International Journal of Innovative Computing, Information and Control
Volume **20**, Number **2**, April **2024**

ICIC International ©2024 ISSN 1349-4198
pp. **461–478**

A NOVEL METHOD OF DIAGONAL-INNER OUTER FACTORIZATION

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Received June 2023; revised November 2023

Abstract. *In this study, we introduce a novel approach to diagonal-inner outer factorization for bi-proper and strictly proper systems. Unlike traditional methods, our strategy is rooted in the fundamental concepts of transfer function and enhanced with a design method utilizing state-space representation. The pivotal contribution of this work is the development of a methodology that yields a diagonal inner function, offering substantial advancements in control and system analysis. In the illustrative numerical examples provided, the application of the proposed method is delineated, showcasing its practicality. These examples serve to validate the theoretical foundations and emphasize the effectiveness of the introduced methodology. This advancement in diagonal-inner outer factorization stands as a significant contribution to the field, offering enhanced tools for system and control engineers and opening avenues for further research and innovation.*
Keywords: Inner-outer factorizations, Inner function, Outer function, Riccati equations

1. **Introduction.** In numerous control problems, the inner-outer factorization serves as a mathematical procedure that partitions a matrix or operator into two distinctive elements: an inner function and an outer function. This strategic technique is employed to simplify the architecture of a matrix or operator, further promoting efficient numerical computations. Inner-outer factorization finds widespread utilization across several fields, including but not limited to numerical linear algebra, optimization, and signal processing. This method also plays a significant role in various other mathematical and engineering

DOI: 10.24507/ijicic.20.02.461

domains. Notable instances of inner-outer factorizations comprise the Cholesky factorization and QR factorization. QR factorization given matrix is factored into the product of two matrices: a unitary or orthogonal matrix (Q) and an upper triangular matrix (R) [1]. Given its utility, the inner-outer factorization method for stable transfer function matrices has proven to be an efficacious instrument for the analysis and synthesis of robust controllers, as well as the processing and communication of data [2, 3, 4, 5]. This technique is renowned for computer computations based on the state-space representation [4, 5, 6, 8, 9, 10, 11, 12].

The inner-outer factorization serves as a computational paradigm in the context of H_2 and H_∞ control systems, offering the potential to fragment a stable and proper system $G(s)$ into inner and outer functions [2, 3, 4, 5, 8]. In a related development, Dewild and Veen devised an approach for the inversion of infinite systems of linear equations which are represented by a discrete time-varying dynamical system. Their method extrapolates on the classical matrix inversion theory by deploying inner-outer factorizations, which operate analogously to QR-factorization in linear algebra. The algorithms they developed to this effect, coined ‘square root’ algorithms, not only avoid the requirement for multiple eigenvalue determination but are also efficient and linear in the volume of data [13]. In a separate study, Gu scrutinized inner-outer factorization for strictly proper transfer matrices, providing characterizations to the solution of this specific factorization problem and devising a computational algorithm for its resolution [14]. Boche and Pohl pioneered a series of algorithms to perform discrete inner-outer, coprime, and spectral factorizations directly on the state-space realizations of discrete systems. Their approach circumvents the standard bilinear transformation, emphasizing the discrete algebraic Riccati equation and state-space realizations for factorizations [15]. Kase and Mutoh proposed an innovative and computationally efficient methodology for factorization that takes account of the specific attributes of the inner and outer components. Employing a recursive zeros dislocation technique, their approach manages generalized Lyapunov equations and is applicable regardless of whether G is proper or of full column/row rank, expanding the applicability of existing techniques to arbitrary rational matrices while avoiding the use of Riccati equations [16]. In a unique take on the matter, Helmer introduced an inner-outer factorization within non-commutative Hardy algebras $H^\infty(E)$. His approach encapsulates various algebraic structures, centralizing around the Hardy algebra. It harnesses a general version of the Wold decomposition and factors a vector within the underlying Hilbert space, followed by an element of the algebra’s commutant, using duality concepts for W^*-correspondences to foster new factorization approaches within this algebraic framework [17]. Reis and Voigt extended a method to non-square transfer function matrices, contemplating the derivation of the right interactor for the inner-outer factorization. The proposed interactor showcases all-pass properties in discrete time, with all its zeros located at the origin. However, the authors addressed the zeros assignment of the interaction, considering the instability of the origin in continuous-time systems [18]. Frazho and Ran, in their note, utilized operator methods to address a rational inner-outer factorization problem for wide functions. They engaged Wiener-Hopf operators, Hankel operators, and invariant subspaces for the backward shift, hoping to yield significant insight into the inner-outer factorization problem [19]. Recent studies can be used to improve the method, Huang et al. examined optimal controllers for model predictive control, focusing on theoretical analysis and practical application. Theoretically, it employs variation analysis and linear matrix inequality for controller design and stability analysis. Practically, it validates the theories using two simulation examples, applying an online subgradient descent algorithm for constrained optimization problems [20].

Given the considerable significance and wide-ranging applications of the inner-outer factorization problem, numerous techniques have been explored extensively [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 15, 16, 18]. In these methods, the inner function is typically not a diagonal function. Despite the ability to control many systems even when the inner function is non-diagonal, the control problem may become more challenging under such conditions. Conversely, systems become more tractable in terms of control when the inner function is a diagonal function. Moreover, the diagonal function can streamline the analysis and design of control algorithms. However, the examination of a diagonal-inner outer factorization method remains absent in the literature.

