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MODELLING FOR MECHANICAL ELEMENTS OF T-SHAPED CROSS-SECTIONAL BEAMS WITH STRAIGHT HAUNCHES: PART II

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Abstract. *The first part of this paper shows a model for T-shaped cross-sectional beams with straight haunches under a uniformly distributed load that considers shear and bending deformations to obtain the fixed-end moments, carry-over and stiffness factors. Now, this paper shows a model for the T-shaped beams with straight haunches under a concentrated load localized anywhere of the beam that takes account of bending and shear deformations to obtain the fixed-end moments, which is the main part of this investigation. The methodology used is the same procedure applied in Part I. The traditional model considers only the bending deformations and the shear deformations are neglected, and some authors consider bending and shear deformations for some proportions that are shown in tables. A numerical example has been developed to observe the application of the proposed model, and a comparison among the proposed approach that considers bending and shear deformations against the traditional model that considers bending deformations only is presented in the tables and graphics. A significant advantage is that the fixed-end moments can be obtained for any T-shaped beams with straight haunches using the model proposed in this paper.*

Keywords: T-shaped beams, Fixed-end moments factors, Carry-over factors, Stiffness factors, Concentrated load, Bending and shear deformations

1. **Introduction.** The main concern of structural engineering is to propose suitable methods that are reliable to satisfactorily modeling non-prismatic cross-sectional members, in such a way that is having certainty in the determination of the mechanical elements, such as deformations and displacements that allow this type of structural members to be properly designed.

The reinforced concrete non-prismatic beams have the following advantages with respect to prismatic beams: 1) Substantially increase lateral stiffness; 2) Promote the more efficient use of concrete and longitudinal reinforcement steel; 3) Reduce the weight of the building; 4) Facilitate the placement of electrical, air conditioning and sanitary installations in the building.

Between the years of 1950 to 1960 several design aids are developed, such as those presented by Guldan [1], and the most popular tables published by the PCA (Portland Cement Association) in 1958 "Handbook" [2].

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The most relevant documents dealing the topic of structural members with non-prismatic cross section are discussed in Part I [3-26].

The bibliographic review shows that the closest studies for the T-shaped beams with straight haunches are the tables provided the fixed-end moments, carry-over and stiffness factors for beams of sections in shape of "I" and "T" [7]. These tables for the T-shaped beams consider the flange width "b_f" is equal to the web thickness "b_w" more 16 times flange thickness "t_f", the relationship between the beam length and the effective depth is equal to 10, the relationship between the flange thickness "t_f" and the effective depth is equal to 1/4, the relationship between the web thickness "b_w" and the effective depth is equal to 3/4, and the height of the straight haunches is the same in the two ends. Therefore, there are no papers on the subject with the level of current knowledge that provide a way to obtain the fixed-end moments, carry-over and stiffness factors for T-shaped beams with straight haunches for the general case.

This paper shows a mathematical model for T-shaped cross-sectional beams with straight haunches under a concentrated load localized anywhere of the beam considering the bending and shear deformations to find the fixed-end moments. The methodology used is the same procedure applied in Part I. The traditional model only considers the bending deformations, and other authors consider bending and shear deformations for proportions restricted that are shown in tables. A numerical example has been developed to observe the application of the proposed model, and a comparison between the proposed approaches that considers bending and shear deformations against the traditional model that considers bending deformations only are presented in the tables and graphics to observe differences.

The paper is organized as follows. Section 2 shows the formulation of the mathematical model for the equations of fixed-end moments factors. Section 3 describes the application of the proposed model. Section 4 is dedicated to the results through the comparison of the two models, the proposed model (PM), and the traditional model (TM). Section 5 presents the conclusions.

2. **Formulation of the Mathematical Model.** Figure 1 shows in detail the asymmetric T-shaped beam under a concentrated load where the flange width "b_f", flange thickness "t_f", web thickness "b_w" are constant, k is the location of load, GC is center of gravity of the cross section, L_1 and L_2 are the lengths of the haunches, and the web height "d_x" that varies with the linear shape at the ends, and the beam in the center has a constant section.

The geometrical properties of the cross section of the T-shaped beam are shown in Equations (1) to (12) of Part I.

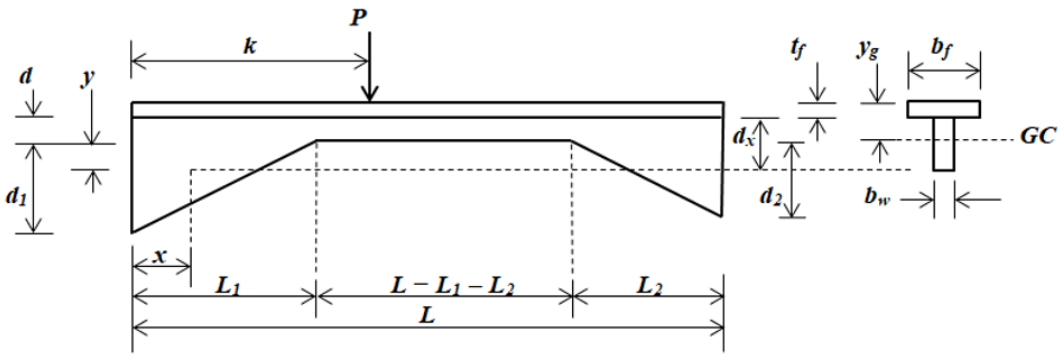


FIGURE 1. T-shaped beam with straight haunches under a concentrated load

Figure 2(a) shows a beam "AB" under a concentrated load located on anywhere of the beam and fixed at its ends for three different cases (Case 1: The load P is located from $0 \leq x \leq L_1$, Case 2: The load P is localized from $L_1 \leq x \leq L - L_2$, Case 3: The load P is located of $L - L_2 \leq x \leq L$) is obtained by the principle of superposition (sum of the individual rotations in each support "A" and "B"). The moments according to the sign are considered: counterclockwise is positive, and clockwise is negative. In Figure 2(b) for all three cases is presented the same simply supported beam at its ends under the load applied to obtaining the rotations "β_{A1}" and "β_{B1}" for case 1, "β_{A2}" and "β_{B2}" for the case 2, "β_{A3}" and "β_{B3}" for case 3. In Figure 2(c) for all three cases is shown the moment "M_{AB}" applied in the support "A" that generates the rotations "β_{A12}" and "β_{B12}" for case 1, "β_{A22}" and "β_{B22}" for the case 2, "β_{A32}" and "β_{B32}" for case 3. In Figure 2(d) for all three cases is presented the moment "M_{BA}" applied in the support "B" that generates the rotations "β_{A13}" and "β_{B13}" for case 1, "β_{A23}" and "β_{B23}" for the case 2, "β_{A33}" and "β_{B33}" for case 3 [27-32].

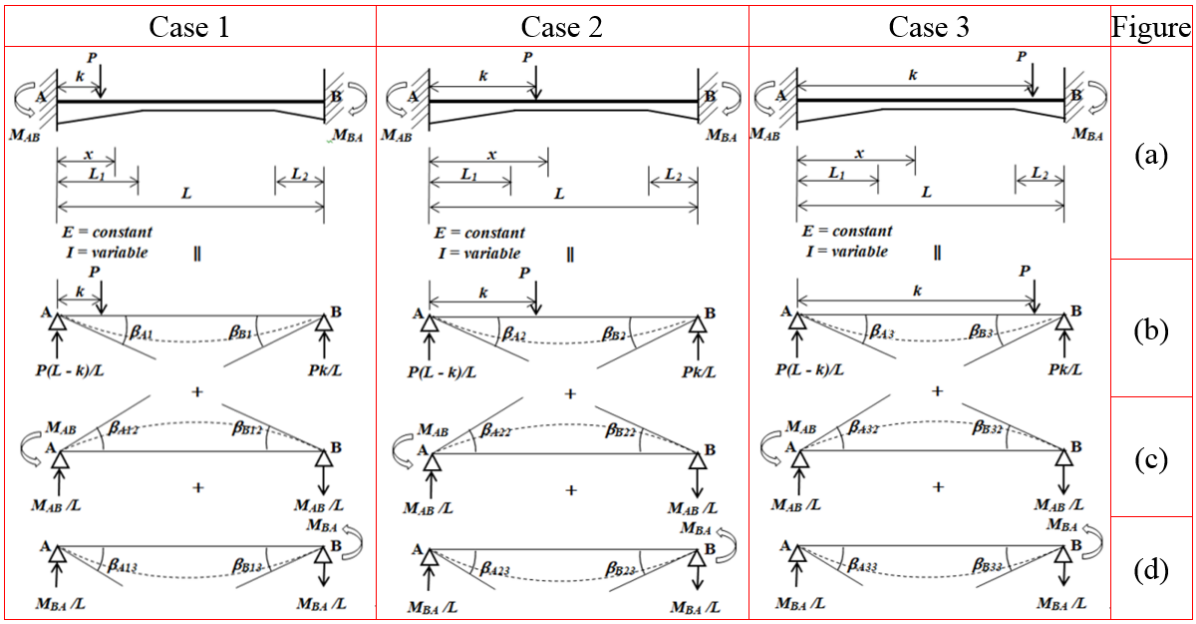


FIGURE 2. T-shaped beam with fixed supports

Now, by the principle of superposition and using the beam compatibility equations for the rotations in each support obtain the following equations:

To $0 \leq x \leq L_1$:

$$\beta_{A1} - \beta_{A12} + \beta_{A13} = 0 \tag{1}$$

$$\beta_{B1} - \beta_{B12} + \beta_{B13} = 0 \tag{2}$$

To $L_1 \leq x \leq L - L_2$:

$$\beta_{A2} - \beta_{A22} + \beta_{A23} = 0 \tag{3}$$

$$\beta_{B2} - \beta_{B22} + \beta_{B23} = 0 \tag{4}$$

To $L - L_2 \leq x \leq L$:

$$\beta_{A3} - \beta_{A32} + \beta_{A33} = 0 \tag{5}$$

$$\beta_{B3} - \beta_{B32} + \beta_{B33} = 0 \tag{6}$$

Figure 2(a) for the three cases is analyzed to obtain the rotations (β_{A1} and β_{B1}, β_{A2} and β_{B2}, β_{A3} and β_{B3}) and the bending and shear deformations are taken into account due to the loads applied on beam.

