	9.7870	9.7870	1.0000	0.5354	0.6912	1.2910	0.5354	0.6912	1.2910	6.0561	9.1117	1.5046	6.0561	9.1117	1.5046
$0.4L$	0.5	11.1785	11.1217	0.9949	10.2491	10.3029	1.0052	0.4966	0.6525	1.3139	0.4229	0.5473	1.2942	4.4242	5.9118	1.3362	5.1945	7.0484	1.3569
	1.0	10.9278	10.8822	0.9958	9.2661	9.3058	1.1122	0.5725	0.7687	1.3427	0.4380	0.5713	1.3043	5.2976	7.7330	1.4597	6.9246	10.4042	1.5025
	1.5	10.8777	10.8821	1.0004	8.6383	8.6349	0.9984	0.6297	0.8570	1.3610	0.4470	0.5847	1.3081	5.9905	9.4490	1.5773	8.4388	13.8495	1.6412
	2.0	10.9082	10.9844	1.0070	8.2044	8.1495	0.9933	0.6737	0.9250	1.3730	0.4529	0.5927	1.3087	6.5525	11.0486	1.6862	9.7459	17.2416	1.7691
$0.6L$	0.5	11.4713	11.2856	0.9838	10.3459	10.5331	1.0181	0.4969	0.6724	1.3532	0.3837	0.5056	1.3177	4.5095	6.0168	1.3342	5.8406	8.0010	1.3699
	1.0	11.4010	11.0920	0.9729	9.3096	9.6057	1.0318	0.5808	0.8266	1.4232	0.3720	0.5014	1.3478	5.4913	8.0462	1.4653	8.5735	13.2649	1.5472
	1.5	11.4959	11.1072	0.9662	8.5891	8.9353	1.0403	0.6501	0.9636	1.4822	0.3620	0.4949	1.3671	6.3034	10.1135	1.6045	11.3195	19.6908	1.7395
	2.0	11.6579	11.2237	0.9628	8.0568	8.4135	1.0443	0.7083	1.0845	1.5311	0.3541	0.4883	1.3790	6.9891	12.2358	1.7507	13.9824	27.1754	1.9435
$0.8L$	0.5	11.8702	11.6673	0.9829	10.5863	10.7895	1.0192	0.4645	0.6490	1.3972	0.3534	0.4794	1.3565	4.7638	6.3180	1.3263	6.2608	8.5535	1.3662
	1.0	12.0235	11.6514	0.9691	9.6756	10.0353	1.0372	0.5097	0.7817	1.5336	0.3193	0.4589	1.4372	6.1163	8.8293	1.4436	9.7621	15.0403	1.5407
	1.5	12.2640	11.7479	0.9579	9.0269	9.5030	1.0527	0.5374	0.9021	1.6786	0.2913	0.4411	1.5142	7.3760	11.5299	1.5632	13.6080	23.5818	1.7330
	2.0	12.5291	11.8895	0.9490	8.5370	9.0988	1.0658	0.5530	1.0128	1.8315	0.2680	0.4259	1.5892	8.5627	14.4233	1.6844	17.6656	34.2962	1.9414
$L_1 = 0.4L$																			
$0.2L$	0.5	10.2491	10.3029	1.0052	11.1785	11.1217	0.9949	0.4229	0.5473	1.2942	0.4966	0.6525	1.3139	5.1945	7.0484	1.3569	4.4242	5.9118	1.3362
	1.0	9.2661	9.3058	1.0043	10.9279	10.8822	0.9958	0.4380	0.5713	1.3043	0.5725	0.7687	1.3427	6.9246	10.4042	1.5025	5.2976	7.7330	1.4597
	1.5	8.6383	8.6349	0.9996	10.8777	10.8821	1.0004	0.4470	0.5847	1.3081	0.6297	0.8570	1.3610	8.4388	13.8495	1.6412	5.9905	9.4490	1.5773
	2.0	8.2044	8.1495	0.9933	10.9082	10.9844	1.0070	0.4529	0.5927	1.3087	0.6737	0.9250	1.3730	9.7459	17.2416	1.7691	6.5525	11.0486	1.6862
$0.4L$	0.5	10.5927	10.5927	1.0000	10.5927	10.5927	1.0000	0.4574	0.6067	1.3264	0.4574	0.6067	1.3264	5.2972	7.3091	1.3798	5.2972	7.3091	1.3798
	1.0	9.8633	9.8633	1.0000	9.8633	9.8633	1.0000	0.4998	0.6822	1.3649	0.4998	0.6822	1.3649	7.2185	11.3623	1.5741	7.2185	11.3623	1.5741
	1.5	9.4272	9.4272	1.0000	9.4272	9.4272	1.0000	0.5299	0.7370	1.3908	0.5299	0.7370	1.3908	8.9896	16.0717	1.7878	8.9896	16.0717	1.7878
	2.0	9.1421	9.1421	1.0000	9.1421	9.1421	1.0000	0.5523	0.7779	1.4085	0.5523	0.7779	1.4085	10.5957	21.3549	2.0154	10.5957	21.3549	2.0154
$0.6L$	0.5	10.9097	10.7934	0.9893	10.7027	10.8247	1.0114	0.4516	0.6192	1.3711	0.4117	0.5588	1.3573	5.4215	7.4522	1.3746	5.9468	8.2576	1.3886
	1.0	10.3819	10.1638	0.9790	9.9385	10.1717	1.0235	0.4857	0.7132	1.4684	0.4142	0.5947	1.4358	7.5871	11.8676	1.5642	8.8981	14.2322	1.5995
	1.5	10.0902	9.7849	0.9697	9.4235	9.7517	1.0348	0.5055	0.7893	1.5614	0.4099	0.6181	1.5079	9.7165	17.2636	1.7767	11.9811	22.0461	1.8401
	2.0	9.9157	9.5364	0.9617	9.0488	9.4558	1.0450	0.5161	0.8525	1.6518	0.4027	0.6342	1.5749	11.7809	23.6720	2.0094	15.0994	31.8193	2.1073