In this paper, we propose a novel method of diagonal-inner outer factorization. Our proposed methodology elucidates an inner function through the use of a straightforward diagonal function. The research exhibits interest in applying bi-proper transfer function and strictly proper transfer function for factorization. The structure of this paper has been thoughtfully divided into two distinctive parts for comprehensive understanding.

In the initial segment, we offer a detailed explanation regarding the problem formulation for bi-proper systems in Section 2. Our proposed methodology introduces a unique way of executing diagonal-inner outer factorization for bi-proper systems, which integrates the idea of the inverse system and inner transformation in Section 3. The effectiveness of this proposed method is vividly demonstrated through a variety of numerical examples illustrated in Section 7.1. In the second segment, our focus is concentrated on the problem formulation for strictly proper systems in Section 4. Consequently, the innovative method of diagonal-inner outer factorization for strictly proper systems is introduced Section 5. After that, we explain state space design methods for the diagonal-inner outer factorization in Section 6. In the last example, we present several numerical examples that effectively illustrate the method, and these examples can be found in Section 7.2. Section 8 gives concluding remarks.

2. **Problem Formulation for Bi-Proper Systems.** Consider a linear time-invariant system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^m$ is the output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times m}$. It is assumed that all eigenvalues of A are in the open left half plane, (A, B) is stabilizable, (C, A) is detectable and

$$\text{rank } D = m. \tag{2}$$

In addition, it is assumed that the system in (1) has no zero on the imaginary axis, that is,

$$\text{rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = m \quad \forall s = j\omega \ (-\infty < \omega < \infty). \tag{3}$$

It is assumed that unstable zeros of the system in (1) are row unstable zeros. That is, all unstable zeros z of $G(s)$ are located as

$$\text{rank} \begin{bmatrix} A - zI & B \\ C_i & D_i \end{bmatrix} < m \quad (i = 1, \ldots, m), \tag{4}$$

where

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix} \tag{5}$$

and

$$D = \begin{bmatrix} D_1 \\ \vdots \\ D_m \end{bmatrix}. \tag{6}$$

The transfer function from $u(s)$ to $y(s)$ in (1) is denoted by

$$G(s) = C(sI - A)^{-1}B + D = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathbb{RH}_\infty^{m\times m}(s). \tag{7}$$

The problem considered in this paper is to factorize $G(s)$ in (7) as

$$G(s) = G_i(s)G_o(s), \tag{8}$$

where $G_i(s) \in \mathbb{R}^{m\times m}(s)$ is a diagonal-inner function and $G_o(s) \in \mathbb{R}^{m\times m}(s)$ is an outer function.

Having elaborated on the intricate aspects of problem formulation for bi-proper systems, we are now equipped with a comprehensive understanding of the challenges and opportunities inherent in these systems. The complexities and specificities outlined in this section serve as the foundational premise for our proposed solutions.

Moving into Section 3, we will introduce the idea of diagonal-inner outer factorization, a pioneering methodology tailored to address the nuanced challenges identified in the bi-proper systems. The diagonal-inner outer factorization emerges as a pragmatic approach, bridging the theoretical constructs and practical implementations, effectively managing the problematics elucidated herein.

3. Diagonal-Inner Outer Factorization for Bi-Proper Systems.

In this section, an idea for a method of diagonal-inner outer factorization for $G(s)$ is explained.

This method adapts the combination of inverse system and inner transformation. From (2), there exists

$$\bar{G}(s) = G^{-1}(s) = \left[\begin{array}{c|c} A - BD^{-1}C & -BD^{-1} \\ \hline D^{-1}C & D^{-1} \end{array}\right] \equiv \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]. \tag{9}$$

$\bar{G}(s)$ can be factorized as

$$\bar{G}(s) = \begin{bmatrix} \bar{G}_1(s) & \dots & \bar{G}_m(s) \end{bmatrix}, \tag{10}$$

where $\bar{G}_i(s) \in \mathbb{RH}_\infty^m$ $(i = 1, \dots, m)$. Minimum realization of $\bar{G}_i(s)$ $(i = 1, \dots, m)$ in (10) is denoted by

$$\bar{G}_i(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] \quad (i = 1, \dots, m), \tag{11}$$

where $\bar{A}_i \in \mathbb{R}^{n_i\times n_i}$ $(i = 1, \dots, m)$, $\bar{B}_i \in \mathbb{R}^{n_i\times 1}$ $(i = 1, \dots, m)$, $\bar{C}_i \in \mathbb{R}^{m_i\times 1}$ $(i = 1, \dots, m)$ and $\bar{D}_i \in \mathbb{R}^{m_i\times 1}$ $(i = 1, \dots, m)$. Using $K_i \in \mathbb{R}^{1\times n_i}$ $(i = 1, \dots, m)$ to make $\bar{A}_i - \bar{B}_iK_i$ have no eigenvalue in the closed right half plane, we have

$$\left[\begin{array}{c|c} \bar{A}_i - \bar{B}_iK_i & \bar{B}_i \\ \hline \bar{C}_i - \bar{D}_iK_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array}\right]^{-1} = \bar{G}_i(s)G_{Ki}(s) \quad (i = 1, \dots, m), \tag{12}$$

where

$$G_{Ki}(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array}\right]^{-1} = \left[\begin{array}{c|c} \bar{A}_i - \bar{B}_i K_i & \bar{B}_i \\ \hline -K_i & 1 \end{array}\right] \quad (i = 1, \dots, m). \tag{13}$$