The rotations for non-prismatic members are obtained by Equations (15) and (16) of Part I.

The shear forces and moments located in anywhere on the beam for the three cases according to Figure 2(b) are

To $0 \leq x \leq k$:

$$V_x = \frac{P(L-k)}{L} \tag{7}$$

$$M_x = \frac{P(L-k)x}{L} \tag{8}$$

To $k \leq x \leq L$:

$$V_x = \frac{Pk}{L} \tag{9}$$

$$M_x = \frac{Pk(L-x)}{L} \tag{10}$$

Substituting Equations (7) to (10) into Equations (15) and (16) of Part I, the rotations (β_{A1} and β_{B1}, β_{A2} and β_{B2}, β_{A3} and β_{B3}) are obtained as follows:

To $0 \leq x \leq L_1$:

$$\begin{aligned}\beta_{A1} = &-\frac{P}{GL^2}\left[\int_0^k \frac{(L-k)}{A_{sx1}}dx - \int_k^{L_1} \frac{k}{A_{sx1}}dx - \int_{L_1}^{L-L_2} \frac{k}{A_{sx2}}dx - \int_{L-L_2}^{L} \frac{k}{A_{sx3}}dx\right] \\ &+ \frac{P}{EL^2}\left[\int_0^k \frac{(L-k)(L-x)x}{I_{x1}}dx + \int_k^{L_1} \frac{k(L-x)^2}{I_{x1}}dx + \int_{L_1}^{L-L_2} \frac{k(L-x)^2}{I_{x2}}dx \right. \\ &\left. + \int_{L-L_2}^{L} \frac{k(L-x)^2}{I_{x3}}dx\right]\end{aligned} \tag{11}$$

$$\begin{aligned}\beta_{B1} = &\frac{P}{GL^2}\left[\int_0^k \frac{(L-k)}{A_{sx1}}dx - \int_k^{L_1} \frac{k}{A_{sx1}}dx - \int_{L_1}^{L-L_2} \frac{k}{A_{sx2}}dx - \int_{L-L_2}^{L} \frac{k}{A_{sx3}}dx\right] \\ &+ \frac{P}{EL^2}\left[\int_0^k \frac{(L-k)x^2}{I_{x1}}dx + \int_k^{L_1} \frac{k(L-x)x}{I_{x1}}dx + \int_{L_1}^{L-L_2} \frac{k(L-x)x}{I_{x2}}dx \right. \\ &\left. + \int_{L-L_2}^{L} \frac{k(L-x)x}{I_{x3}}dx\right]\end{aligned} \tag{12}$$

To $L_1 \leq x \leq L - L_2$:

$$\begin{aligned}\beta_{A2} = &-\frac{P}{GL^2}\left[\int_0^{L_1} \frac{(L-k)}{A_{sx1}}dx + \int_{L_1}^k \frac{(L-k)}{A_{sx2}}dx - \int_k^{L-L_2} \frac{k}{A_{sx2}}dx - \int_{L-L_2}^{L} \frac{k}{A_{sx3}}dx\right] \\ &+ \frac{P}{EL^2}\left[\int_0^{L_1} \frac{(L-k)(L-x)x}{I_{x1}}dx + \int_{L_1}^k \frac{(L-k)(L-x)x}{I_{x2}}dx \right. \\ &\left. + \int_k^{L-L_2} \frac{k(L-x)^2}{I_{x2}}dx + \int_{L-L_2}^{L} \frac{k(L-x)^2}{I_{x3}}dx\right]\end{aligned} \tag{13}$$

$$\begin{aligned}\beta_{B2} = &\frac{P}{GL^2}\left[\int_0^{L_1} \frac{(L-k)}{A_{sx1}}dx + \int_{L_1}^k \frac{(L-k)}{A_{sx2}}dx - \int_k^{L-L_2} \frac{k}{A_{sx2}}dx - \int_{L-L_2}^{L} \frac{k}{A_{sx3}}dx\right] \\ &+ \frac{P}{EL^2}\left[\int_0^{L_1} \frac{(L-k)x^2}{I_{x1}}dx + \int_{L_1}^k \frac{(L-k)x^2}{I_{x2}}dx + \int_k^{L-L_2} \frac{k(L-x)x}{I_{x2}}dx\right.\end{aligned}$$

$$+\int_{L-L_2}^{L}\frac{k(L-x)x}{I_{x3}}dx\Bigg] \tag{14}$$

To $L-L_2 \le x \le L$:

$$\begin{aligned}\beta_{A3} = &-\frac{P}{GL^2}\left[\int_0^{L_1}\frac{(L-k)}{A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{(L-k)}{A_{sx2}}dx+\int_{L-L_2}^{k}\frac{(L-k)}{A_{sx3}}dx-\int_k^L\frac{k}{A_{sx3}}dx\right]\\ &+\frac{P}{EL^2}\left[\int_0^{L_1}\frac{(L-k)(L-x)x}{I_{x1}}dx+\int_{L_1}^{L-L_2}\frac{(L-k)(L-x)x}{I_{x2}}dx\right.\\ &\left.+\int_{L-L_2}^{k}\frac{(L-k)(L-x)x}{I_{x3}}dx+\int_k^L\frac{k(L-x)^2}{I_{x3}}dx\right]\end{aligned} \tag{15}$$

$$\begin{aligned}\beta_{B3} = &\frac{P}{GL^2}\left[\int_0^{L_1}\frac{(L-k)}{A_{sx1}}dx+\int_{L_1}^{L-L_2}\frac{(L-k)}{A_{sx2}}dx+\int_{L-L_2}^{k}\frac{(L-k)}{A_{sx3}}dx-\int_k^L\frac{k}{A_{sx3}}dx\right]\\ &+\frac{P}{EL^2}\left[\int_0^{L_1}\frac{(L-k)x^2}{I_{x1}}dx+\int_{L_1}^{L-L_2}\frac{(L-k)x^2}{I_{x2}}dx+\int_{L-L_2}^{k}\frac{(L-k)x^2}{I_{x3}}dx\right.\\ &\left.+\int_k^L\frac{k(L-x)x}{I_{x3}}dx\right]\end{aligned} \tag{16}$$

The rotations (β_{A12} and β_{B12}, β_{A22} and β_{B22}, β_{A32} and β_{B32}) on the beam according to Figure 2(c) are

$$\beta_{A12} = \beta_{A22} = \beta_{A32} = \beta_{A2} \text{ (Part I)} \tag{17}$$

$$\beta_{B12} = \beta_{B22} = \beta_{B32} = \beta_{B2} \text{ (Part I)} \tag{18}$$

The rotations (β_{A13} and β_{B13}, β_{A23} and β_{B23}, β_{A33} and β_{B33}) on the beam according to Figure 2(d) are:

$$\beta_{A13} = \beta_{A23} = \beta_{A33} = \beta_{A3} \text{ (Part I)} \tag{19}$$

$$\beta_{B13} = \beta_{B23} = \beta_{B33} = \beta_{B3} \text{ (Part I)} \tag{20}$$

Now, analyze the three cases individually for corresponding intervals.

Case 1: Equation (11), and Equations (23) and (27) of Part I that correspond to support "A" are substituted into Equation (1) to obtain the moment "M_{AB}", and Equation (12), and Equations (24) and (28) of Part I that correspond to support "B" are substituted into Equation (2) to obtain the moment "M_{BA}" for the load "P" located in the interval $0 \le x \le L_1$. Subsequently, the generated equations are solved to obtain the values of "M_{AB}" and "M_{BA}".

Case 2: Equation (13), and Equations (23) and (27) of Part I that correspond to support "A" are substituted into Equation (3) to obtain the moment "M_{AB}", and Equation (14), and Equations (24) and (28) of Part I that correspond to support "B" are substituted into Equation (4) to obtain the moment "M_{BA}" for the load "P" located in the interval $L_1 \le x \le L-L_2$. Subsequently, the generated equations are solved to obtain the values of "M_{AB}" and "M_{BA}".

Case 3: Equation (15), and Equations (23) and (27) of Part I that correspond to support "A" are substituted into Equation (5) to obtain the moment "M_{AB}", and Equation (16), and Equations (24) and (28) of Part I that correspond to support "B" are substituted into Equation (6) to obtain the moment "M_{BA}" for the load "P" located in the interval $L-L_2 \le x \le L$. Subsequently, the generated equations are solved to obtain the values of "M_{AB}" and "M_{BA}".

