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A STUDY OF A SUPERLINEAR PARABOLIC DIRICHLET PROBLEM WITH UNKNOWN COEFFICIENT

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Abstract. *In this paper, we purpose to investigate the inverse problem of a superlinear parabolic Dirichlet equation with a supplementary integral over determination condition. In this connection, we use the energy inequality for the solvability of direct problem and the fixed point technique for the inverse problem. More particularly, the present paper is devoted to studying the existence and uniqueness of the solution for the inverse problem of the superlinear parabolic Dirichlet equation with integral condition of second type by reducing the problem to fixed point principle. This target would be achieved by applying the energy inequality method.*

Keywords: Superlinear parabolic Dirichlet equation, Inverse nonlinear problem, Nonlocal integral condition, Fixed point theorem, Energy inequality method

1. Introduction.

Inverse parabolic differential equations are commonly used in the fields of engineering and science for simulating physical processes. These equations describe various processes in viscous fluid flow, filtration of liquids, gas dynamics, heat conduction, elasticity, biological species, chemical reactions, environmental pollution, etc. Inverse problems for parabolic equations fulfilling nonlocal integral over determination condition were initially explored for many equations with coefficients independent of time and subject to certain boundary conditions of the first and third kind, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Under a final over determination condition, Kamynin [13] proved the unique solvability of the inverse problem of the right-hand side of a parabolic equation

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with the leading coefficient dependent on time and space variable. Afterward, Kamynin [14] discussed the existence of the solution to the initial-boundary value problem for the parabolic equation.

In this paper, we intend to investigate the one-of-kind solvability of the inverse problem of determining a pair of function $\{u(x,t), f(t)\}$ satisfying the following parabolic equation:

$$u_t - a\Delta u + bu + c|u|^p u = f(t)h(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

with the initial condition

$$u(x,0) = \varphi(x), \quad x \in \Omega, \tag{2}$$

subject to the boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

and the nonlocal over determination condition

$$\int_\Omega v(x)u(x,t)dx = E(t), \quad t \in (0,T), \tag{4}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, and the functions h, φ, E and v are known.

It should be noted that the solution of the inverse problem comes typically in the form of the integral condition (4). Many authors have explored the theory of the existence and uniqueness of the investigation problem, see [15, 16, 17, 18, 19, 20, 21, 22, 23, 24]. Based on these prior studies and in order to further develop these theories and works, the present paper is devoted to studying the existence and the uniqueness for the inverse problem with integral condition of the second type by reducing the problem to fixed point principle. This would be achieved by applying the energy inequality method.

The remaining of this paper is organized in the following manner. In the next section, we display very short preliminaries. Section 3 formulates first the main problem, and then it explores the existence and uniqueness of the solution for the direct problem with the use of the energy inequality method. Section 4 investigates the unique solvability of the inverse problem, followed by Section 5 that concludes the main results of this work.

2. **Preliminaries.** In this section, we review very short preliminaries and opening remarks that would pave the way to introduce our findings. For this purpose, let us define the function g^* as follows:

$$g^*(t) = \int_\Omega v(x)h(x,t)dx,$$

where $v(x)$, $h(x,t)$ are two functions and $g \in L^1(0,T)$ in which g^* is defined over $Q = \Omega \times (0,T)$ such that $\Omega \subset \mathbb{R}^n$. On the other hand, we recall below a very useful well-known inequality called the Cauchy's ε-inequality, which can be defined as follows:

$$2|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2, \quad a,b \in \mathbb{R}.$$

In the same regard, we recall in what follows the so-called Gronwall's lemma, which would be very significant in the upcoming sections.

Lemma 2.1. *(Gronwall's lemma) Let $f \in L^\infty(0,T)$, $g \in L^1(0,T)$ and $f(t) \geq 0$, $g(t) \geq 0$. If the inequality*

$$f(t) \leq c + \int_0^\tau f(s)g(s)ds,$$

is satisfied, then we have

$$f(t) \leq ce^{\int_0^\tau g(s)ds}.$$

3. **Existence and Uniqueness of a Direct Problem.** Here, in this part, we intend to apply the energy inequality method to studying the solution of problem (1)-(3). In particular, we aim to prove the existence and the uniqueness of the strong solution of direct main problem, see [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35] to track the same procedure we use. More specifically, the desired proof is based on the energy inequality and the density of the operator's range generated by the abstract formulation of the problem, which would yield difficulty in choosing the multiplier. However, we will attempt to formulate the main desired problem in what follows. To do so, we consider the following superlinear parabolic Dirichlet problem:

$$(P) \qquad \begin{cases} u_t - a\Delta u + bu + c|u|^p u = f(x,t), & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

$$\mathcal{L}u = u_t - a\Delta u + bu + c|u|^p u = f(x,t), \tag{5}$$

with the initial condition

$$l_1 u = u(x,0) = \varphi(x), \quad x \in \Omega, \tag{6}$$

and with the Dirichlet boundary condition

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{7}$$

In fact, the problem stated above is defined over the domain $Q = \Omega \times (0,T)$ such that $T > 0$ and $\Omega \subset \mathbb{R}^n$. Herein, $f(x,t)$ and $u_0(x)$ are two given functions, and a, b, c are some given constants that verify the following hypothesis:

$$A1 : a \geq 0,\ b \geq 0,\ c \geq 0.$$

Besides, the operator L is defined from E to F, where E is a Banach space, which includes all functions $u(x,t)$ having the following finite norms:

$$\|u\|_E^2 = \|u\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u\|^2_{L(Q)} + \|u\|^2_{L(Q)} + \|u\|^{p+2}_{L^{p+2}(Q)}.$$

In the same regard, F defined above is the Hilbert space, which consists of all elements $\mathcal{F} = (f, \varphi)$ and equipped according to the norm

$$\|\mathcal{F}\|_F^2 = \|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)}.$$

In the following content, we begin by establishing an a priori estimate for problem resolution. For this purpose, we introduce the next theoretical result.

Theorem 3.1. *If the assumption $A1$ is satisfied, then for any function $u \in D(L)$, there exists a positive constant C that is independent of u such that*

$$\|u\|_E \leq C\|\mathcal{F}\|_F,$$

where $D(L)$ is the domain of the definition of the operator L, which is defined by

$$D(L) = \left\{u/u, u_t, \nabla u, \Delta u \in L^2(Q), u \in L^{p+2}(Q)\right\}.$$

Proof: To prove this result, we use the scalar product in $L^2(Q)$ of (5) and the operator $Mu = u$, where $Q = \Omega \times (0,T)$. This would imply

$$\begin{aligned} \langle \mathcal{L}u, Mu\rangle_{L^2(Q)} &= \langle u_t, u\rangle_{L^2(Q)} - a\langle \Delta u, u\rangle_{L^2(Q)} + \langle bu, u\rangle_{L^2(Q)} + \langle c|u|^p u, u\rangle_{L^2(Q)} \\ &= \langle f, u\rangle_{L^2(Q)}. \end{aligned} \tag{8}$$

Consequently, we have

$$\langle u_t, u\rangle_{L^2(Q)} = \int_Q u_t(t,x)u(t,x)dtdx = \frac{1}{2}\int_Q \frac{d}{dt}u^2(t,x)dtdx.$$

This immediately yields

$$\langle u_t, u\rangle_{L^2(Q)} = \frac{1}{2}\|u(.,\tau)\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} \tag{9}$$

and

$$-a\langle \Delta u, u\rangle_{L^2(Q)} = -a\int_Q \Delta u(t,x)u(t,x)dtdx.$$

Hence, we obtain

$$-a\langle \Delta u, u\rangle_{L^2(Q)} = a\|\nabla u\|^2_{L^2(Q)}, \tag{10}$$

$$\langle bu, u\rangle_{L^2(Q)} = b\int_Q u^2(x,t)dxdt = b\|u\|^2_{L^2(Q)} \tag{11}$$

and

$$\langle c|u|^p u, u\rangle_{L^2(Q)} = c\int_Q |u(x,t)|^p u^2(x,t)dtdx = c\|u\|^{p+2}_{L^{p+2}(Q)}. \tag{12}$$

By substituting (9)-(12) into (8), we get

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)} - \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} = \int_Q f(t,x)u(t,x)dtdx.$$

If one estimates the last term of the right hand side using $\left(|ab| \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}\right)$, we get

$$\begin{aligned}&\frac{1}{2}\|u(\cdot,\tau)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)}\\ &\leq \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)} + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^T \|u\|^2_{L^2(\Omega)}dt.\end{aligned}$$