This yields

$$\hat{G}(s) = \bar{G}(s) G_K(s), \tag{14}$$

where

$$\hat{G}(s) = \left[\begin{array}{ccc} \bar{C}_1 \left(sI - \bar{A}_1 + \bar{B}_1 K_1\right)^{-1} \bar{B}_1 + \bar{D}_1 & \dots & \bar{C}_m \left(sI - \bar{A}_m + \bar{B}_m K_m\right)^{-1} \bar{B}_m + \bar{D}_m \end{array} \right], \tag{15}$$

$$\begin{aligned} G_K(s) &= \text{diag} \left[\begin{array}{ccc} \left\{1 + K_1 \left(sI - \bar{A}_1\right)^{-1} \bar{B}_1\right\}^{-1} & \dots & \left\{1 + K_m \left(sI - \bar{A}_m\right)^{-1} \bar{B}_m\right\}^{-1} \end{array} \right] \\ &= \left[\begin{array}{ccc|ccc} \bar{A}_1 - \bar{B}_1 K_1 & & 0 & \bar{B}_1 & & \\ & \ddots & & & \ddots & \\ 0 & & \bar{A}_m - \bar{B}_m K_m & 0 & & \bar{B}_m \\ \hline -K_1 & & 0 & 1 & & \\ & \ddots & & & \ddots & \\ 0 & & -K_m & 0 & & 1 \end{array}\right] \\ &= \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] \end{aligned} \tag{16}$$

and

$$\bar{G}(s) = \left[\begin{array}{ccc} \bar{G}_1(s) & \dots & \bar{G}_m(s) \end{array} \right]. \tag{17}$$

From (9) and (14), we have

$$G(s) = G_K(s) \hat{G}^{-1}(s). \tag{18}$$

From (15), $\hat{G}(s)$ is stable and of minimum phase. If K_i $(i = 1, \dots, m)$ are settled to make $G_K(s)$ inner function, $G(s)$ is factorized as

$$G(s) = G_i(s) G_o(s), \tag{19}$$

where $G_i(s)$ is an inner function and given by

$$G_i(s) = G_K(s) \tag{20}$$

and $G_o(s)$ is an outer function and given by

$$G_o(s) = G_i^{-1}(s) G(s) = \hat{G}^{-1}(s). \tag{21}$$

It is obvious that $G_K(s)$ in (16) is a diagonal-inner function. In addition, from the assumption that all unstable zeros of $G(s)$ satisfy (4) and (21), $G_o(s)$ is obviously stable and of minimum-phase.

The rest is to obtain K_i $(i = 1, \dots, m)$ to make $G_K(s)$ an inner function. On a design method for K_i $(i = 1, \dots, m)$, we have following theorem.

Theorem 3.1. *If K_i $(i = 1, \dots, m)$ are settled by*

$$K_i = \bar{B}_i^T X_i \quad (i = 1, \dots, m), \tag{22}$$

where $X_i \geq 0$ $(j = 1, \dots, m)$ is the unique solution of the Riccati equation

$$X_i\bar{A}_i + \bar{A}_i^T X_i - X_i\bar{B}_i\bar{B}_i^T X_i = 0 \quad (i = 1, \ldots, m), \tag{23}$$

then

$$G_K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] \in \mathbb{RH}_\infty^{m\times m} \tag{24}$$

is an inner function.

In order to prove this theorem, the following lemma is required.

Lemma 3.1. [21] *Let*

$$U_j(s) = \left[\begin{array}{c|c} A_u & B_u \\ \hline C_u & D_u \end{array}\right] \in \mathbb{RH}_\infty^{p\times q}, \tag{25}$$

where $p \geq q$*, assume that* $\operatorname{rank} D_u = q$*,* (C_u, A_u) *is detectable. If there exist* $X = 0$ *satisfying*

$$XA_u + A_u^T X + C_u^T C_u = 0, \tag{26}$$

$$D_u C_u + B_u^T X = 0, \tag{27}$$

$$D_u^T D_u = I, \tag{28}$$

then A_u *is stable and* $U_j(s)$ *is an inner function.*

Using Lemma 3.1, we show the proof of Theorem 3.1.

Proof: We show that $G_{Ki}(s)$ $(i = 1, \ldots, m)$ in (47) satisfies conditions in Lemma 3.1. From (13), (C_{Ki}, A_i) is detectable and $\operatorname{rank} D_i = 1$. From Lemma 3.1, if $G_{Ki}(s)$ $(i = 1, \ldots, m)$ satisfies (26), (27) and (28), then $G_{Ki}(s)$ $(i = 1, \ldots, m)$ is an inner function. If $G_{Ki}(s)$ $(i = 1, \ldots, m)$ is an inner function, $G_K(s)$ in (16) is a diagonal inner function.

The rest is to prove that $G_{Ki}(s)$ $(i = 1, \ldots, m)$ in (13) satisfies (26), (27) and (28). By substitution of $G_{Ki}(s)$ in (13) to (26), we have (23). From (22), $G_i(s)$ holds (27). In addition $D_{Ki} = 1$ $(i = 1, \ldots, m)$, (28) is satisfied. Hence, $G_{Ki}(s)$ satisfies all conditions in (26), (27) and (28).