The equations of the moments for the three cases are presented as follows:

$$M_{AB} = \frac{PL}{m_{AB}} \tag{21}$$

$$M_{BA} = \frac{PL}{m_{BA}} \tag{22}$$

where m_{AB} and m_{BA} are the fixed-end moment factors.

The equations for fixed-end moments "M_{AB}" and "M_{BA}" are shown in Appendix.

3. **Numerical Example of the Proposed Model.** Figure 3 illustrates a three-span continuous highway bridge for a reinforced concrete T-shaped beam with straight haunches. The first and third beams (A-B and C-D) have a length of 12.00 m, and have two asymmetrical straight haunches. The second beam (B-C) has a length of 15.00 m, and has two straight haunches, and the haunches are perfectly symmetrical. Figure 4 shows the critical position of the live loads for each stretch of the highway bridge, taking account of the live loads that provide the specifications for the design of bridges [33]. Constant data over all the cross section are $\nu = 0.20$ for concrete, $b_f = 1.50$ m, $t_f = 0.30$ m, $b_w = 0.50$ m. For obtaining the final moments, the proposed model developed through matrix methods is used.

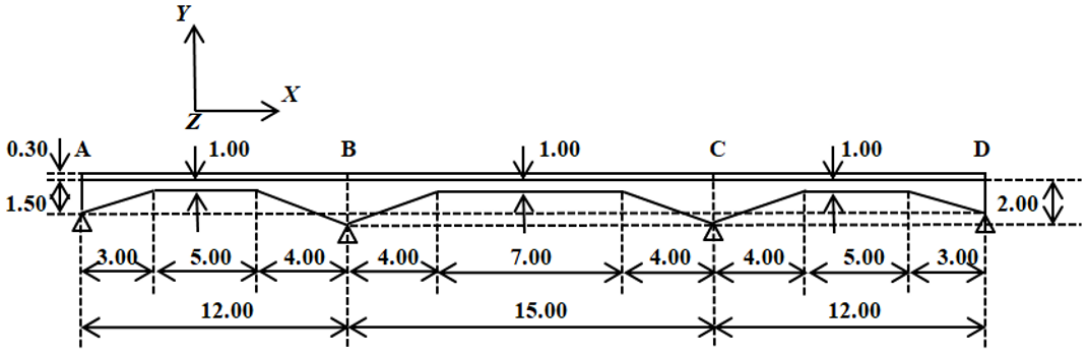
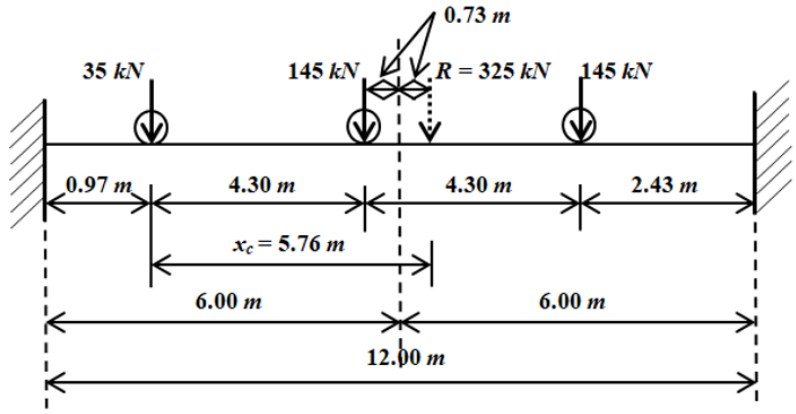


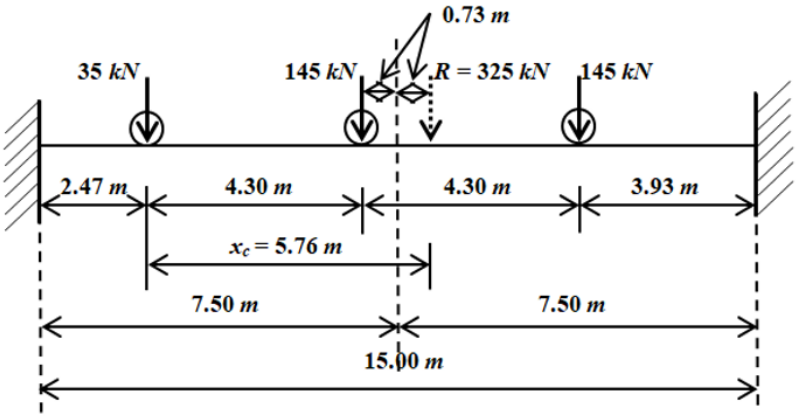
FIGURE 3. Non-prismatic reinforced concrete T-shaped beam

For the beam A-B: $L_1 = 3.00$ m; $L_2 = 4.00$ m; $d = 1.00$ m; $d_1 = 0.50$ m; $d_2 = 1.00$ m; $L = 12.00$ m. To the load $P = 35$ kN, $k = 0.97$ m, and is localized of $0 \leq x \leq L_1$, the fixed-end moments are $M_{AB1} = 29.4629$ kN-m and $M_{BA1} = 3.5294$ kN-m. To the load $P = 145$ kN, $k = 5.27$ m, and is found of $L_1 \leq x \leq L - L_2$, the fixed-end moments are $M_{AB2} = 168.8641$ kN-m and $M_{BA2} = 277.9787$ kN-m. To the load $P = 145$ kN, $k = 9.57$ m, and is located of $L - L_2 \leq x \leq L$, the fixed-end moments are $M_{AB3} = 41.6909$ kN-m and $M_{BA3} = 275.4334$ kN-m. The total fixed-end moments are $M_{ABT} = 240.0179$ kN-m and $M_{BAT} = 556.9415$ kN-m. The carry-over factors by Part I of this paper are obtained: $C_{AB} = 0.6968$ and $C_{BA} = 0.5348$. The stiffness factors by Part I of this paper are obtained: $k_{AB} = 6.2319$ and $k_{BA} = 8.1197$. The absolute stiffnesses are $K_{AB} = 6.2319EI/L$ and $K_{BA} = 8.1197EI/L$.

For the beam B-C: $L_1 = 4.00$ m; $L_2 = 4.00$ m; $d = 1.00$ m; $d_1 = 1.00$ m; $d_2 = 1.00$ m; $L = 15.00$ m. To the load $P = 35$ kN, $k = 2.47$ m, and is localized of $0 \leq x \leq L_1$, the fixed-end moments are $M_{BC1} = 71.4921$ kN-m and $M_{CB1} = 9.7183$ kN-m. To the load $P = 145$ kN, $k = 6.77$ m, and is found of $L_1 \leq x \leq L - L_2$, the fixed-end moments are $M_{BC2} = 376.2745$ kN-m and $M_{CB2} = 290.9268$ kN-m. To the load $P = 145$ kN, $k = 11.07$ m, and is located of $L - L_2 \leq x \leq L$, the fixed-end moments are $M_{BC3} = 102.6699$ kN-m and $M_{CB3} = 397.9590$ kN-m. The total fixed-end moments are $M_{BCT} = 550.4365$ kN-m



(a) To first and third light (A-B and C-D)



(b) To second light (B-C)

FIGURE 4. Critical position of the loads

and $M_{CBT} = 698.6041$ kN-m. The carry-over factors by Part I of this paper are obtained: $C_{BC} = 0.6441$ and $C_{CB} = 0.6441$. The stiffness factors by Part I of this paper are obtained: $k_{BC} = 7.9295$ and $k_{CB} = 7.9295$. The absolute stiffnesses are $K_{BC} = 7.9295EI/L$ and $K_{CB} = 7.9295EI/L$.

For the beam C-D: $L_1 = 4.00$ m; $L_2 = 3.00$ m; $d = 1.00$ m; $d_1 = 1.00$ m; $d_2 = 0.50$ m; $L = 12.00$ m. To the load $P = 35$ kN, $k = 0.97$ m, and is localized of $0 \leq x \leq L_1$, the fixed-end moments are $M_{CD1} = 30.7106$ kN-m and $M_{DC1} = 1.9945$ kN-m. To the load $P = 145$ kN, $k = 5.27$ m, and is found of $L_1 \leq x \leq L - L_2$, the fixed-end moments are $M_{CD2} = 344.3930$ kN-m and $M_{DC2} = 178.4415$ kN-m. To the load $P = 145$ kN, $k = 9.57$ m, and is located of $L - L_2 \leq x \leq L$, the fixed-end moments are $M_{CD3} = 73.6830$ kN-m and $M_{DC3} = 244.5827$ kN-m. The total fixed-end moments are $M_{CDT} = 448.7866$ kN-m and $M_{DCT} = 425.0187$ kN-m. The carry-over factors by Part I of this paper are obtained: $C_{CD} = 0.5348$ and $C_{DC} = 0.6968$. The stiffness factors by Part I of this paper are obtained: $k_{CD} = 8.1197$ and $k_{DC} = 6.2319$. The absolute stiffnesses are $K_{CD} = 8.1197EI/L$ and $K_{DC} = 6.2319EI/L$.

The stiffness matrix of the beam "A-B" is

$$K_{AB} = \begin{bmatrix} k_{11}^{AB} & k_{12}^{AB} \\ k_{21}^{AB} & k_{22}^{AB} \end{bmatrix} = \begin{bmatrix} 6.2319 & 4.3424 \\ 4.3424 & 8.1197 \end{bmatrix} \frac{EI_{x2}}{L}$$

where $k_{11}^{AB} = K_{AB}$; $k_{22}^{AB} = K_{BA}$; $k_{12}^{AB} = C_{AB}K_{AB}$; $k_{21}^{AB} = C_{BA}K_{BA}$; $k_{12}^{AB} = k_{21}^{AB}$.