Thus, using Gronwall's lemma yields

$$\begin{aligned}&\frac{1}{2}\|u(.,\tau)\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)}\\ &\leq \frac{c'}{2\varepsilon}\|f\|^2_{L^2(Q)} + \frac{c'}{2}\|\varphi\|^2_{L^2(\Omega)},\end{aligned}$$

where

$$c' = \exp\left(\frac{\varepsilon T}{2}\right).$$

It should be noted here that the right hand side of the last estimate is independent of τ, and so we can replace the left hand side by its upper bound with respect to τ from 0 to T. This means

$$\|u(\cdot,\tau)\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u\|^2_{L^2(Q)} + \|u\|^2_{L^2(Q)} + \|u\|^{p+2}_{L^{p+2}(Q)} \leq C\left(\|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)}\right),$$

where

$$C = \frac{\max\left(\frac{c'}{2}, \frac{c'}{2\varepsilon}\right)}{\min\left(\frac{1}{2}, a, b, c\right)}.$$

So, we have

$$\|u\|_E \leq C\|Lu\|_F. \tag{13}$$

□

Herein, the range of the operator L is denoted by $R(L)$. However, because we do not know anything about $R(L)$ except that $R(L) \subset F$, we have to extend the operator L. Thus, the estimate (13) holds for the extension and its range is the whole space F. As a result of this discussion, we state and prove the following proposition.

Proposition 3.1. *The operator $L : E \longrightarrow F$ has a closure.*

Proof: Let $(u_n)_{n\in\mathbb{N}} \subset D(L)$ be a sequence in which

$$u_n \longrightarrow 0 \text{ in } E$$

and

$$Lu_n \longrightarrow (f, \varphi). \tag{14}$$

Herein, we must demonstrate that

$$f \equiv 0, \quad \varphi \equiv 0 \text{ in } F.$$

Herein, the convergence of u_n to 0 in E causes

$$u_n \longrightarrow 0 \text{ in } D'(Q). \tag{15}$$

The relationship (15) is regarded very complicated in accordance with the continuity derivation of $D'(Q)$ in $D'(Q)$ and the continuity distribution of the function $|u_n|^q u_n$. This means

$$\mathcal{L}u_n \longrightarrow 0 \text{ in } D'(Q). \tag{16}$$

In addition, the convergence of Lu_n to f in $L^2(Q)$ yields

$$\mathcal{L}u_n \longrightarrow f \text{ in } D'(Q). \tag{17}$$

Hence, we can deduce from (16) and (17) that $f \equiv 0$. This is because we know the limit in $D'(Q)$ is unique. However, it can be generated from (14) that

$$l_1 u_n \longrightarrow \varphi \text{ in } L^2(\Omega).$$

On the other hand, we can have

$$\|u_n\|_E = \|u_n(\cdot,\tau)\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\nabla u_n\|^2_{L^2(Q)} + \|u_n\|^2_{L^2(Q)} + \|u_n\|^{p+2}_{L^{p+2}(Q)}.$$

This consequently implies

$$\|u_n\|_E \geq \|u_n(x,0)\|^2_{L^2(\Omega)},$$

and so

$$\|u_n\|_E \geq \|\varphi\|^2_{L^2(\Omega)}.$$

Now, since we have

$$u_n \longrightarrow 0 \text{ in } E,$$

then we can obtain

$$\|u_n\|^2_E \longrightarrow 0 \text{ in } \mathbb{R}.$$

Consequently, we get

$$0 \geq \|\varphi\|^2_{L^2(\Omega)}.$$

Therefore, one might deduce

$$\varphi \equiv 0,$$

which accordingly implies the desired result. □

Definition 3.1. *A solution to the operator equation*

$$\overline{L}u = F$$

is known as a strong solution to problem (5)-(7).

In light of the above definition, we may extend a priori estimate to strong solutions, i.e., we define the following estimate:

$$\|u\|_E \leq C \left\|\overline{L}u\right\|_F, \quad \forall u \in D\left(\overline{L}\right), \tag{18}$$

where L is the closure of this operator, and D is the domain of definition of L.