We have thus proved Theorem 3.1. □

This method can apply to the bi-proper system $G(s)$. However, this method cannot apply to strictly proper systems. Next, we present a diagonal-inner outer factorization for strictly proper systems.

4. **Problem Formulation for Strictly Proper Systems.** This section provides an innovative problem formulation tailored for strictly proper systems. We identify and address specific challenges, incorporating a refined set of variables and methodologies that yield clarity with these systems.

Consider a linear time-invariant systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}, \tag{29}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^m$ is the output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$. It is assumed that all eigenvalues of A are in the open left half plane, (A, B) is stabilizable, (C, A) is detectable, the system in (29) satisfies

$$\operatorname{rank} G(s) = m. \tag{30}$$

In addition, it is assumed that the system in (1) has no zero on the imaginary axis, that is,

$$\operatorname{rank}\begin{bmatrix} A-sI & B \\ C & 0 \end{bmatrix} = m \quad \forall s = j\omega \ (-\infty < \omega < \infty). \tag{31}$$

It is assumed that unstable zeros of the system in (29) are row unstable zeros. That is, all unstable zeros z of $G(s)$ are located as

$$\operatorname{rank}\begin{bmatrix} A-zI & B \\ C_i & 0 \end{bmatrix} < m \quad (i = 1, \ldots, m), \tag{32}$$

where

$$C = \begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}. \tag{33}$$

The transfer function from $u(s)$ to $y(s)$ in (29) is denoted by

$$G(s) = C(sI-A)^{-1}B \in \mathbb{R}^{m\times m}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0_{m\times m} \end{array}\right] \tag{34}$$

The system in (29) is assumed to satisfy

$$\operatorname{rank} \Phi = m, \tag{35}$$

where

$$\Phi = \begin{bmatrix} B_1^T \left(A^T\right)^{\alpha_1 - 1} C^T \\ \vdots \\ B_m^T \left(A^T\right)^{\alpha_m - 1} C^T \end{bmatrix}, \tag{36}$$

$$B = \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix} \quad (B_i \in \mathbb{R}^n,\ i = 1, \ldots, m) \tag{37}$$

and

$$\alpha_i = \min\left(j \left| B_i^T \left(A^T\right)^{j-1} C^T \neq 0;\ j = 1, \ldots, n\right.\right) \quad (i = 1, \ldots, m). \tag{38}$$

An additional problem in this paper is to clarify diagonal-inner outer factorization of the form in (8), where $G_i(s) \in \mathbb{R}^{m\times m}(s)$ is a diagonal-inner function and $G_o(s) \in \mathbb{R}^{m\times m}(s)$ is an outer function.

5. **Diagonal-Inner Outer Factorization for Strictly Proper Systems.**

In this section, we unveil a specialized diagonal inner-outer factorization, optimized for strictly proper systems. The innovation lies in its significance in methods and offering tangible improvements in the system $G(s)$ in (29).

According to [22], if (35) is satisfied, there exists $\bar{G}(s)$ satisfying

$$\bar{G}(s)G(s) = Q(s) \tag{39}$$

and

$$\operatorname{rank} \bar{D} = m, \tag{40}$$

where

$$\bar{G}(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right], \tag{41}$$

$\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times m}$, $\bar{C} \in \mathbb{R}^{m\times n}$ and $\bar{D} \in \mathbb{R}^{m\times m}$ and

$$Q(s) = \operatorname{diag}\left\{ \frac{1}{(1+sT_1)^{\alpha_1}} \ \cdots \ \frac{1}{(1+sT_m)^{\alpha_m}} \right\} \tag{42}$$

and

$$T_i > 0 \in \mathbb{R} \quad (i = 0, \dots, m). \tag{43}$$

$\bar{G}(s)$ satisfying (39) is factorized by

$$\bar{G}(s) = \left[\begin{array}{ccc} \bar{G}_1(s) & \dots & \bar{G}_m(s) \end{array} \right], \tag{44}$$

where $\bar{G}_i(s) \in \mathbb{RH}_\infty^m$ $(i = 1, \dots, m)$. Minimum realization of $\bar{G}_i(s)$ $(i = 1, \dots, m)$ in (44) is denoted by

$$\bar{G}_i(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array} \right] \quad (i = 1, \dots, m), \tag{45}$$

where $\bar{A}_i \in \mathbb{R}^{n_i \times n_i}$ $(i = 1, \dots, m)$, $\bar{B}_i \in \mathbb{R}^{n_i \times 1}$ $(i = 1, \dots, m)$, $\bar{C}_i \in \mathbb{R}^{m_i \times 1}$ $(i = 1, \dots, m)$ and $\bar{D}_i \in \mathbb{R}^{m_i \times 1}$ $(i = 1, \dots, m)$. Using $K_i \in \mathbb{R}^{1 \times n_i}$ $(i = 1, \dots, m)$ to make $\bar{A}_i - \bar{B}_i K_i$ $(i = 1, \dots, m)$ have no eigenvalue in the closed right half plane, we have

$$\left[\begin{array}{c|c} \bar{A}_i - \bar{B}_i K_i & \bar{B}_i \\ \hline \bar{C}_i - \bar{D}_i K_i & \bar{D}_i \end{array} \right] = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array} \right] \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array} \right]^{-1} = \bar{G}_i(s) G_{Ki}(s), \tag{46}$$

where

$$G_{Ki}(s) = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline K_i & 1 \end{array} \right]^{-1} = \left[\begin{array}{c|c} \bar{A}_i - \bar{B}_i K_i & \bar{B}_i \\ \hline -K_i & 1 \end{array} \right]. \tag{47}$$