Table 1 (k_{BA}: $L_1 = 0.4L$, $L_2 = 0.6L$, $d_2 = 2d$) and 110.73% in Table 2 (k_{BA}: $L_1 = 0.4L$, $L_2 = 0.6L$, $d_2 = 2d$).

Therefore, the results show that the major differences are presented in Table 2 with the parameter $d = 0.20L$.

5. **Conclusions.** Traditional methods used for the variable cross section members are by the Simpson's rule to obtain the rotations or some other technique to perform numerical integration, and other authors present some tables considering the bending deformations and shear, but are limited to certain relationships and also the heights of the haunches are the same at both ends.

The main findings of this investigation are the following.

1) For the fixed-end moment factors: When the fixed-end moment factors are greater in one support than the opposite end, the traditional model is greater in one support, and the proposed model is greater in the opposite end. When the straight haunches of beams are symmetrically, the fixed-end moment factors are the same for the traditional model and the proposed model, and the fixed-end moment factors are influenced by the web depth in straight haunches; i.e., if the straight haunches volume is increased, the factor is greater in the same support.

2) For the carry-over factors: The proposed model is smaller compared to the traditional model in all cases, this is because the lower stiffness is presented in the proposed model and shear deformations are considered.

3) For the stiffness factors: The proposed model is smaller than the traditional model in all cases, this is because the beams have higher rotations and displacements, and the stiffness is obtained from the rotations and displacements, and the proposed model considers the bending and shear deformations.

4) If the parameter "d/L" is higher, the differences are greater in the traditional model compared to the proposed model for the fixed-end moments, carry-over, and stiffness factors, because the factors for the proposed model are reduced.

5) If the parameter "d/L" for the traditional model is modified, the fixed-end moments, carry-over, and stiffness factors do not change their values.

The benefits of non-prismatic beams are as follows.

1) The T-shaped cross-sectional beams with straight haunches have been used in various structures, including buildings and bridges.

2) With the beams being tapered, the architects would be able to create and implement novel aesthetic architectural designs.

3) Structural engineers could seek for the optimum low weight-high strength systems through a redistribution of materials and beams shape.

The fixed-end moments, the carry-over, and the stiffness factors of a member will be helpful for the matrix methods of structural analysis (see application example) and also for the method of moment distribution (Hardy Cross method).

The proposed model is a more suitable model for structural engineering analysis and also is adapted to real conditions, since the shear forces and bending moments act in any structures type. Therefore, the bending and shear deformations are present.

Therefore, the proposed model is valid and is not restricted to certain dimensions or proportions as shown to other authors, and also the bending and shear deformations are considered.

Suggestions for future research are as follows:

1) Studying other types of non-constant cross-section;

2) Studying other types of haunches such as parabolic haunches;

3) Studying other types of loads.