The stiffness matrix of the beam "B-C" is

$$K_{BC} = \begin{bmatrix} k_{11}^{BC} & k_{12}^{BC} \\ k_{21}^{BC} & k_{22}^{BC} \end{bmatrix} = \begin{bmatrix} 7.9295 & 5.1074 \\ 5.1074 & 7.9295 \end{bmatrix} \frac{EI_{x2}}{L}$$

where $k_{11}^{BC} = K_{BC}$; $k_{22}^{BC} = K_{CB}$; $k_{12}^{BC} = C_{BC}K_{BC}$; $k_{21}^{BC} = C_{CB}K_{CB}$; $k_{12}^{BC} = k_{21}^{BC}$.

The stiffness matrix of the beam "C-D" is

$$K_{CD} = \begin{bmatrix} k_{11}^{CD} & k_{12}^{CD} \\ k_{21}^{CD} & k_{22}^{CD} \end{bmatrix} = \begin{bmatrix} 8.1197 & 4.3424 \\ 4.3424 & 6.2319 \end{bmatrix} \frac{EI_{x2}}{L}$$

where $k_{11}^{CD} = K_{CD}$; $k_{22}^{CD} = K_{DC}$; $k_{12}^{CD} = C_{CD}K_{CD}$; $k_{21}^{CD} = C_{DC}K_{DC}$; $k_{12}^{CD} = k_{21}^{CD}$.

General stiffness matrix "K_G" of the continuous beam is

$$K_G = \begin{bmatrix} k_{11}^{AB} & k_{12}^{AB} & 0 & 0 \\ k_{21}^{AB} & k_{22}^{AB} + k_{11}^{BC} & k_{12}^{BC} & 0 \\ 0 & k_{21}^{BC} & k_{22}^{BC} + k_{11}^{CD} & k_{12}^{CD} \\ 0 & 0 & k_{21}^{CD} & k_{22}^{CD} \end{bmatrix}$$

$$= \begin{bmatrix} 6.2319 & 4.3424 & 0 & 0 \\ 4.3424 & 16.0492 & 5.1074 & 0 \\ 0 & 5.1074 & 16.0492 & 4.3424 \\ 0 & 0 & 4.3424 & 6.2319 \end{bmatrix} \frac{EI_{x2}}{L}$$

Fixed-end moments of the beams (phase 1) are

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} +240.0179 \\ -556.9415 \end{bmatrix}; \quad \begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} +550.4365 \\ -698.6041 \end{bmatrix}; \quad \begin{bmatrix} M_{CD} \\ M_{DC} \end{bmatrix} = \begin{bmatrix} +448.7866 \\ -425.0187 \end{bmatrix}$$

The vector of effective moments that act on the continuous beam is

$$\begin{bmatrix} M_A \\ M_B \\ M_C \\ M_D \end{bmatrix} = \begin{bmatrix} -240.0179 \\ +556.9415 - 550.4365 \\ +698.6041 - 448.7866 \\ +425.0187 \end{bmatrix} = \begin{bmatrix} -240.0179 \\ +6.5050 \\ +249.8175 \\ +425.0187 \end{bmatrix}$$

Force-displacement relationship is

$$[P] = [K][d]$$

where $[P]$ is the vector of effective moments that acts on the continuous beam, $[K]$ is the general stiffness matrix, and $[d]$ is the vector of displacements.

$$\begin{bmatrix} -240.0179 \\ +6.5050 \\ +249.8175 \\ +425.0187 \end{bmatrix} = \begin{bmatrix} 6.2319 & 4.3424 & 0 & 0 \\ 4.3424 & 16.0492 & 5.1074 & 0 \\ 0 & 5.1074 & 16.0492 & 4.3424 \\ 0 & 0 & 4.3424 & 6.2319 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} \beta_A \\ \beta_B \\ \beta_C \\ \beta_D \end{bmatrix}$$

The solution of the system is

$$\begin{bmatrix} \beta_A \\ \beta_B \\ \beta_C \\ \beta_D \end{bmatrix} = \begin{bmatrix} -50.64926072 \\ +17.41507643 \\ -10.38761306 \\ +75.43860956 \end{bmatrix} \frac{L}{EI_{x2}}$$

The mechanical elements associated to the analysis moments (phase 2) are

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} k_{11}^{AB} & k_{12}^{AB} \\ k_{21}^{AB} & k_{22}^{AB} \end{bmatrix} \begin{bmatrix} \beta_A \\ \beta_B \end{bmatrix} = \begin{bmatrix} 6.2319 & 4.3424 \\ 4.3424 & 8.1197 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} -50.64926072 \\ +17.41507643 \end{bmatrix} \frac{L}{EI_{x2}}$$
$$= \begin{bmatrix} -240.0179 \\ -78.5342 \end{bmatrix}$$
$$\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} k_{11}^{BC} & k_{12}^{BC} \\ k_{21}^{BC} & k_{22}^{BC} \end{bmatrix} \begin{bmatrix} \beta_B \\ \beta_C \end{bmatrix} = \begin{bmatrix} 7.9295 & 5.1074 \\ 5.1074 & 7.9295 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} +17.41507643 \\ -10.38761306 \end{bmatrix} \frac{L}{EI_{x2}}$$
$$= \begin{bmatrix} +85.0392 \\ +6.5772 \end{bmatrix}$$
$$\begin{bmatrix} M_{CD} \\ M_{DC} \end{bmatrix} = \begin{bmatrix} k_{11}^{CD} & k_{12}^{CD} \\ k_{21}^{CD} & k_{22}^{CD} \end{bmatrix} \begin{bmatrix} \beta_C \\ \beta_D \end{bmatrix} = \begin{bmatrix} 8.1197 & 4.3424 \\ 4.3424 & 6.2319 \end{bmatrix} \frac{EI_{x2}}{L} \begin{bmatrix} -10.38761306 \\ +75.43860956 \end{bmatrix} \frac{L}{EI_{x2}}$$
$$= \begin{bmatrix} +243.2403 \\ +425.0187 \end{bmatrix}$$

The final moments are (sum of phases 1 and 2)

$$\begin{bmatrix} M_{AB} \\ M_{BA} \end{bmatrix} = \begin{bmatrix} +240.0179 \\ -556.9415 \end{bmatrix} + \begin{bmatrix} -240.0179 \\ -78.5342 \end{bmatrix} = \begin{bmatrix} 0 \\ -635.4757 \end{bmatrix}$$
$$\begin{bmatrix} M_{BC} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} +550.4365 \\ -698.6041 \end{bmatrix} + \begin{bmatrix} +85.0392 \\ +6.5772 \end{bmatrix} = \begin{bmatrix} +635.4757 \\ -692.0269 \end{bmatrix}$$
$$\begin{bmatrix} M_{CD} \\ M_{DC} \end{bmatrix} = \begin{bmatrix} +448.7866 \\ -425.0187 \end{bmatrix} + \begin{bmatrix} +243.2403 \\ +425.0187 \end{bmatrix} = \begin{bmatrix} +692.0269 \\ 0 \end{bmatrix}$$

Therefore, these final moments are the moments for the design of the beams.

4. **Results.** Tables 1 and 2 present the comparison of the proposed model against the traditional model. The proposed model (PM) considers the bending and shear deformations, and the traditional model (TM) only takes account of the bending deformations and without considering the shear deformations. Tables show the fixed-end moments factors (m_{AB} and m_{BA}) for a beam subjected to a concentrated load located anywhere on the beam. Table 1 for $L = 10d \rightarrow d = 0.10L$, $b_f = b_w + 16t_f = 0.475L$, $b_w = 0.75d = 0.075L$, $t_f = 0.25d = 0.025L$, $d_1 = d_2$, $\nu = 0.20$ of concrete. Table 2 for $L = 5d \rightarrow d = 0.20L$, $b_f = b_w + 16t_f = 0.95L$, $b_w = 0.75d = 0.15L$, $t_f = 0.25d = 0.05L$, $d_1 = d_2$, $\nu = 0.20$ of concrete. The values shown in Table 1 of the proposed model are identical to those in the tables presented in Appendix B by Tena-Colunga [7].

Another way to verify the proposed model is as follows.

1) Substituting "$k = 0.5L$" and "$L_1 = L_2 = 0L$" or "$d_1 = d_2 = 0$" into Equations (23) and (24) of the Appendix, the fixed-end moments factors are obtained "$m_{AB} = m_{BA} = 8$". Thus, fixed-end moments are "$M_{AB} = M_{BA} = PL/8$".

2) Substituting "$k = 0.5L$" and "$L_1 = L_2 = 0L$" or "$d_1 = d_2 = 0$" into Equations (25) and (26) of the Appendix, the fixed-end moments factors are obtained "$m_{AB} = m_{BA} = 8$". Thus, fixed-end moments are "$M_{AB} = M_{BA} = PL/8$".

3) Substituting "$k = 0.5L$" and "$L_1 = L_2 = 0L$" or "$d_1 = d_2 = 0$" into Equations (27) and (28) of the Appendix, the fixed-end moments factors are obtained "$m_{AB} = m_{BA} = 8$". Thus, fixed-end moments are "$M_{AB} = M_{BA} = PL/8$".