This yields

$$\hat{G}(s) = \bar{G}(s) G_K(s), \tag{48}$$

where

$$\begin{aligned} &\hat{G}(s) \\ &= \left[\begin{array}{ccc} \bar{C}_1 \left(sI - \bar{A}_1 + \bar{B}_1 K_i \right)^{-1} \bar{B}_1 + \bar{D}_1 & \dots & \bar{C}_m \left(sI - \bar{A}_m + \bar{B}_m K_m \right)^{-1} \bar{B}_m + \bar{D}_m \end{array} \right], \end{aligned} \tag{49}$$

$$\begin{aligned} G_K(s) &= \operatorname{diag} \left[\begin{array}{ccc} \left\{ 1 + K_1 \left(sI - \bar{A}_1 \right)^{-1} \bar{B}_1 \right\}^{-1} & \dots & \left\{ 1 + K_m \left(sI - \bar{A}_m \right)^{-1} \bar{B}_m \right\}^{-1} \end{array} \right] \\ &= \left[\begin{array}{ccc|ccc} \bar{A}_1 - \bar{B}_1 K_1 & & 0 & \bar{B}_1 & & \\ & \ddots & & & \ddots & \\ 0 & & \bar{A}_m - \bar{B}_m K_m & 0 & & \bar{B}_m \\ \hline -K_1 & & 0 & 1 & & \\ & \ddots & & & \ddots & \\ 0 & & -K_m & 0 & & 1 \end{array} \right] \\ &= \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right] \end{aligned} \tag{50}$$

and

$$\bar{G}(s) = \left[\begin{array}{ccc} \bar{G}_1(s) & \dots & \bar{G}_m(s) \end{array} \right]. \tag{51}$$

From (49), we have

$$G(s) = G_K(s) \hat{G}^{-1}(s) Q(s). \tag{52}$$

From the discussion in Section 3 and Theorem 3.1, if K_i $(i = 1, \ldots, m)$ is settled by (22), $G_K(s)$ in (50) is a diagonal-inner function. From (15), $\hat{G}(s)$ is stable and of minimum phase. Therefore, $\hat{G}^{-1}(s)Q(s)$ is stable and of minimum phase. Since $G(s)$ is strictly proper and $G_K(s)$ is bi-proper, $\hat{G}^{-1}(s)Q(s)$ is strictly proper. Thus, $\hat{G}^{-1}(s)Q(s)$ is an outer function. In this way, $G(s)$ is factorized as

$$G(s) = G_i(s)G_o(s), \tag{53}$$

where $G_i(s)$ is an inner function and given by

$$G_i(s) = G_K(s) \tag{54}$$

and $G_o(s)$ is an outer function and given by

$$G_o(s) = G_i^{-1}G(s) = \hat{G}^{-1}(s)Q(s). \tag{55}$$

In addition, from the assumption that all unstable zeros of $G(s)$ satisfy (32), and (55), $G_o(s)$ is obviously stable and of minimum-phase.

6. **State Space Design Method of Diagonal-Inner Outer Factorization for Strictly Proper Systems.** In this section, we explain state space design method for the diagonal-inner outer factorization described in the previous section. Illustrated through concrete equations to use for an example in the next section, this method showcases its calculation efficiency in deploying for system.

According to [22] and easy calculations, the state space representation of $\bar{G}(s)$ satisfying (39) and (40) is written by

$$\bar{G}(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right] = \left[\begin{array}{c|c} A - \Psi\hat{\Phi}^T C & \Psi\hat{\Phi}^T \\ \hline P\hat{\Phi}^T C & P\hat{\Phi}^T \end{array}\right], \tag{56}$$

where

$$\beta_{ij} = \begin{Bmatrix} \alpha_i \\ j \end{Bmatrix} (T_i)^{-j} \quad (i = 1, \ldots, m; j = 1, \ldots, \alpha_i), \tag{57}$$

$$\Phi\hat{\Phi} = I_m, \tag{58}$$

$$P = \mathrm{diag}\left\{ \begin{array}{ccc} \beta_{1\alpha_1} & \ldots & \beta_{m\alpha_m} \end{array} \right\} \tag{59}$$

and

$$\Psi_i = \left[\begin{array}{ccc} A^{\alpha_1}B_1 + \cdots + \beta_{1\alpha_1}B_1 & \ldots & A^{\alpha_m}B_m + \cdots + \beta_{m\alpha_m}B_m \end{array} \right] \quad (j = 1, \ldots, m). \tag{60}$$

From [22], we find $\bar{G}(s)$ in (56). In addition, from the assumption in (35) and (58), $\bar{D}$ in (56) satisfies

$$\mathrm{rank}\ \bar{D} = \mathrm{rank}\ P\hat{\Phi}^T \tag{61}$$

$$= m. \tag{62}$$

Thus, it is confirmed that $\bar{G}(s)$ in (56) satisfies (39) and (40).

K_i $(i = 1, \ldots, m)$ in (47) is obtained using Theorem 3.1.

7. **Numerical Example.** In this section, we show numerical examples to illustrate the effectiveness of the proposed method.

