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THE TANH METHOD AND THE $\left(\frac{G'}{G}\right)$-EXPANSION METHOD FOR SOLVING THE SPACE-TIME CONFORMABLE FZK AND FZZ EVOLUTION EQUATIONS

Ahmed Anber[1], Iqbal Jebril[2,*], Zoubir Dahmani[3], Nabil Bedjaoui[4] and Abdelkader Lamamri[3]

[1]Department of Mathematics
University of Sciences and the Technology of Oran (USTO)
Oran 31000, Algeria
ahmed.anber@univ-usto.dz

[2]Department of Mathematics
Al Zaytoonah University of Jordan
Amman 11733, Jordan
*Corresponding author: i.jebril@zuj.edu.jo

[3]Department of Mathematics
Saad Dahlab Blida 1 University
Blida 09000, Algeria
{ dahmani_zoubir; a.lamamri }@univ-blida.dz

[4]Laboratoire Amiénois de Mathématique Fondamentale et Appliquée (LAMFA)
University of Picardie Jules Vernes
Amiens 80039, France
nabil.bedjaoui@u-picardie.fr

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Abstract. *In this paper, we apply the $\left(\frac{G'}{G}\right)$-expansion and the Tanh methods to finding exact traveling wave solutions of the space and time fractional Zakharov-Kuznetsov (FZK) equation and the space and time fractional Zoomeron (FZZ) equation using the conformable fractional derivatives. For the FZK equation, we obtain six solutions with the Tanh method and three solutions with the $\left(\frac{G'}{G}\right)$-expansion method. On the other hand, for the FZZ equation, we find one solution with the Tanh method and three solutions with the second method. From this study, we observe that the $\left(\frac{G'}{G}\right)$-expansion method and the Tanh method are not equivalent, it is shown that the Tanh method is more effective than the $\left(\frac{G'}{G}\right)$-expansion method, for solving fractional differential equation in special conditions.*

Keywords: Conformable fractional derivative, Tanh method, $\left(\frac{G'}{G}\right)$-expansion method, Traveling wave solution, FZK equation, FZZ equation

1. **Introduction.** Fractional differential equations (FDEs for short) have attracted significant attention in recent years due to their applications in various fields such as physics, engineering, and biology [1, 2, 6, 19, 20, 21, 28, 29]. These equations contain derivatives of non-integer order, and therefore, they generalize the classical differential equations. One of the most interesting aspects of FDEs is the existence of traveling wave solutions, which play a crucial role in understanding of the dynamics of many physical phenomena [30, 31, 32, 33, 34].

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Finding analytical solutions to FDEs is a challenging task due to the non-locality of the fractional derivatives. Therefore, several methods have been proposed to solve these equations.

In this paper, we consider the Tanh and $\left(\frac{G'}{G}\right)$-expansion numerical methods to solve some classes of FDEs. The Tanh method is a powerful and efficient method for solving nonlinear differential equations, including FDEs, see for instance [3, 12, 16, 24]. This method is based on the assumption that the solution can be expressed in terms of hyperbolic functions. By substituting this ansatz into the FDE and solving, for the coefficients, one can obtain analytical expressions for the traveling wave solutions.

The $\left(\frac{G'}{G}\right)$-expansion method is another technique for solving nonlinear differential equations, which has been widely applied to FDEs [9, 10, 11, 18, 25]. This method is based on the idea of expanding the solution as a power series of a function $G(x)$, which satisfies a linear differential equation. By substituting the series into the associated FDE and solving, for the coefficients, one can obtain analytical expressions for the traveling wave solutions.

In addition to the above two applied methods, we will also use the Khalil conformable fractional derivatives in our proposed FDEs [8]. These derivatives are relatively new concept in the field of FDEs and have been shown to have some advantages over other types of fractional derivatives [26].

The main aim of this paper is to employ the Tanh method and the $\left(\frac{G'}{G}\right)$-expansion method to find traveling wave solutions for the following two FDEs.