4) If the fixed-end moments for the traditional model are verified with the well-known equations for uniform cross sections with a concentrated load located anywhere on the

TABLE 1. Fixed-end moments factors for $d = 0.1L$

L_2	d_2/d	$M_{AB} = PL/m_{AB}$; $M_{BA} = PL/m_{BA}$																											
		$k = 0.1L$						$k = 0.3L$						$k = 0.5L$						$k = 0.7L$						$k = 0.9L$					
		m_{AB}			m_{BA}			m_{AB}			m_{BA}			m_{AB}			m_{BA}			m_{AB}			m_{BA}			m_{AB}			m_{BA}		
		PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM
$L_1 = 0.1L$																													
$0.1L$	0.5	11.7233	11.3379	0.9671	111.1111	163.9344	1.4754	6.3052	6.1538	0.9760	15.2207	16.2075	1.0648	7.4738	7.4738	1.0000	7.4738	7.4738	1.0000	15.2207	16.2075	1.0648	6.3052	6.1538	0.9760	111.1111	163.9344	1.4754	11.7233	11.3379	0.9671
	1.0	11.2360	10.8696	0.9674	136.9863	232.5581	1.6977	6.0024	5.8411	0.9731	15.4321	16.6113	1.0764	7.2359	7.2359	1.0000	7.2359	7.2359	1.0000	15.4321	16.6113	1.0764	6.0024	5.8411	0.9731	136.9863	232.5581	1.6977	11.2360	10.8696	0.9674
	1.5	10.9529	10.6045	0.9682	161.2903	312.5000	1.9375	5.8241	5.6593	0.9717	15.6250	16.9492	1.0847	7.0972	7.0972	1.0000	7.0972	7.0972	1.0000	15.6250	16.9492	1.0847	5.8241	5.6593	0.9717	161.2903	312.5000	1.9375	10.9529	10.6045	0.9682
	2.0	10.7759	10.4493	0.9697	185.1852	400.0000	2.1600	5.7110	5.5463	0.9712	15.7978	17.2117	1.0895	7.0126	7.0126	1.0000	7.0126	7.0126	1.0000	15.7978	17.2117	1.0895	5.7110	5.5463	0.9712	185.1852	400.0000	2.1600	10.7759	10.4493	0.9697
$0.3L$	0.5	11.8343	11.4025	0.9635	90.9091	133.3333	1.4667	6.5274	6.3654	0.9752	12.7389	13.4953	1.0594	8.0580	8.1037	1.0057	6.4020	6.3654	0.9943	17.4216	18.8679	1.0830	5.6818	5.5279	0.9729	112.3596	163.9344	1.4590	11.8064	11.3766	0.9636
	1.0	11.3895	10.9529	0.9617	97.0874	163.9344	1.6885	6.3776	6.2073	0.9733	11.4155	12.1507	1.0644	8.3542	8.4818	1.0153	5.5556	5.4915	0.9885	20.4082	23.0415	1.1290	5.0226	4.8567	0.9670	140.8451	238.0952	1.6905	11.3250	10.8932	0.9619
	1.5	11.1359	10.7066	0.9614	104.1667	200.0000	1.9200	6.3211	6.1425	0.9717	10.5820	11.2867	1.0666	8.7032	8.9206	1.0250	5.0454	4.9579	0.9827	23.7530	28.1690	1.1859	4.6296	4.4563	0.9626	172.4138	322.5806	1.8710	11.0375	10.6157	0.9618
	2.0	10.9649	10.5374	0.9610	111.1111	243.9024	2.1951	6.2972	6.1162	0.9713	10.0301	10.7066	1.0674	9.0416	9.3545	1.0346	4.7103	4.6041	0.9775	27.3224	34.2466	1.2534	4.3725	4.1964	0.9597	204.0816	434.7826	2.1304	10.8460	10.4493	0.9634
$0.5L$	0.5	11.8765	11.4416	0.9634	81.9672	123.4568	1.5062	6.6225	6.4516	0.9742	11.9048	12.6743	1.0646	8.3056	8.3195	1.0017	6.2228	6.2112	0.9981	17.7620	19.0114	1.0703	5.8582	5.7012	0.9732	108.6957	153.8462	1.4154	12.0919	11.6144	0.9605
	1.0	11.4811	11.0132	0.9592	78.7402	136.9863	1.7397	6.5876	6.3939	0.9706	9.9010	10.5932	1.0699	8.9127	9.0009	1.0099	5.1894	5.1493	0.9923	20.8333	22.9885	1.1034	5.2466	5.0761	0.9675	129.8701	204.0816	1.5714	11.7371	11.1982	0.9541
	1.5	11.2613	10.7643	0.9559	76.9231	153.8462	2.0000	6.6579	6.4475	0.9684	8.5616	9.1575	1.0696	9.6712	9.8814	1.0217	4.5269	4.4563	0.9844	24.2131	27.6243	1.1409	4.8591	4.6729	0.9617	149.2537	263.1579	1.7632	11.4943	10.9290	0.9508
	2.0	11.1359	10.6045	0.9523	75.1880	172.4138	2.2931	6.7659	6.5574	0.9692	7.6220	8.1169	1.0649	10.5152	10.9290	1.0394	4.0683	3.9714	0.9762	27.7778	33.0033	1.1881	4.5914	4.3956	0.9574	169.4915	333.3333	1.9667	11.3379	10.7411	0.9474
$L_1 = 0.3L$																													
$0.1L$	0.5	11.8064	11.3766	0.9636	112.3596	163.9344	1.4590	5.6818	5.5279	0.9729	17.4216	18.8679	1.0830	6.4020	6.3654	0.9943	8.0580	8.1037	1.0057	12.7389	13.4953	1.0594	6.5274	6.3654	0.9752	90.9091	133.3333	1.4667	11.8343	11.4025	0.9635
	1.0	11.3250	10.8932	0.9619	140.8451	238.0952	1.6905	5.0226	4.8567	0.9670	20.4082	23.0415	1.1290	5.5556	5.4915	0.9885	8.3542	8.4818	1.0153	11.4155	12.1507	1.0644	6.3776	6.2073	0.9733	97.0874	163.9344	1.6885	11.3895	10.9529	0.9617
	1.5	11.0375	10.6157	0.9618	172.4138	322.5806	1.8710	4.6296	4.4563	0.9626	23.7530	28.1690	1.1859	5.0454	4.9579	0.9827	8.7032	8.9206	1.0250	10.5820	11.2867	1.0666	6.3211	6.1425	0.9717	104.1667	200.0000	1.9200	11.1359	10.7066	0.9614
	2.0	10.8460	10.4493	0.9634	204.0816	434.7826	2.1304	4.3725	4.1964	0.9597	27.3224	34.2466	1.2534	4.7103	4.6041	0.9775	9.0416	9.3545	1.0346	10.0301	10.7066	1.0674	6.2972	6.1162	0.9713	111.1111	243.9024	2.1951	10.9649	10.5374	0.9610
$0.3L$	0.5	11.9332	11.4679	0.9610	92.5926	133.3333	1.4400	5.8651	5.7045	0.9726	14.5138	15.6006	1.0749	6.8681	6.8681	1.0000	6.8681	6.8681	1.0000	14.5138	15.6006	1.0749	5.8651	5.7045	0.9726	92.5926	133.3333	1.4400	11.9332	11.4679	0.9610
	1.0	11.5340	11.0254	0.9559	100.0000	163.9344	1.6393	5.2938	5.1073	0.9648	14.7710	16.4474	1.1135	6.3131	6.3131	1.0000	6.3131	6.3131	1.0000	14.7710	16.4474	1.1135	5.2938	5.1073	0.9648	100.0000	163.9344	1.6393	11.5340	11.0254	0.9559
	1.5	11.2867	10.7527	0.9527	107.5269	204.0816	1.8980	4.9358	4.7304	0.9584	15.4321	17.8571	1.1571	5.9952	5.9952	1.0000	5.9952	5.9952	1.0000	15.4321	17.8571	1.1571	4.9358	4.7304	0.9584	107.5269	204.0816	1.8980	11.2867	10.7527	0.9527
	2.0	11.1235	10.5820	0.9513	116.2791	250.0000	2.1500	4.6904	4.4683	0.9526	16.2602	19.6850	1.2106	5.7904	5.7904	1.0000	5.7904	5.7904	1.0000	16.2602	19.6850	1.2106	4.6904	4.4683	0.9526	116.2791	250.0000	2.1500	11.1235	10.5820	0.9513
$0.5L$	0.5	11.9904	11.5075	0.9597	84.0336	123.4568	1.4691	5.9488	5.7737	0.9706	13.5318	14.6199	1.0804	7.0771	7.0472	0.9958	6.6711	6.7024	1.0047	14.8148	15.7729	1.0647	6.0569	5.8962	0.9735	89.2857	126.5823	1.4177	12.2399	11.7096	0.9567
	1.0	11.6414	11.0865	0.9523	81.9672	138.8889	1.6944	5.4407	5.2329	0.9618	12.6904	14.2248	1.1209	6.7069	6.6578	0.9927	5.8824	5.9277	1.0077	15.1515	16.4474	1.0855	5.5586	5.3792	0.9677	91.7431	144.9275	1.5797	11.9904	11.3636	0.9477
	1.5	11.4548	10.8460	0.9469	80.0000	153.8462	1.9231	5.1361	4.8972	0.9535	12.2100	14.2045	1.1633	6.5746	6.5147	0.9909	5.3447	5.3937	1.0092	15.7729	17.4825	1.1084	5.2383	5.0403	0.9622	94.3396	163.9344	1.7377	11.8483	11.1359	0.9399
	2.0	11.3379	10.6838	0.9423	78.7402	172.4138	2.1897	4.9358	4.6685	0.9458	11.9048	14.4092	1.2104	6.5617	6.5062	0.9915	4.9456	4.9875	1.0085	16.5016	18.7266	1.1348	5.0100	4.7916	0.9564	96.1538	185.1852	1.9259	11.7647	10.9649	0.9320