7.1. **Numerical example of a bi-proper system.** Let us consider the problem of the diagonal-inner outer factorization for the system in

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 3 & -8 & -5 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u(t) \end{cases} \tag{63}$$

using the method in Section 3. The transfer function $G(s)$ of the system in (63) is given by

$$G(s) = \begin{bmatrix} \dfrac{s^2-1}{s^2+5s+6} & \dfrac{s-1}{s^2+9s+20} \\ \dfrac{1}{s+5} & \dfrac{s+4}{s+5} \end{bmatrix}. \tag{64}$$

$\bar{G}(s)$ satisfying (9) is written by

$$\bar{G}(s) = \left[\begin{array}{ccccc|cc} -5 & 8 & 5 & 0 & -6 & -1 & 0 \\ -3 & 5 & 5 & 0 & -6 & -1 & 0 \\ 0 & 0 & -4 & -1 & 1 & 0 & -1 \\ -3 & 8 & 5 & -5 & -6 & -1 & 0 \\ 0 & 0 & 0 & -1 & -4 & 0 & -1 \\ \hline 3 & -8 & -5 & 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]. \tag{65}$$

$\bar{G}(s)$ is factorized as (10). $\bar{G}_1(s)$ and $\bar{G}_2(s)$ are given by

$$\bar{G}_1(s) = \left[\begin{array}{ccccc|c} -5 & 8 & 5 & 0 & -6 & -1 \\ -3 & 5 & 5 & 0 & -6 & -1 \\ 0 & 0 & -4 & -1 & 1 & 0 \\ -3 & 8 & 5 & -5 & -6 & -1 \\ 0 & 0 & 0 & -1 & -4 & 0 \\ \hline 3 & -8 & -5 & 0 & 6 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

and

$$\bar{G}_2(s) = \left[\begin{array}{cccc|c} -0.94260 & 0 & 0 & 0 & -0.50650 \\ 0 & -5.72300 & 0 & 0 & 0.81630 \\ 0 & 0 & -3.66700 & 0.51850 & -0.87820 \\ 0 & 0 & -0.51850 & -3.66700 & -0.09770 \\ \hline 0.47390 & -0.41790 & 0.40220 & 0.67190 & 0 \\ -0.15500 & -0.24250 & -1.27000 & -0.03965 & 1 \end{array}\right].$$

From (22), K_i $(i = 1, 2)$ are calculated by

$$K_1 = \begin{bmatrix} 2 & -4 & -2 & 0 & 2 \end{bmatrix}, \tag{66}$$

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}. \tag{67}$$

Using above mentioned parameters and (13), $G_K(s)$ is given by

$$G_K(s) = \left[\begin{array}{cc|cc} 1 & 0 & 1.51200 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1.32300 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{68}$$

From (15) $\hat{G}_j$ $(j = 1, 2)$ is given by

$$\hat{G}_1(s) = \left[\begin{array}{ccccc|c} -5.72300 & 0 & 0 & 0 & 0 & -0.29270 \\ 0 & -3.6670 & 0.51850 & 0 & 0 & -0.01287 \\ 0 & -0.51850 & -3.66700 & 0 & 0 & -0.06238 \\ 0 & 0 & 0 & -0.94260 & 0 & 24.97000 \\ 0 & 0 & 0 & 0 & -1 & 24.40000 \\ \hline 0.73250 & -0.34930 & -1.47700 & 1.56700 & -1.47500 & 0 \\ 0.42510 & 2.32400 & 0.81800 & -0.51250 & 0.49170 & 1 \end{array}\right],$$

$$\hat{G}_2(s) = \left[\begin{array}{cccc|c} -0.94260 & 0 & 0 & 0 & -0.50650 \\ 0 & -5.72300 & 0 & 0 & 0.81630 \\ 0 & 0 & -3.66700 & 0.51850 & -0.87820 \\ 0 & 0 & -0.51850 & -3.66700 & -0.09770 \\ \hline 0.47390 & -0.41790 & 0.40220 & 0.67190 & 0 \\ -0.15500 & -0.24250 & -1.27000 & -0.03965 & 1 \end{array}\right].$$

Thus, the diagonal-inner outer factorization for $G(s)$ in (63) is obtained as

$$G_i(s) = \begin{bmatrix} \dfrac{s-1}{s+1} & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$G_o(s) = \left[\begin{array}{ccccc|cc} -2 & 0 & 0 & 0 & 0 & -0.65690 & 0 \\ 0 & -3 & 0 & 0 & 0 & 1.37000 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0.74450 \\ 0 & 0 & 0 & -5 & 0 & 2.12100 & -3.23800 \\ 0 & 0 & 0 & 0 & -5 & -1.70400 & 4.13110 \\ \hline -1.52200 & -2.91900 & -4.02900 & 2.10200 & 2.61600 & 1 & 0 \\ 0 & 0 & 0 & 0.74810 & 0.34430 & 0 & 1 \end{array}\right]$$

$$= \begin{bmatrix} \dfrac{s^4+12s^3+46s^2+60s+25}{s^4+15s^3+81s^2+185s+150} & \dfrac{s^2+6s+5}{s^3+14s^2+65s+100} \\ \dfrac{s+5}{s^2+10s+25} & \dfrac{s^2+9s+20}{s^2+10s+25} \end{bmatrix}. \quad (69)$$

From (69), $G_i(s)$ is obviously a diagonal-inner function. Since $G_o(s)$ in (69) is stable and zeros of $G_o(s)$ in (69) are located in -5.7230, -0.9426, -1, $-3.6672 + 0.5185j$ and $-3.6672 - 0.5185i$, $G_o(s)$ in (69) is an outer function.