1) The conformable space-time fractional Zakharov-Kuznetsov (FZK) equation:

$$\begin{aligned} &T_t^{\alpha}u(t,x,y,z) + Au(t,x,y,z)T_x^{\beta_1}u(t,x,y,z) + T_x^{2\beta_1}u(t,x,y,z) + T_y^{2\beta_2}u(t,x,y,z) \\ &+ T_z^{2\beta_3}u(t,x,y,z) = 0, \end{aligned} \tag{1}$$

where, $T_x^{\beta_1}$, $T_y^{\beta_2}$, $T_z^{\beta_3}$, T_t^{α} are the conformable fractional derivatives in the sense of Khalil et al. [8], with α $(0 < \alpha \le 1)$ and β $(0 < \beta_i \le 1, i \in \{1,2,3\})$ as parameters describing the order of the conformable time fractional and the conformable space fractional, respectively, and A is constant. When $\alpha = \beta_i = 1$, Equation (1) corresponds to the Zakharov-Kuznetsov equation. The FZK equation describes the propagation of long-amplitude ion-acoustic waves in a plasma with dispersion and dissipation [22].

2) The conformable space-time fractional Zoomeron (FZZ) equation:

$$\begin{aligned} &T_t^{2\alpha}u(t,x,y)\left(\frac{T_x^{\beta_1}T_y^{\beta_2}u(t,x,y)}{u(t,x,y)}\right) - T_x^{2\beta_1}u(t,x,y)\left(\frac{T_x^{\beta_1}T_y^{\beta_2}u(t,x,y)}{u(t,x,y)}\right) \\ &+ 2T_t^{\alpha}u(t,x,y)\big(T_x^{\beta_1}u^2(t,x,y)\big) = 0. \end{aligned} \tag{2}$$

The FZZ equation models the propagation of light pulses in a nonlinear fiber-optic medium with higher-order dispersion and nonlinearity [23].

As important connections between all the paragraphs of our paper, we shall present the following three important comments.

Common Mathematical Methods: In our work, the $\left(\frac{G'}{G}\right)$-expansion method and the Tanh method are applied to both the FZK and FZZ equations.

These methods are two different techniques used to find exact solutions to differential equations. By using the same methods for both equations, one can potentially draw parallels and compare the solutions that we obtain. This may lead to insight into the behavior of these different physical systems modeled by the FZK and FZZ equations.

Fractional Calculus and Conformable Fractional Derivatives: Both the above two equations involve fractional calculus and, in particular, conformable fractional derivatives. So, we are interested in a branch of mathematics that extends traditional calculus to non-integer orders, making it suitable for modeling complex physical phenomena. By applying these fractional calculus techniques to both the FZK and FZZ equations, the interested reader is essentially using a common framework for his analysis.

Physical Phenomena: The FZK and FZZ equations likely model physical systems with certain similarities or interrelated aspects. By studying these equations together, researcher may be interested in understanding how similar mathematical methods and techniques can be used to describe different physical phenomena. This can provide a more comprehensive understanding of the underlying physics and the connection between these systems.

In summary of the above three paragraphs and for a possible connection between the parts of our paper, the study of the FZK and FZZ equations lies in the shared mathematical methods, fractional calculus techniques, and the potential for gaining insights into related physical phenomena by applying similar approaches to their analysis.

At the end of this section, the remaining paper can be organized as follows. In Section 2, we will provide a brief review of conformable fractional derivatives. In Section 3, we will introduce the Tanh method and the $\left(\frac{G'}{G}\right)$-expansion method. In Section 4, we will apply these two methods for solving the space-time conformable FZK equation. The space-time conformable FZZ equation will also be studied by the two methods in this section. Finally, in Section 5, we will summarize our results and provide some concluding remarks.

2. **Conformable Fractional Derivatives.** In this section, we just recall two definitions with small properties on Khalil conformable fractional derivatives, see [8].