Corollary 3.1. *The range of the operator* $\overline{L}$ *is closed in* F *and equals the closure of* $R(L)$*, that is,*

$$R\left(\overline{L}\right) = \overline{R(L)}.$$

Proof: First, we intend to demonstrate the uniqueness of the solution if it exists. To do so, we let u_1 and u_2 be two solutions, and $\eta = u_1 - u_2$. Accordingly, η satisfies the following problem:

$$(P') \qquad \begin{cases} \eta_t - a\Delta\eta + |u_1|^p u_1 - |u_2|^p u_2 + b\eta = 0, & (x,t) \in \Omega \times (0,T), \\ \eta(x,0) = 0, & x \in \Omega, \\ \eta(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{19}$$

in which the following assertion is held:

$$\eta_t - a\Delta\eta + |u_1|^p u_1 - |u_2|^p u_2 + b\eta = 0, \qquad (x,t) \in \Omega \times (0,T). \tag{20}$$

Herein, we use the scalar product in $L^2(\Omega)$ of (20) and η to obtain

$$\int_\Omega \eta_t(t,x)\eta(t,x)dx - a\int_\Omega \Delta\eta(t,x)\eta(t,x)dx$$
$$+ c\int_\Omega \left(|u_1|^p u_1 - |u_2|^p u_2\right)(u_1 - u_2)dx + b\int_\Omega \eta^2(t,x)dx = 0.$$

Consequently, we can have

$$\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} + a\|\nabla\eta\|^2_{L^2(\Omega)} + c\int_\Omega \left(|u_1|^p u_1 - |u_2|^p u_2\right)(u_1 - u_2)dx + b\|\eta\|^2_{L^2(\Omega)} = 0. \tag{21}$$

Since $|\lambda|^p\lambda$ is a monotone function in λ, the last term of the left hand side of (21) will be non negative. It follows from (21) that

$$\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

which implies

$$\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

for all $t \in (0,T)$. In other words, we have $\eta(t) = 0$, which shows the uniqueness issue of the solution, i.e., $u_1(t) = u_2(t)$.

In light of the previous discussion, we are ready now to prove Corollary 3.1. For this purpose, we let $z \in \overline{R(L)}$. Then, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ in $R(L)$ such that $\lim_{n\longrightarrow+\infty} z_n = z$. So, as $(z_n)_{n\in\mathbb{N}}$ in $R(L)$, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(L)$ such that $Lu_n = z_n$. Now, let ε, $n \geq n_0$, and let $m, m' \in \mathbb{N}$, $m \geq m'$ such that u_m and $u_{m'}$ are two solutions, i.e.,

$$Lu_m = f \text{ and } Lu_{m'} = f.$$

Herein, we assume that $y = u_m - u_{m'}$, then y satisfies the following problem:

$$(P'') \qquad \begin{cases} y_t - a\Delta y + |u_m|^p u_m - |u_{m'}|^p u_{m'} + by = 0, & (x,t) \in \Omega \times (0,T), \\ y(x,0) = 0, & x \in \Omega, \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

Now, by applying the same procedure used to prove the uniqueness issue, we get $y = 0$. This immediately implies

$$0 \leq \|u_m(t) - u_{m'}(t)\|_E \leq 0, \tag{22}$$

for all $t \in (0,T)$. In other words, we can have

$$\lim_{m,m'\longrightarrow+\infty} \|u_m(t) - u_{m'}(t)\|_E = 0 \Longleftrightarrow \forall\varepsilon \geq 0, \quad \exists n_0 \in \mathbb{N}\backslash\forall m, m' \geq n_0$$

$$\|u_m(t) - u_{m'}(t)\|_E \leq \varepsilon.$$

Thus, we can conclude that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in E, whereby E itself represents a Banach space. Therefore, there exists $u \in E$ such that

$$\lim_{n\longrightarrow+\infty} u_n = u.$$

Now, we use the definition of $\overline{L}$ that says $\lim_{n\longrightarrow+\infty} u_n = u$ in E if

$$\lim_{n\longrightarrow+\infty} u_n = \lim_{n\longrightarrow+\infty} z_n = z,$$

then

$$\lim_{n\longrightarrow+\infty} \overline{L}u_n = z$$

as $\overline{L}$ is closed, and so we have $\overline{L}u = z$. This makes the function u satisfy

$$u \in D\left(\overline{L}\right),\ \overline{L}u = z.$$

Then, $z \in R\left(\overline{L}\right)$, and so we have

$$\overline{R(L)} \subset R\left(\overline{L}\right).$$

Furthermore, due to the fact that $R\left(\overline{L}\right)$ is a Banach subspace, we conclude that $\overline{L}$ is closed. From this point of view, it is still necessary to show that the opposing party has been included. To this end, we let z in $R\left(\overline{L}\right)$. Then, there exists a sequence of $(z_n)_n$ in F consisting of the element of the set $R\left(\overline{L}\right)$ such that $\lim_{n\longrightarrow+\infty} z_n = z$. As a consequence, there exists a corresponding sequence $(v_n)_n \subset D\left(\overline{L}\right)$ such that $\overline{L}v_n = z_n$. On the other hand, we have $(v_n)_n$ is a cauchy sequence in F. Actually, this infer can be yielded by applying the same procedure used to prove that $(u_n)_n$ is a cauchy sequence in the previous steps. Consequently, there exists $v \in E$