TABLE 2. Fixed-end moments factors for $d = 0.2L$

L_2	d_2/d	$M_{AB} = PL/m_{AB}$; $M_{BA} = PL/m_{BA}$																											
		$k = 0.1L$						$k = 0.3L$						$k = 0.5L$						$k = 0.7L$						$k = 0.9L$					
		m_{AB}			m_{BA}			m_{AB}			m_{BA}			m_{AB}			m_{BA}			m_{AB}			m_{BA}			m_{AB}			m_{BA}		
		PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM	PM	TM	TM/PM
$L_1 = 0.1L$																													
$0.1L$	0.5	12.6582	11.3379	0.8957	64.9351	163.9344	2.5246	6.6711	6.1538	0.9225	13.4409	16.2075	1.2058	7.4738	7.4738	1.0000	7.4738	7.4738	1.0000	13.4409	16.2075	1.2058	6.6711	6.1538	0.9225	64.9351	163.9344	2.5246	12.6582	11.3379	0.8957
	1.0	12.1065	10.8696	0.8978	72.9927	232.5581	3.1860	6.3735	5.8411	0.9165	13.4048	16.6113	1.2392	7.2359	7.2359	1.0000	7.2359	7.2359	1.0000	13.4048	16.6113	1.2392	6.3735	5.8411	0.9165	72.9927	232.5581	3.1860	12.1065	10.8696	0.8978
	1.5	11.7509	10.6045	0.9024	80.6452	312.5000	3.8750	6.1996	5.6593	0.9128	13.4409	16.9492	1.2610	7.0972	7.0972	1.0000	7.0972	7.0972	1.0000	13.4409	16.9492	1.2610	6.1996	5.6593	0.9128	80.6452	312.5000	3.8750	11.7509	10.6045	0.9024
	2.0	11.5207	10.4493	0.9070	87.7193	400.0000	4.5600	6.0864	5.5463	0.9113	13.4953	17.2117	1.2754	7.0126	7.0126	1.0000	7.0126	7.0126	1.0000	13.4953	17.2117	1.2754	6.0864	5.5463	0.9113	87.7193	400.0000	4.5600	11.5207	10.4493	0.9070
$0.3L$	0.5	12.8370	11.4025	0.8883	54.0541	133.3333	2.4667	6.8918	6.3654	0.9236	11.4025	13.4953	1.1835	7.9554	8.1037	1.0186	6.4767	6.3654	0.9828	15.0150	18.8679	1.2566	6.0386	5.5279	0.9154	68.0272	163.9344	2.4098	12.8041	11.3766	0.8885
	1.0	12.3609	10.9529	0.8861	53.4759	163.9344	3.0656	6.7522	6.2073	0.9193	10.1937	12.1507	1.1920	8.1169	8.4818	1.0450	5.6915	5.4915	0.9649	16.6389	23.0415	1.3848	5.3908	4.8567	0.9009	79.3651	238.0952	3.0000	12.2699	10.8932	0.8878
	1.5	12.0627	10.7066	0.8876	54.3478	200.0000	3.6800	6.6890	6.1425	0.9183	9.4697	11.2867	1.1919	8.3195	8.9206	1.0723	5.2247	4.9579	0.9489	18.3150	28.1690	1.5380	4.9975	4.4563	0.8917	90.0901	322.5806	3.5806	11.9190	10.6157	0.8907
	2.0	11.8483	10.5374	0.8894	55.8659	243.9024	4.3659	6.6622	6.1162	0.9180	8.9928	10.7066	1.1906	8.5179	9.3545	1.0982	4.9188	4.6041	0.9360	20.0000	34.2466	1.7123	4.7371	4.1964	0.8859	102.0408	434.7826	4.2609	11.6550	10.4493	0.8966
$0.5L$	0.5	12.9199	11.4416	0.8856	48.5437	123.4568	2.5432	7.0126	6.4516	0.9200	10.5820	12.6743	1.1977	8.2645	8.3195	1.0067	6.2500	6.2112	0.9938	15.6250	19.0114	1.2167	6.2073	5.7012	0.9185	67.1141	153.8462	2.2923	13.2275	11.6144	0.8780
	1.0	12.5000	11.0132	0.8811	43.6681	136.9863	3.1370	6.9881	6.3939	0.9150	8.8028	10.5932	1.2034	8.7566	9.0009	1.0279	5.2687	5.1493	0.9773	17.7305	22.9885	1.2966	5.6022	5.0761	0.9061	76.3359	204.0816	2.6735	12.8866	11.1982	0.8690
	1.5	12.2549	10.7643	0.8784	40.9836	153.8462	3.7538	7.0472	6.4475	0.9149	7.6923	9.1575	1.1905	9.3284	9.8814	1.0593	4.6533	4.4563	0.9577	19.8807	27.6243	1.3895	5.2138	4.6729	0.8963	85.4701	263.1579	3.0789	12.6582	10.9290	0.8634
	2.0	12.0919	10.6045	0.8770	39.5257	172.4138	4.3621	7.1378	6.5574	0.9187	6.9396	8.1169	1.1696	9.9206	10.9290	1.1016	4.2355	3.9714	0.9376	22.0751	33.0033	1.4950	4.9456	4.3956	0.8888	93.4579	333.3333	3.5667	12.4533	10.7411	0.8625
$L_1 = 0.3L$																													
$0.1L$	0.5	12.8041	11.3766	0.8885	68.0272	163.9344	2.4098	6.0386	5.5279	0.9154	15.0150	18.8679	1.2566	6.4767	6.3654	0.9828	7.9554	8.1037	1.0186	11.4025	13.4953	1.1835	6.8918	6.3654	0.9236	54.0541	133.3333	2.4667	12.8370	11.4025	0.8883
	1.0	12.2699	10.8932	0.8878	79.3651	238.0952	3.0000	5.3908	4.8567	0.9009	16.6389	23.0415	1.3848	5.6915	5.4915	0.9649	8.1169	8.4818	1.0450	10.1937	12.1507	1.1920	6.7522	6.2073	0.9193	53.4759	163.9344	3.0656	12.3609	10.9529	0.8861
	1.5	11.9190	10.6157	0.8907	90.0901	322.5806	3.5806	4.9975	4.4563	0.8917	18.3150	28.1690	1.5380	5.2247	4.9579	0.9489	8.3195	8.9206	1.0723	9.4697	11.2867	1.1919	6.6890	6.1425	0.9183	54.3478	200.0000	3.6800	12.0627	10.7066	0.8876
	2.0	11.6550	10.4493	0.8966	102.0408	434.7826	4.2609	4.7371	4.1964	0.8859	20.0000	34.2466	1.7123	4.9188	4.6041	0.9360	8.5179	9.3545	1.0982	8.9928	10.7066	1.1906	6.6622	6.1162	0.9180	55.8659	243.9024	4.3659	11.8483	10.5374	0.8894
$0.3L$	0.5	13.0039	11.4679	0.8819	56.4972	133.3333	2.3600	6.2266	5.7045	0.9162	12.6904	15.6006	1.2293	6.8681	6.8681	1.0000	6.8681	6.8681	1.0000	12.6904	15.6006	1.2293	6.2266	5.7045	0.9162	56.4972	133.3333	2.3600	13.0039	11.4679	0.8819
	1.0	12.5628	11.0254	0.8776	58.1395	163.9344	2.8197	5.6657	5.1073	0.9014	12.4844	16.4474	1.3174	6.3131	6.3131	1.0000	6.3131	6.3131	1.0000	12.4844	16.4474	1.3174	5.6657	5.1073	0.9014	58.1395	163.9344	2.8197	12.5628	11.0254	0.8776
	1.5	12.2850	10.7527	0.8753	60.6061	204.0816	3.3673	5.3163	4.7304	0.8898	12.5945	17.8571	1.4178	5.9952	5.9952	1.0000	5.9952	5.9952	1.0000	12.5945	17.8571	1.4178	5.3163	4.7304	0.8898	60.6061	204.0816	3.3673	12.2850	10.7527	0.8753
	2.0	12.0627	10.5820	0.8772	63.6943	250.0000	3.9250	5.0761	4.4683	0.8803	12.8700	19.6850	1.5295	5.7904	5.7904	1.0000	5.7904	5.7904	1.0000	12.8700	19.6850	1.5295	5.0761	4.4683	0.8803	63.6943	250.0000	3.9250	12.0627	10.5820	0.8772
$0.5L$	0.5	13.0890	11.5075	0.8792	51.0204	123.4568	2.4198	6.3291	5.7737	0.9122	11.7647	14.6199	1.2427	7.1327	7.0472	0.9880	6.6138	6.7024	1.0134	13.2275	15.7729	1.1924	6.4020	5.8962	0.9210	55.8659	126.5823	2.2658	13.4409	11.7096	0.8712
	1.0	12.7389	11.0865	0.8703	48.0769	138.8889	2.8889	5.8411	5.2329	0.8959	10.7066	14.2248	1.3286	6.7935	6.6578	0.9800	5.8072	5.9277	1.0208	13.3511	16.4474	1.2319	5.8997	5.3792	0.9118	56.1798	144.9275	2.5797	13.2626	11.3636	0.8568
	1.5	12.5156	10.8460	0.8666	46.5116	153.8462	3.3077	5.5432	4.8972	0.8835	10.0806	14.2045	1.4091	6.6667	6.5147	0.9772	5.2743	5.3937	1.0226	13.7552	17.4825	1.2710	5.5679	5.0403	0.9052	57.4713	163.9344	2.8525	13.1406	11.1359	0.8474
	2.0	12.3609	10.6838	0.8643	46.0829	172.4138	3.7414	5.3419	4.6685	0.8739	9.6805	14.4092	1.4885	6.6401	6.5062	0.9798	4.8924	4.9875	1.0194	14.2653	18.7266	1.3127	5.3305	4.7916	0.8989	59.1716	185.1852	3.1296	13.0208	10.9649	0.8421

beam ($M_{AB} = Pk(L-k)^2/L^2$ and $M_{BA} = Pk^2(L-k)/L^2$), these equations fulfill perfectly with the model presented in this paper.