Definition 2.1. *Let* $f:(0,\infty)\to\mathbb{R}$. *The conformable fractional derivative of order* $0<\alpha<1$ *is defined by*

$$(T^\alpha f)(t)=\frac{\partial^\alpha f(t,x)}{\partial t^\alpha}=\lim_{\varepsilon\to 0}\left(\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}\right),\quad t>0,\ 0<\alpha\le 1. \tag{3}$$

Definition 2.2. *The conformable fractional integral of a function* $f:(0,\infty)\to\mathbb{R}$ *of order* $0<\alpha<1$ *is defined as*

$$(I^\alpha f)(t)=\int_0^t \tau^{\alpha-1}f(\tau)d\tau,\quad 0<\alpha\le 1. \tag{4}$$

The following properties are needed.

$$T^\alpha(af(t)+bg(t))=aT^\alpha f(t)+bT^\alpha g(t),\quad \forall a,b\in\mathbb{R}, \tag{5}$$

$$I^\alpha T^\alpha f(t)=f(t)-f(0), \tag{6}$$

$$(T^\alpha f)(t)=t^{1-\alpha}\frac{df(t)}{dt}. \tag{7}$$

3. **Tanh Method and $\left(\frac{G'}{G}\right)$-Expansion Method.** In this section, we present the Tanh and the $\left(\frac{G'}{G}\right)$-expansion methods. For more details on the two methods, one can consult [4, 5, 7, 13, 14, 15, 17, 18, 27].

We consider the following problem:

$$F\left(u,T_t^\alpha u,T_{x_1}^{\beta_1}u,T_{x_2}^{\beta_2}u,T_{x_3}^{\beta_3}u,T_{x_1}^{2\beta_1}u,\ldots\right)=0, \tag{8}$$

where, u is an unknown function of $x_1,x_2,\ldots,x_n$ and t, F is a polynomial that depends on u and its conformable partial fractional derivatives, $T^\alpha u$, $T^{\beta_i}u$ $(i=1,2,\ldots,n)$ are conformable partial fractional derivatives of u, with $0<\alpha,\beta_i\le 1$ and $T^{2\alpha}u=T^\alpha\left(T^\alpha u\right)$.

To search for traveling wave solutions for the above equation, we use the transformation:

$$u(x_1,x_2,\ldots,x_n,t)=U(\xi), \tag{9}$$

where,

$$\xi = \frac{k_1 x_1^{\beta_1}}{\beta_1} + \frac{k_2 x_2^{\beta_2}}{\beta_2} + \cdots + \frac{k_n x_n^{\beta_n}}{\beta_n} - \frac{ct^{\alpha}}{\alpha}, \tag{10}$$

such that, k_i $(i = 1, 2, \ldots, n)$ and c are arbitrary constants.

So, we get the following ODE

$$G\big(U, U', U'', \ldots\big) = 0, \tag{11}$$

which needs to be integrated.

3.1. **Ideas of Tanh method.** The main steps of the Tanh method for constructing the solutions for Equation (8) can be given as follows.

Step 1: By means of the Tanh method, we can choose the following change of variable:

$$Y = \tanh(\mu\xi), \quad \mu \in \mathbb{R}. \tag{12}$$

Therefore, we obtain the following new expressions:

$$\frac{d}{d\xi} = \mu\big(1 - Y^2\big)\frac{d}{dY}, \tag{13}$$

$$\frac{d^2}{d\xi^2} = \mu^2\big(1 - Y^2\big)\left(-2Y\frac{d}{dY} + \big(1 - Y^2\big)\frac{d^2}{dY^2}\right), \tag{14}$$

$$\frac{d^3}{d\xi^3} = \mu^3\big(1 - Y^2\big)\left(2\big(3Y^2 - 1\big)\frac{d}{dY} - 6Y\big(1 - Y^2\big)\frac{d^2}{dY^2} + \big(1 - Y^2\big)^2\frac{d^3}{dY^3}\right). \tag{15}$$

Then, we use the following finite expansion

$$U(\xi) = S(Y) = \sum_{k=0}^{m} a_k Y^k + \sum_{k=1}^{m} b_k Y^{-k}, \tag{16}$$

where m is a positive integer that needs to be determined and a_k, b_k are constants to be determined later.