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Appendix A. Fixed-end moments in support A "M_{AB}" and in support B "M_{BA}" due to a uniformly distributed load are presented as follows:

$$\begin{aligned}
M_{AB} = \frac{wL}{2}\Bigg[&\Bigg\{\int_0^{L_1}\left[\frac{(L-x)^2x}{EI_{x1}} - \frac{(L-2x)}{GA_{sx1}}\right]dx + \int_{L_1}^{L-L_2}\left[\frac{(L-x)^2x}{EI_{x2}} - \frac{(L-2x)}{GA_{sx2}}\right]dx \\
&+ \int_{L-L_2}^{L}\left[\frac{(L-x)^2x}{EI_{x3}} - \frac{(L-2x)}{GA_{sx3}}\right]dx\Bigg\}\Bigg\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right]dx \\
&+ \int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right]dx + \int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right]dx\Bigg\} \\
&- \Bigg\{\int_0^{L_1}\left[\frac{(L-x)x^2}{EI_{x1}} + \frac{(L-2x)}{GA_{sx1}}\right]dx + \int_{L_1}^{L-L_2}\left[\frac{(L-x)x^2}{EI_{x2}} + \frac{(L-2x)}{GA_{sx2}}\right]dx \\
&+ \int_{L-L_2}^{L}\left[\frac{(L-x)x^2}{EI_{x3}} + \frac{(L-2x)}{GA_{sx3}}\right]dx\Bigg\}\Bigg\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}}\right]dx \\
&+ \int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}}\right]dx + \int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}}\right]dx\Bigg\}\Bigg] \Bigg/ \\
&\Bigg[\Bigg\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right]dx + \int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right]dx \\
&+ \int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right]dx\Bigg\}\Bigg\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right]dx
\end{aligned}$$

$$+ \int_{L_1}^{L-L_2} \left[\frac{x^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{x^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right] dx\Bigg\}$$
$$-\Bigg\{\int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}}\right] dx$$
$$+ \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}}\right] dx\Bigg\}^2\Bigg] \quad (37)$$

$$M_{BA} = \frac{wL}{2}\Bigg[\Bigg\{\int_0^{L_1} \left[\frac{(L-x)x^2}{EI_{x1}} + \frac{(L-2x)}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x^2}{EI_{x2}} + \frac{(L-2x)}{GA_{sx2}}\right] dx$$
$$+ \int_{L-L_2}^{L} \left[\frac{(L-x)x^2}{EI_{x3}} + \frac{(L-2x)}{GA_{sx3}}\right] dx\Bigg\}\Bigg\{\int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right] dx$$
$$+ \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right] dx\Bigg\}$$
$$-\Bigg\{\int_0^{L_1} \left[\frac{(L-x)^2x}{EI_{x1}} - \frac{(L-2x)}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2x}{EI_{x2}} - \frac{(L-2x)}{GA_{sx2}}\right] dx$$
$$+ \int_{L-L_2}^{L} \left[\frac{(L-x)^2x}{EI_{x3}} - \frac{(L-2x)}{GA_{sx3}}\right] dx\Bigg\}\Bigg\{\int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}}\right] dx$$
$$+ \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}}\right] dx\Bigg\}\Bigg] \Bigg/$$
$$\Bigg[\Bigg\{\int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right] dx$$
$$+ \int_{L-L_2}^{L} \left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right] dx\Bigg\}\Bigg\{\int_0^{L_1} \left[\frac{x^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right] dx$$
$$+ \int_{L_1}^{L-L_2} \left[\frac{x^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{x^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right] dx\Bigg\}$$
$$-\Bigg\{\int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}}\right] dx$$
$$+ \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}}\right] dx\Bigg\}^2\Bigg] \quad (38)$$

Appendix B. Carry-over factors in support A "C_{AB}" and in support B "C_{BA}" are presented as follows:

$$C_{AB} = \frac{\int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}}\right] dx}{\int_0^{L_1} \left[\frac{x^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{x^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{x^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right] dx} \quad (39)$$

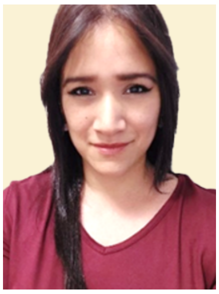
$$C_{BA} = \frac{\int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}}\right] dx}{\int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}}\right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}}\right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}}\right] dx} \quad (40)$$

Appendix C. Stiffness factors in support A "k_{AB}" and in support B "k_{BA}" are presented as follows:

$$\begin{aligned}
k_{AB} = &\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx\right.\\
&\left.+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\Bigg/\left[\left\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.\right.\\
&+\int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx\\
&\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.\\
&\left.+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\\
&-\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx\right.\\
&\left.\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}^2\right]
\end{aligned} \tag{41}$$

$$\begin{aligned}
k_{BA} = &\left\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx\right.\\
&\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\Bigg/\left[\left\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.\right.\\
&+\int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx\\
&\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.\\
&\left.+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\\
&-\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx\right.\\
&\left.\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}^2\right]
\end{aligned} \tag{42}$$

Author Biography



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