One way to verify the continuity of the beam element is

1) Substituting $k = L_1$ into Equations (23) and (25) to find "M_{AB}", and into Equations (24) and (26) to find "M_{BA}". The results of the moments at the fixed-ends are the same.

2) Substituting $k = L - L_2$ into Equations (25) and (27) to find "M_{AB}", and into Equations (26) and (28) to find "M_{BA}". The results of the moments at the fixed-ends are the same.

Results for the fixed-end moment factors are as follows.

1) If the moment is greater at one support than at the opposite end, the traditional model is greater at one support, and the proposed model is greater at the opposite end.

2) The biggest differences in Table 1 are 2.2931 times the traditional model (m_{BA}, $k = 0.1L$, $L_1 = 0.1L$, $L_2 = 0.5L$, $d_2 = 2d$) and 2.1951 times the traditional model (m_{AB}, $k = 0.9L$, $L_1 = 0.3L$, $L_2 = 0.1L$, $d_2 = 2d$).

3) The biggest differences in Table 2 are 4.56 times the traditional model (m_{BA}, $k = 0.1L$, $L_1 = 0.1L$, $L_2 = 0.1L$, $d_2 = 2d$) and 4.56 times the traditional model (m_{AB}, $k = 0.9L$, $L_1 = 0.1L$, $L_2 = 0.1L$, $d_2 = 2d$).

4) Tables 1 and 2 show a relationship of TM/PM = 1.0000 in m_{AB} and m_{BA} for $k = 0.5L$ in $L_1 = 0.1L$, $L_2 = 0.1L$, $d_2/d = 0.5$, 1.0, 1.5, 2.0, also for $k = 0.5L$ in $L_1 = 0.3L$, $L_2 = 0.3L$, $d_2/d = 0.5$, 1.0, 1.5, 2.0.

Therefore, the results show that the major differences are presented in Table 2 with the parameter $d = 0.20L$.

Thus, the fixed-end moments are verified for constant or uniform cross sections.

Tables 1 and 2 show that when the beam is symmetric in load and in geometry, the fixed-end moments are the same in the traditional model and the proposed model, i.e., the shear deformations do not affect the fixed-end moments for symmetrical beams in load and in geometry.

5. **Conclusions.** Traditional methods used for the variable cross section members are by the Simpson's rule to obtain the rotations or some other technique to perform numerical integration, and other authors present some tables considering the bending deformations and shear, but are limited to certain relationships and also the heights of the haunches are the same at both ends.

The main findings of this investigation are the following.

1) If the fixed-end moment factors are greater at one support than at the opposite end, the traditional model is greater at one support, and the proposed model is greater at the opposite end.

2) The fixed-end moments for symmetrical beams in load and in geometry, the fixed-end moment factors are the same for the traditional model and the proposed model, i.e., the shear deformations do not affect the fixed-end moments for symmetrical beams in load and in geometry.

3) The fixed-end moment factors are influenced by the web depth in straight haunches, i.e., if the straight haunches volume is increased, the factor is greater in the same support.

4) If the fixed-end moment factors are greater, the fixed-end moment is smaller, because the factor is located in the denominator.

5) When the parameter "d/L" increases, the differences are greater in the traditional model compared to the proposed model for the fixed-end moments, because the factors for the proposed model are reduced.

6) If the parameter "d/L" for the traditional model is modified, the fixed-end moment factors do not change their values.

7) When the concentrated load is located closer to the supports, the difference in the moments is greater between the traditional model and the proposed model.

The benefits of non-prismatic beams are as follows.

1) The T-shaped cross-sectional beams with straight haunches have been used in various structures, including buildings and bridges.

2) With the beams being tapered, the architects would be able to create and implement novel aesthetic architectural designs.

3) Structural engineers could seek for the optimum low weight-high strength systems through a redistribution of materials and beams shape.

Suggestions for future research are as follows:

1) Studying other types of non-constant cross-section;

2) Studying other types of haunches such as parabolic haunches;

3) Studying other types of loads.

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Appendix. Fixed-end moments in support A "M_{AB}" and in support B "M_{BA}" due to a concentrated load P located from $0 \leq x \leq L_1$ are presented as follows:

$$M_{AB} = P\left[\left\{\int_0^k \left[\frac{(L-k)(L-x)x}{EI_{x1}} - \frac{(L-k)}{GA_{sx1}}\right]dx + \int_k^{L_1}\left[\frac{k(L-x)^2}{EI_{x1}} + \frac{k}{GA_{sx1}}\right]dx + \int_{L_1}^{L-L_2}\left[\frac{k(L-x)^2}{EI_{x2}} + \frac{k}{GA_{sx2}}\right]dx\right.$$

$$
\begin{aligned}
&+\int_{L-L_2}^{L}\left[\frac{k(L-x)^2}{EI_{x3}}+\frac{k}{GA_{sx3}}\right]dx\Bigg\}\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.\\
&\left.+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\\
&-\left\{\int_0^{k}\left[\frac{(L-k)x^2}{EI_{x1}}+\frac{(L-k)}{GA_{sx1}}\right]dx+\int_k^{L_1}\left[\frac{k(L-x)x}{EI_{x1}}-\frac{k}{GA_{sx1}}\right]dx\right.\\
&+\int_{L_1}^{L-L_2}\left[\frac{k(L-x)x}{EI_{x2}}-\frac{k}{GA_{sx2}}\right]dx\\
&\left.+\int_{L-L_2}^{L}\left[\frac{k(L-x)x}{EI_{x3}}-\frac{k}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx\right.\\
&\left.+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}\Bigg]\Bigg/\\
&\left[\left\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx\right.\right.\\
&+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\Bigg\}\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.\\
&\left.+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\\
&-\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx\right.\\
&\left.\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}^2\right] \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
M_{BA}=P&\left[\left\{\int_0^{k}\left[\frac{(L-k)x^2}{EI_{x1}}+\frac{(L-k)}{GA_{sx1}}\right]dx+\int_k^{L_1}\left[\frac{k(L-x)x}{EI_{x1}}-\frac{k}{GA_{sx1}}\right]dx\right.\right.\\
&+\int_{L_1}^{L-L_2}\left[\frac{k(L-x)x}{EI_{x2}}-\frac{k}{GA_{sx2}}\right]dx\\
&\left.+\int_{L-L_2}^{L}\left[\frac{k(L-x)x}{EI_{x3}}-\frac{k}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.\\
&\left.+\int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\\
&-\left\{\int_0^{k}\left[\frac{(L-k)(L-x)x}{EI_{x1}}-\frac{(L-k)}{GA_{sx1}}\right]dx+\int_k^{L_1}\left[\frac{k(L-x)^2}{EI_{x1}}+\frac{k}{GA_{sx1}}\right]dx\right.\\
&+\int_{L_1}^{L-L_2}\left[\frac{k(L-x)^2}{EI_{x2}}+\frac{k}{GA_{sx2}}\right]dx\\
&\left.+\int_{L-L_2}^{L}\left[\frac{k(L-x)^2}{EI_{x3}}+\frac{k}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx\right.\\
&\left.+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}\Bigg]\Bigg/
\end{aligned}
$$

$$\left[\left\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx\right.\right.$$
$$\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.$$
$$\left.+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}$$
$$-\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx\right.$$
$$\left.\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}^2\right] \tag{24}$$