$$\lim_{n\longrightarrow+\infty} v_n = v, \quad v \in E,$$

which implies

$$\lim_{n\longrightarrow+\infty} Lv_n = z.$$

As a consequence, $z \in R\left(\overline{L}\right)$, and hence we can conclude

$$R\left(\overline{L}\right) \subset \overline{R(L)},$$

which completes the proof of the desired result. □

In the following content, we aim to investigate the existence of the solution for the problem at hand. In this connection, we must show that $R(L)$ is dense in F for all $u \in E$ and for any $\mathcal{F} = (f, \varphi) \in F$. In order to achieve this goal, we state and prove the next result.

Theorem 3.2. *Assume that the assumption $A1$ is satisfied. Then for every $\mathcal{F} = (f, \varphi) \in F$, there exists a unique strong solution $u = L^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ to problem (P).*

Proof: To begin the proof of this result, we should demonstrate the density of $R(L)$ in F, i.e., we want to choose $w \in R^{\perp}(L)$, and then prove that $R^{\perp}(L) = 0$, for all $u \in D(L)$ and for the exceptional case, where $D(L)$ is reduced to $D_0(L)$ and where

$$D_0(L) = \{u, u \in D(L) : l_1 u = 0\}.$$

However, with the aim of achieving this goal, we have to verify the validation of the following claim.

Claim: Assume that the condition of Theorem 3.2 is satisfied. If for $w \in L^2(Q)$ and for all $u \in D_0(L)$, we have

$$\int_Q \mathcal{L}u.wdxdt = 0, \tag{23}$$

then, w vanishes almost everywhere in Q.

For the purpose of proving this claim, we should note that the scalar product of F can be defined as

$$(Lu, W)_F = \int_Q \mathcal{L}u.wdxdt, \tag{24}$$

where $W = (w, 0) \in D(L)$. As a result, Equality (23) can be written in the form

$$\begin{aligned} &\int_Q u_t(x,t).w(x,t)dxdt - a\int_Q \Delta u.w(x,t)dxdt + b\int_Q u(x,t)w(x,t)dxdt \\ &+ c\int_Q |u(x,t)|^p u(x,t).w(x,t)dxdt = 0. \end{aligned} \tag{25}$$

Then, setting $w = u$ yields

$$\begin{aligned} &\int_Q u_t(x,t).u(x,t)dxdt - a\int_Q \Delta u.u(x,t)dxdt + b\int_Q u^2(x,t)dxdt \\ &+ c\int_Q |u(x,t)|^{p+2}dxdt = 0. \end{aligned} \tag{26}$$

This consequently gives

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} = 0,$$

i.e.,

$$a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} = -\frac{1}{2}\|u\|^2_{L^2(\Omega)},$$

which means

$$a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} \leq 0.$$

So, we can conclude

$$\|u\|^2_{L^2(Q)} \leq 0. \tag{27}$$

It means that $u \equiv 0$ in Q. Therefore, $w \equiv 0$ in Q, and hence the claim is verified.

Now, we intend to go back to Theorem 3.2. To this end, we should prove the set $R(L)$ is dense in F. For this purpose, we assume the following assumption

$$(Lu, W)_F = \int_Q \mathcal{L}u.wdxdt + \int_\Omega l_1 u w_1 dx = 0 \tag{28}$$

is held, for some $W = (w, w_1) \in R^\perp(L)$ and for all $u \in D(L)$. According to the claim reported above, if we put $u \in D_0(L)$, we can have $\int_Q \mathcal{L}u.wdxdt = 0$, and hence $w \equiv 0$. As a result, assumption (28) becomes

$$\int_\Omega l_1 u w_1 dx = 0, \quad u \in D(L). \tag{29}$$

As a consequence, due to the fact that the range of the trace operator l_1 is dense everywhere in the Hilbert space, then Equality (29) implies that $w_1 = 0$. Thus, we have $W = 0$, which implies $\left(\overline{R(L)} = F\right)$, and therefore the proof of this theorem is completed. $\square$