In this way, it is easy to obtain the diagonal-inner outer factorization using the method in Section 3.

7.2. **Numerical example of a strictly proper system.** Consider the problem of obtaining the diagonal-inner outer factorization for the system in

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -4 & 5 & 0 & 0 & -7 & 0 & 8 & 0 \\ 0 & 0 & -1 & 2 & 0 & 1 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} u(t) \end{cases}. \quad (70)$$

The transfer function $G(s)$ of the system in (70) is given by

$$G(s) = \begin{bmatrix} \dfrac{s-3}{s^2+3s+2} & \dfrac{s-3}{s^2+9s+20} \\ \dfrac{1}{s^2+11s+30} & \dfrac{s+1}{s^2+5s+6} \end{bmatrix}. \tag{71}$$

From (38), since

$$B_1^T C^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{72}$$

and

$$B_2^T C^T = \begin{bmatrix} 1 & 1 \end{bmatrix}, \tag{73}$$

we have

$$\alpha_1 = 1 \tag{74}$$

and

$$\alpha_2 = 1. \tag{75}$$

From (36) and (58), Φ and $\hat{\Phi}$ are given by

$$\Phi = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

and

$$\hat{\Phi} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

T_i $(i = 1, 2)$ are assigned as

$$T_1 = 0.01 \tag{76}$$

and

$$T_2 = 0.02. \tag{77}$$

Equations (57), (59) and (60) yield

$$\beta_{11} = 100, \tag{78}$$

$$\beta_{21} = 50, \tag{79}$$

$$P = \begin{bmatrix} 100 & 0 \\ 0 & 50 \end{bmatrix}, \tag{80}$$

and

$$\Psi = \begin{bmatrix} 99 & 0 \\ 98 & 0 \\ 0 & 48 \\ 0 & 47 \\ 0 & 46 \\ 95 & 0 \\ 0 & 45 \\ 94 & 0 \end{bmatrix}. \tag{81}$$

$\bar{G}(s)$ satisfying (56) is written by

$$\bar{G}(s) = \left[\begin{array}{cccccccc|cc} 395 & -495 & -99 & 198 & 693 & 99 & -792 & -99 & -99 & 99 \\ 392 & -492 & -98 & 196 & 686 & 98 & -784 & -98 & -98 & 98 \\ 0 & 0 & 46 & -96 & 0 & -48 & 0 & 48 & 0 & -48 \\ 0 & 0 & 47 & -97 & 0 & -47 & 0 & 47 & 0 & -47 \\ 0 & 0 & 46 & -92 & -4 & -46 & 0 & 46 & 0 & -46 \\ 380 & -475 & -95 & 190 & 665 & 90 & -760 & -95 & -95 & 95 \\ 0 & 0 & 45 & -90 & 0 & -45 & -5 & 45 & 0 & -45 \\ 376 & -470 & -94 & 188 & 658 & 94 & -752 & -100 & -94 & 94 \\ \hline -400 & 500 & 100 & -200 & -700 & -100 & 800 & 100 & 100 & -100 \\ 0 & 0 & -50 & 100 & 0 & 50 & 0 & -50 & 0 & 50 \end{array}\right]. \tag{82}$$

$\bar{G}(s)$ is factorized as (44), where

$$\bar{G}_1(s) = \left[\begin{array}{cccccccc|c} 395 & -495 & -99 & 198 & 693 & 99 & -792 & -99 & -99 \\ 392 & -492 & -98 & 196 & 686 & 98 & -784 & -98 & -98 \\ 0 & 0 & 46 & -96 & 0 & -48 & 0 & 48 & 0 \\ 0 & 0 & 47 & -97 & 0 & -47 & 0 & 47 & 0 \\ 0 & 0 & 46 & -92 & -4 & -46 & 0 & 46 & 0 \\ 380 & -475 & -95 & 190 & 665 & 90 & -760 & -95 & -95 \\ 0 & 0 & 45 & -90 & 0 & -45 & -5 & 45 & 0 \\ 376 & -470 & -94 & 188 & 658 & 94 & -752 & -100 & -94 \\ \hline -400 & 500 & 100 & -200 & -700 & -100 & 800 & 100 & 100 \\ 0 & 0 & -50 & 100 & 0 & 50 & 0 & -50 & 0 \end{array}\right]$$

and

$$\bar{G}_2(s) = \left[\begin{array}{ccccccc|c} -50 & 0 & 0 & 0 & 0 & 0 & 0 & 11.2100 \\ 0 & -100 & 0 & 0 & 0 & 0 & 0 & -132.7000 \\ 0 & 0 & -5.8040 & 2.9330 & 0 & 0 & 0 & 3.4240 \\ 0 & 0 & -2.9330 & -5.8040 & 0 & 0 & 0 & 4.8860 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0.6454 \\ 0 & 0 & 0 & 0 & 0 & -3.6960 & 0.4947 & 0.5945 \\ 0 & 0 & 0 & 0 & 0 & -0.4947 & -3.6960 & -0.6110 \\ \hline 0 & -76.1300 & -3.0280 & 2.4780 & 0 & -1.6620 & 0.01126 & -100 \\ -200 & 0 & -0.4264 & -1.5550 & 3.1890 & -0.07548 & 0.2792 & 50 \end{array}\right]. \tag{83}$$