Fixed-end moments in support A "M_{AB}" and in support B "M_{BA}" due to a concentrated load P located from $L_1 \leq x \leq L-L_2$ are presented as follows:

$$M_{AB}=P\left[\left\{\int_0^{L_1}\left[\frac{(L-k)(L-x)x}{EI_{x1}}-\frac{(L-k)}{GA_{sx1}}\right]dx\right.\right.$$
$$+\int_{L_1}^{k}\left[\frac{(L-k)(L-x)x}{EI_{x2}}-\frac{(L-k)}{GA_{sx2}}\right]dx+\int_{k}^{L-L_2}\left[\frac{k(L-x)^2}{EI_{x2}}+\frac{k}{GA_{sx2}}\right]dx$$
$$\left.+\int_{L-L_2}^{L}\left[\frac{k(L-x)^2}{EI_{x3}}+\frac{k}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.$$
$$\left.+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}$$
$$-\left\{\int_0^{L_1}\left[\frac{(L-k)x^2}{EI_{x1}}+\frac{(L-k)}{GA_{sx1}}\right]dx+\int_{L_1}^{k}\left[\frac{(L-k)x^2}{EI_{x2}}+\frac{(L-k)}{GA_{sx2}}\right]dx\right.$$
$$+\int_{k}^{L-L_2}\left[\frac{k(L-x)x}{EI_{x2}}-\frac{k}{GA_{sx2}}\right]dx$$
$$\left.+\int_{L-L_2}^{L}\left[\frac{k(L-x)x}{EI_{x3}}-\frac{k}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx\right.$$
$$\left.\left.+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}\right]\Bigg/$$
$$\left[\left\{\int_0^{L_1}\left[\frac{(L-x)^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx\right.\right.$$
$$\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}\left\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx\right.$$
$$\left.+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\right\}$$
$$-\left\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx\right.$$
$$\left.\left.+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\right\}^2\right] \tag{25}$$

$$
\begin{aligned}
M_{BA} = P \Bigg[& \Bigg\{ \int_0^{L_1} \left[\frac{(L-k)x^2}{EI_{x1}} + \frac{(L-k)}{GA_{sx1}} \right] dx + \int_{L_1}^{k} \left[\frac{(L-k)x^2}{EI_{x2}} + \frac{(L-k)}{GA_{sx2}} \right] dx \\
& + \int_{k}^{L-L_2} \left[\frac{k(L-x)x}{EI_{x2}} - \frac{k}{GA_{sx2}} \right] dx \\
& + \int_{L-L_2}^{L} \left[\frac{k(L-x)x}{EI_{x3}} - \frac{k}{GA_{sx3}} \right] dx \Bigg\} \Bigg\{ \int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx \\
& + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}} \right] dx \Bigg\} \\
& - \Bigg\{ \int_0^{L_1} \left[\frac{(L-k)(L-x)x}{EI_{x1}} - \frac{(L-k)}{GA_{sx1}} \right] dx \\
& + \int_{L_1}^{k} \left[\frac{(L-k)(L-x)x}{EI_{x2}} - \frac{(L-k)}{GA_{sx2}} \right] dx + \int_{k}^{L-L_2} \left[\frac{k(L-x)^2}{EI_{x2}} + \frac{k}{GA_{sx2}} \right] dx \\
& + \int_{L-L_2}^{L} \left[\frac{k(L-x)^2}{EI_{x3}} + \frac{k}{GA_{sx3}} \right] dx \Bigg\} \Bigg\{ \int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}} \right] dx \\
& + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}} \right] dx \Bigg\} \Bigg] \Bigg/ \\
& \Bigg[\Bigg\{ \int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx \\
& + \int_{L-L_2}^{L} \left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}} \right] dx \Bigg\} \Bigg\{ \int_0^{L_1} \left[\frac{x^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx \\
& + \int_{L_1}^{L-L_2} \left[\frac{x^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{x^2}{EI_{x3}} + \frac{1}{GA_{sx3}} \right] dx \Bigg\} \\
& - \Bigg\{ \int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}} \right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}} \right] dx \\
& + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}} \right] dx \Bigg\}^2 \Bigg]
\end{aligned}
\tag{26}
$$

Fixed-end moments in support A "M_{AB}" and in support B "M_{BA}" due to a concentrated load P located from $L - L_2 \leq x \leq L$ are presented as follows:

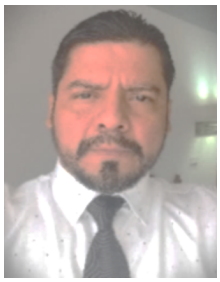
$$
\begin{aligned}
M_{AB} = P \Bigg[& \Bigg\{ \int_0^{L_1} \left[\frac{(L-k)(L-x)x}{EI_{x1}} - \frac{(L-k)}{GA_{sx1}} \right] dx \\
& + \int_{L_1}^{L-L_2} \left[\frac{(L-k)(L-x)x}{EI_{x2}} - \frac{(L-k)}{GA_{sx2}} \right] dx \\
& + \int_{L-L_2}^{k} \left[\frac{(L-k)(L-x)x}{EI_{x3}} - \frac{(L-k)}{GA_{sx3}} \right] dx \\
& + \int_{k}^{L} \left[\frac{k(L-x)^2}{EI_{x3}} + \frac{k}{GA_{sx3}} \right] dx \Bigg\} \Bigg\{ \int_0^{L_1} \left[\frac{x^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx \\
& + \int_{L_1}^{L-L_2} \left[\frac{x^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{x^2}{EI_{x3}} + \frac{1}{GA_{sx3}} \right] dx \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&- \left\{ \int_0^{L_1} \left[\frac{(L-k)x^2}{EI_{x1}} + \frac{(L-k)}{GA_{sx1}} \right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-k)x^2}{EI_{x2}} + \frac{(L-k)}{GA_{sx2}} \right] dx \right. \\
&+ \int_{L-L_2}^{k} \left[\frac{(L-k)x^2}{EI_{x3}} + \frac{(L-k)}{GA_{sx3}} \right] dx \\
&\left. + \int_k^{L} \left[\frac{k(L-x)x}{EI_{x3}} - \frac{k}{GA_{sx3}} \right] dx \right\} \left\{ \int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}} \right] dx \right. \\
&\left. + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}} \right] dx \right\} \Bigg] \Bigg/ \\
&\left[\left\{ \int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx \right. \right. \\
&+ \int_{L-L_2}^{L} \left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}} \right] dx \Bigg\} \left\{ \int_0^{L_1} \left[\frac{x^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx \right. \\
&+ \int_{L_1}^{L-L_2} \left[\frac{x^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{x^2}{EI_{x3}} + \frac{1}{GA_{sx3}} \right] dx \Bigg\} \\
&- \left\{ \int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}} \right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}} \right] dx \right. \\
&\left. \left. + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}} \right] dx \right\}^2 \right] \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
M_{BA} = P &\left[\left\{ \int_0^{L_1} \left[\frac{(L-k)x^2}{EI_{x1}} + \frac{(L-k)}{GA_{sx1}} \right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-k)x^2}{EI_{x2}} + \frac{(L-k)}{GA_{sx2}} \right] dx \right. \right. \\
&+ \int_{L-L_2}^{k} \left[\frac{(L-k)x^2}{EI_{x3}} + \frac{(L-k)}{GA_{sx3}} \right] dx \\
&\left. + \int_k^{L} \left[\frac{k(L-x)x}{EI_{x3}} - \frac{k}{GA_{sx3}} \right] dx \right\} \left\{ \int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx \right. \\
&\left. + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)^2}{EI_{x3}} + \frac{1}{GA_{sx3}} \right] dx \right\} \\
&- \left\{ \int_0^{L_1} \left[\frac{(L-k)(L-x)x}{EI_{x1}} - \frac{(L-k)}{GA_{sx1}} \right] dx \right. \\
&+ \int_{L_1}^{L-L_2} \left[\frac{(L-k)(L-x)x}{EI_{x2}} - \frac{(L-k)}{GA_{sx2}} \right] dx \\
&\left. + \int_{L-L_2}^{k} \left[\frac{(L-k)(L-x)x}{EI_{x3}} - \frac{(L-k)}{GA_{sx3}} \right] dx \right. \\
&\left. + \int_k^{L} \left[\frac{k(L-x)^2}{EI_{x3}} + \frac{k}{GA_{sx3}} \right] dx \right\} \left\{ \int_0^{L_1} \left[\frac{(L-x)x}{EI_{x1}} - \frac{1}{GA_{sx1}} \right] dx \right. \\
&\left. + \int_{L_1}^{L-L_2} \left[\frac{(L-x)x}{EI_{x2}} - \frac{1}{GA_{sx2}} \right] dx + \int_{L-L_2}^{L} \left[\frac{(L-x)x}{EI_{x3}} - \frac{1}{GA_{sx3}} \right] dx \right\} \Bigg] \Bigg/ \\
&\left[\left\{ \int_0^{L_1} \left[\frac{(L-x)^2}{EI_{x1}} + \frac{1}{GA_{sx1}} \right] dx + \int_{L_1}^{L-L_2} \left[\frac{(L-x)^2}{EI_{x2}} + \frac{1}{GA_{sx2}} \right] dx \right. \right.
\end{aligned}
$$

$$+\int_{L-L_2}^{L}\left[\frac{(L-x)^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\Bigg\}\Bigg\{\int_0^{L_1}\left[\frac{x^2}{EI_{x1}}+\frac{1}{GA_{sx1}}\right]dx$$
$$+\int_{L_1}^{L-L_2}\left[\frac{x^2}{EI_{x2}}+\frac{1}{GA_{sx2}}\right]dx+\int_{L-L_2}^{L}\left[\frac{x^2}{EI_{x3}}+\frac{1}{GA_{sx3}}\right]dx\Bigg\}$$
$$-\Bigg\{\int_0^{L_1}\left[\frac{(L-x)x}{EI_{x1}}-\frac{1}{GA_{sx1}}\right]dx+\int_{L_1}^{L-L_2}\left[\frac{(L-x)x}{EI_{x2}}-\frac{1}{GA_{sx2}}\right]dx$$
$$+\int_{L-L_2}^{L}\left[\frac{(L-x)x}{EI_{x3}}-\frac{1}{GA_{sx3}}\right]dx\Bigg\}^2\Bigg] \tag{28}$$

Author Biography



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