4. **Unique Solvability of the Inverse Problem.** In this part, the unique solvability of the inverse problem is addressed. In order to attain this objective, we assume that the functions appearing in the data for the problem are measurable and satisfy the following conditions:

$$(H) \qquad \begin{cases} h \in C(0,T,L^2(\Omega)), v \in V = \{v, \nabla v \in L^2(\Omega), v \in L^{p+2}(\Omega)\}, \ E \in W_2^2(0,T), \\ \|h(x,t)\| \le m; \ |g^*(t)| \ge r > 0, \ \text{for } r \in \mathbb{R}, (x,t) \in Q, \\ \varphi(x) \in W_2^1(\Omega). \end{cases}$$

The correspondence between f and u can be viewed as one way to specify the following linear operator:

$$A : L^2(0,T) \longrightarrow L^2(0,T), \tag{30}$$

which is defined by

$$Af(t) = \frac{1}{g^*}\left\{a\int_\Omega \nabla u \nabla v dx + \int_\Omega |u|^p uv dx\right\}. \tag{31}$$

In this case, it is reasonable to refer to the second kind linear equation for the function f over the space $L^2(0,T)$, that is,

$$f = Af + \mu, \tag{32}$$

where

$$\mu = \frac{E' + bE}{g^*}. \tag{33}$$

In view of the above lines, we introduce the next theoretical result that aims to demonstrate the unique solvability of the inverse problem.

Theorem 4.1. *Suppose that the data function of the inverse problem (1)-(4) satisfies Condition (H). Then, the following statements are equivalent:*

1) If the inverse problem (1)-(4) is solvable, then Equation (32) is solvable;
2) If Equation (32) has a solution and the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ holds, then the inverse problem (1)-(4) has a solution.

Proof:

1) Suppose that the problem (1)-(4) is solvable with a solution of the form $\{u, f\}$. Now, by multiplying Equation (1) by v and then integrating the result over Ω, we get

$$\int_\Omega u_t v(x)dx + a\int_\Omega \nabla u \nabla v dx + b\int_\Omega u(x,t)v(x)dx + \int_\Omega |u|^p uv(x)dx = f(t)g^*(t). \tag{34}$$

Using (4) and (30) yields

$$\frac{E' + bE}{g^*} + Af = f.$$

This means that f solves Equation (32), and hence the result holds.

2) By considering the given assumption, we deduce that Equation (31) has a solution, say f. Now, by substituting f into Equation (1), then the resulting relation (1)-(3) can be treated as a direct problem having a unique solution. Thus, it remains for us to prove that u satisfies also the integral over determination (4). To do so, it should be noted that Equation (34) can yield

$$\frac{d}{dt}\int_\Omega u(x,t)v(x)dx + a\int_\Omega \nabla u \nabla v dx + b\int_\Omega u(x,t)v(x)dx + c\int_\Omega |u|^p u(x,t)v(x)dx$$
$$= f(t)g^*(t). \tag{35}$$

On the other hand, being a solution of Equation (32), the function u satisfies the following relation:

$$E' + bE + a\int_\Omega \nabla u\nabla v dx + b\int_\Omega uvdx + c\int_\Omega |u|^p uvdx = f(t)g^*(t). \tag{36}$$

Consequently, subtracting (35) from (36) immediately yields

$$\frac{d}{dt}\int_\Omega uvdx + b\int_\Omega uvdx = E' + bE. \tag{37}$$

By integrating the previous differential equation coupled with taking account of the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$, we conclude that u satisfies the integral condition (4). Therefore, we infer that $\{u, f\}$ is the solution of the inverse problem (1)-(4), as required. □

In what follows, we state and prove some properties in connection of the operator A. These properties are formulated as certain theoretical aspects for completeness.