From (22), K_i $(i = 1, 2)$ are calculated by

$$K_1 = \begin{bmatrix} -6 & 6 & 0 & 0 & -6 & 0 & 6 & 0 \end{bmatrix} \tag{84}$$

and

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{85}$$

Using above mentioned parameters and (50), G_K is given by

$$G_K(s) = \left[\begin{array}{cc|cc} -3 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]. \tag{86}$$

Thus, the diagonal-inner outer factorization for $G(s)$ in (70) is obtained as

$$G_i(s) = \begin{bmatrix} \frac{s-3}{s+3} & 0 \\ 0 & 1 \end{bmatrix}$$

From (42), Q is given by

$$Q(s) = \left[\begin{array}{cc|cc} -100 & 0 & 8 & 0 \\ 0 & 50 & 0 & 8 \\ \hline 12.5 & 0 & 0 & 0 \\ 0 & 6.25 & 0 & 0 \end{array}\right]. \tag{87}$$

From (49), the inverse of $\hat{G}(s)$ is given by

$$\hat{G}^{-1}(s) = \left[\begin{array}{cccccccc|cc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1.1120 & 0 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3.8790 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0290 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 2.1200 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -1.0960 & -0.4194 \\ 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 2.0770 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 1.0060 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 2.3940 & 0 \\ \hline -1.7800 & 0 & -0.8943 & 0.1769 & 0.8943 & 0 & 1.7890 & 0 & 0.01 & 0.02 \\ 0 & 0.4847 & 0 & -0.4527 & 0 & -0.4526 & 0 & 0.3969 & 0 & 0.02 \end{array}\right].$$

From (55), $G_o(s)$ is obtained as

$$G_o(s) = \left[\begin{array}{cccccccc|cc} -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.554 \\ 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.978 \\ 0 & 0 & -6 & 0 & 0 & 0 & 0 & 0 & -1.069 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0.5571 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0.0003145 & 0.554 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & -1.207 & 0.0006272 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -0.5538 & -0.2159 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -0.5375 \\ \hline 1.805 & 0 & 0 & 3.59 & 3.61 & 0.0009405 & 1.806 & -0.7255 & 0 & 0 \\ 0 & 2.045 & 0.9351 & 0 & 0.0009378 & -0.8282 & -0.0001021 & 1.861 & 0 & 0 \end{array}\right]$$

$$= \begin{bmatrix} \frac{s^3+13s^2+55s+75}{s^4+13s^3+57s^2+95s+50} & \frac{s^4+12s^3+51s^2+92s+60}{s^5+18s^4+125s^3+416s^2+660s+400} \\ \frac{1.001s^2+7.005s+10.01}{s^4+18s^3+117s^2+320s+300} & \frac{s^4+13s^3+57s^2+95s+50}{s^5+17s^4+111s^3+347s^2+520s+300} \end{bmatrix}. \tag{88}$$

From (87), $G_i(s)$ is obviously a diagonal-inner function. Since $G_o(s)$ in (88) is stable and zeros of $G_o(s)$ in (88) are located in $-5.8043 + 2.9331j$, $-5.8043 - 2.9331j$, -1, $-3.6957 + 0.4947j$, $-3.6957 - 0.4947j$, -3, $G_o(s)$ in (87) is an outer function.

In this way, the diagonal-inner outer factorization can be readily attained by utilizing the methodology delineated in Section 5.

8. **Conclusions.** In this paper, we propose a method of diagonal-inner outer factorization for bi-proper systems and strictly proper systems. Comparisons between our method and past research [10, 14, 16] are summarized in Table 1. It is evident that a number of studies

Table 1. Comparison of inner-outer factorization method

Referent	**Strictly proper**	**Bi-proper**	**Diagonal-inner function**
Kase and Mutoh [16]	✓	✗	✗
Gu [14]	✓	✓	✗
Varga [10]	✓	✓	✗
This paper	✓	✓	✓

have engaged in inner-outer factorization, predominantly focusing on the factorization of strictly proper systems. There also exists a subset of research that delves into the factorization of bi-proper systems. However, a discernible gap in the literature is the absence of work incorporating the factorization method yielding a diagonal-inner function. Hence, the exploration and articulation of diagonal-inner function constitute a significant contribution and distinction of the present research.

The application adoption of this method is expected to be a pivotal tool in augmenting the performance of control systems within industrial manufacturing contexts. By refining the control algorithms, there is a substantial potential for industries to realize enhanced operational efficiency. The integration with artificial intelligence and machine learning is anticipated, facilitating advanced system control and adaptability. Furthermore, this integration is foreseen to extend to emerging Industrial Internet of Things (IIoT) applications, engendering real-time control and optimized performance.

Consequently, basic idea for diagonal-inner outer factorization is explained based on transfer function. In addition, we present a design method using state-space representation. The proposed method provides a diagonal inner function. The numerical example is also depicted in order to illustrate the effectiveness of the proposed method. Application of this paper will be presented in another article.

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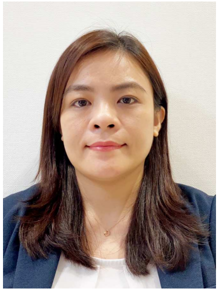

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