Lemma 4.1. *Suppose that Condition (H1) holds, then there exists a positive δ for which the operator A is a contracting operator in $L^2(0,T)$.*

Proof: Based on (31), we can get the following estimate:

$$\begin{aligned}
|Af(t)|^2 &\le \left|\frac{1}{g^*}\left\{a\int_\Omega \nabla u\nabla vdx + \int_\Omega |u|^p uvdx\right\}\right|^2\\
&\le \frac{2}{r^2}\left[a^2\left(\int_\Omega \nabla u\nabla vdx\right)^2 + \left(\int_\Omega |u|^p|uv|dx\right)^2\right]\\
&\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \left(\int_\Omega |u|^{p+1}|v|dx\right)^2\right]\\
&\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \|u\|^{2(p+1)}_{L^{p+2}(\Omega)}\|v\|^2_{L^{p+2}(\Omega)}\right]\\
&\le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \|u\|^{p+2}_{L^{p+2}(\Omega)}\|u\|^{p}_{L^{p+2}(\Omega)}\|v\|^2_{L^{p+2}(\Omega)}\right].
\end{aligned}$$

Now, we suppose that $\|u\|^p_{L^\infty(0,T,L^{p+2}(\Omega))} = \gamma(t) \ge 0$. Then, we have

$$|Af(t)|^2 \le \frac{2}{r^2}\left[a^2\|\nabla u\|^2_{L^2(\Omega)}\|\nabla v\|^2_{L^2(\Omega)} + \|u\|^{p+2}_{L^{p+2}(\Omega)}\gamma(t)\|v\|^2_{L^{p+2}(\Omega)}\right].$$

As a result, integrating the above assertion over $(0,T)$ yields

$$\begin{aligned}
&\int_0^T |Af(t)|^2 dt\\
&\le \frac{2}{r^2}\max\left(a^2\|\nabla v\|^2_{L^2(\Omega)}, \gamma(t)\|v\|^2_{L^{p+2}(\Omega)}\right)\left(\int_0^T \|\nabla u\|^2_{L^2}d\tau + \int_0^T \|u(\cdot,\tau)\|^{p+2}_{L^{p+2}(\Omega)}d\tau\right).
\end{aligned} \tag{38}$$

Thus, we obtain

$$\|Af\|^2_{L^2(0,T)} \le K\left(\int_0^T \|\nabla u\|^2_{L^2}d\tau + \int_0^T \|u(\cdot,\tau)\|^{p+2}_{L^{p+2}(\Omega)}d\tau\right),$$

where

$$K = \frac{2}{r^2}\max\left(a^2\|\nabla v\|^2_{L^2(\Omega)}, \gamma(t)\|v\|^2_{L^{p+2}(\Omega)}\right).$$

By multiplying both side of (1) by u in $L^2(Q)$, and then integrating the resulting expression by parts, we get

$$\int_Q u_t u dxdt - a\int_Q \Delta u u dxdt + b\int_Q u^2 dxdt + \int_Q |u|^p u^2 dxdt = \int_Q f(t)h(x,t)u dxdt, \quad (39)$$

where $(x,t) \in \Omega \times (0,T)$. Accordingly, we can obtain

$$\begin{aligned} &\frac{1}{2}\|u\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + b\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} \\ \leq\ &\frac{m^2}{2\varepsilon}\|f\|^2_{L^2(0,T)} + \frac{\varepsilon}{2}\|u\|^2_{L^2(Q)} + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} \end{aligned} \quad (40)$$

and so, we can have

$$\frac{1}{2}\|u\|^2_{L^2(\Omega)} + a\|\nabla u\|^2_{L^2(Q)} + \left(b - \frac{\varepsilon}{2}\right)\|u\|^2_{L^2(Q)} + c\|u\|^{p+2}_{L^{p+2}(Q)} \leq \frac{m^2}{2\varepsilon}\|f\|^2_{L^2(0,T)} + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}, \quad (41)$$

where $0 < \varepsilon < 2b$. Passing to the maximum and omitting some terms yield

$$\int_0^T \left(\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^{p+2}_{L^{p+2}(\Omega)}\right) \leq M''\|f\|^2_{L^2(0,T)}, \quad (42)$$

where

$$M'' = \frac{\frac{m^2}{2\varepsilon}}{\min(a,c)}.$$

Therefore, we obtain

$$\|Af\|_{L^2(0,T)} \leq \delta\|f\|_{L^2(0,T,L^2(0,T))}, \quad (43)$$

where $\delta = \sqrt{kM''}$. As a result of the preceding, there exists a positive δ such that

$$\delta \leq 1, \quad (44)$$

which demonstrates that the operator A has a contracting mapping on $(L^2(0,T), L^2(0,T))$ and this completes the proof. □

Theorem 4.2. *Assume that Condition (H) and the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ are satisfied. Then the following assertions are true.*

1) The following approximations

$$f_{n+1} = \mathcal{A}f_n \quad (45)$$

converge to f in $L^2((0,T), L^2(0,T))$-norm with any initial iteration f_0 in $L^2(0,T)$, and for the operator $\mathcal{A}$.

2) The solution $\{u, f\}$ of the inverse problem (1)-(4) exists and is unique.

Proof:

1) To prove this result, we use the following nonlinear operator

$$\mathcal{A} : L^2(0,T) \longrightarrow L^2((0,T), L^2(0,T)),$$

which is defined by

$$\mathcal{A}f = Af + \frac{E' + bE}{g^*}, \quad (46)$$

where the operator A and the function g^* arise from (31). From (45) it follows that (32) can be written as

$$f = \mathcal{A}f. \quad (47)$$

Now, we have to demonstrate that the operator $\mathcal{A}$ has a fixed point in the space $L^2((0,T), L^2(0,T))$. To do so, we have to observe that according to the relationship

$$\mathcal{A}f_1 - \mathcal{A}f_2 = Af_1 - Af_2 = A(f_1 - f_2),$$

we can infer, based on the estimate (43), the following assertions:

$$\|\mathcal{A}f_1 - \mathcal{A}f_2\|_{L^2(0,T)} = \|Af_1 - Af_2\|_{L^2(0,T)} \leq \delta \|f_1 - f_2\|_{L^2((0,T),L^2(0,T))}, \tag{48}$$

in which $\mathcal{A}$ is a contracting mapping on $L^2((0,T), L^2(0,T))$ based on (44) and (47). As a result, $\mathcal{A}$ has a unique fixed point f in $L^2((0,T), L^2(0,T))$, and hence the successive approximations (45) converge to f in $L^2((0,T), L^2(0,T))$-norm, which is independent of the initial iteration $f_0 \in L^2((0,T), L^2(0,T))$.

2) Actually, based on the previous discussion, we can infer that Equations (47) and thus (32) have a unique solution f in $L^2((0,T), L^2(0,T))$. On the other hand, the existence of a solution of the inverse problem (1)-(4) is confirmed by Theorem 4.1. So, it remains to prove that this solution is unique. So, by the proof of contrary, we suppose that there are two distinct solutions $\{u_1, f_1\}$ and $\{u_2, f_2\}$ for the main inverse problem. Consequently, we first assert that $f_1 \neq f_2$ almost everywhere on $(0,T)$. If $f_1 = f_2$, then by applying the uniqueness theorem of the corresponding direct problem (5)-(7), we get $u_1 = u_2$ almost everywhere in Q. As both pairs have verified (35), we conclude that the functions f_1 and f_2 acquired two distinct solutions of Equation (47), which contradicts the uniqueness of the solution of Equation (47). This completes the proof of the desired result. □

Corollary 4.1. *If the conditions of Theorem 4.2 are satisfied, then the solution f depends continuously with respect to the data μ of Equation (32).*

Proof: Assume that μ and ϑ are two sets of data that satisfy the assumptions of Theorem 4.2. Let f and g be two solutions of Equation (32) that corresponds respectively to μ and ϑ. According to (32), we can have

$$f = Af + \mu, \quad g = Ag + \vartheta.$$

Now, let us begin by estimating the difference $f - g$. Then, by using (43), we obtain

$$\|f - g\|_{L^2((0,T),L^2(0,T))} = \|(Af_1 + \mu) - (Ag + \vartheta)\|_{L^2(0,T)} = \|A(f-g) + (\mu - \vartheta)\|_{L^2(0,T)}.$$

Accordingly, we can get

$$\|f - g\|_{L^2((0,T),L^2(0,T))} \leq \frac{1}{1-\delta} \|\mu - \vartheta\|_{L^2(0,T)}.$$

As a result, the proof of the corollary is finished. □

5. **Conclusion.** In this paper, the existence and uniqueness of the solutions for the inverse problem of a superlinear parabolic Dirichlet equation has been investigated with a supplementary integral over determination condition. The energy inequality method has been successfully used for the solvability of direct problem. The fixed point technique has been also used for the inverse problem. Thus, we believe that the existence and uniqueness of the solutions for the superlinear elliptic partial differential equations and systems can be explored in a similar manner to this study. This would be left to the future for further consideration.

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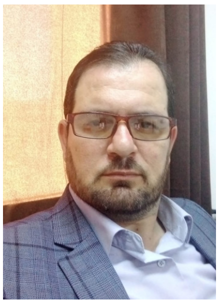

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