The parameter m is usually obtained by balancing between the maximum order nonlinear term and the derivative of the maximum order appearing in Equation (11), and we get

$$U^r \to rm \quad U^{(r)} \to m + r. \tag{17}$$

Step 2: We substitute Equation (12) together with Equation (16) in Equation (11) and using Maple, we find μ, a_k $(k = 0, 1, 2, \ldots, m)$, b_k $(k = 1, 2, \ldots, m)$.

Step 3: We insert the values that have been found in Step 2 into Equation (16) along with Equation (12), and we construct closed-form traveling wave solutions of Equation (11) from which we find the solutions of Equation (8).

3.2. **Idea of $\left(\frac{G'}{G}\right)$-expansion method.** The main steps of $\left(\frac{G'}{G}\right)$-expansion method can be given as follows.

Step 1: Suppose that the solutions of Equation (11) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'}{G}\right)^i, \ a_m \neq 0, \tag{18}$$

where, a_i are constants, m is positive integer, and $G = G(\xi)$ satisfies the following ODE:

$$G'' + \lambda G' + \mu G = 0, \tag{19}$$

where, λ and μ are constants to be determined later.

Step 2: By substituting Equations (18) and (19) into Equation (11), and collecting all terms with the same order of $\left(\frac{G'}{G}\right)$ together, then by setting each coefficient to zero, we obtained a set of algebraic equations for a_i, c, k_1, k_2 and k_3.

Step 3: We solve the system of algebraic equations obtained in Step 2, for a_i, $i = 0, 1, \ldots, m$, c, k_1, k_2 and k_3. Then, we substitute a_i $(i = 0, 1, \ldots, m)$, c, k_1, k_2, k_3 and the solutions of Equation (19) into Equation (18), and we can obtain the set of the solutions of Equation (8).

4. **New Traveling Waves Solutions.** In this section, we apply the above proposed method to finding traveling wave solutions for the FZK and FZZ equations.

4.1. **Space-time conformable FZK equation.** We consider the following version of the space-time conformable FZK equation:

$$\begin{aligned} &T_t^{\alpha}u(t,x,y,z) + Au(t,x,y,z)T_x^{\beta_1}u(t,x,y,z) + T_x^{2\beta_1}u(t,x,y,z) \\ &+ T_y^{2\beta_2}u(t,x,y,z) + T_z^{2\beta_3}u(t,x,y,z) = 0, \end{aligned} \tag{20}$$

where, $T_x^{\beta_1}, T_y^{\beta_2}, T_z^{\beta_3}, T_t^{\alpha}$ are the conformable fractional derivatives, with $0 < \alpha, \beta_1, \beta_2, \beta_3 \leq 1$ and A is constant.

We take

$$u(t,x,y,z) = U(\xi), \quad \xi = \frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} + \frac{k_3 z^{\beta_3}}{\beta_3} - \frac{ct^{\alpha}}{\alpha}, \tag{21}$$

where, k_1, k_2, k_3, c are constants.

Therefore,

$$\begin{aligned} &T_t^{\alpha}u = -cU_\xi, \quad T_x^{\beta_1}u = k_1U_\xi, \quad T_x^{2\beta_1}u = k_1^2U_{\xi\xi}, \quad T_y^{2\beta_2}u = k_2^2U_{\xi\xi}, \\ &T_z^{2\beta_3}u = k_3^2U_{\xi\xi}. \end{aligned} \tag{22}$$

Substituting Equations (21) and (22) into Equation (20), we have

$$-cU_\xi + Ak_1UU_\xi + k_1^2U_{\xi\xi} + k_2^2U_{\xi\xi} + k_3^2U_{\xi\xi} = 0;$$

therefore, we obtain

$$kU_\xi - cU + k'U^2 = 0, \tag{23}$$

where, $k = k_1^2 + k_2^2 + k_3^2$ and $k' = \frac{1}{2}Ak_1$.

4.1.1. *Traveling waves using Tanh method.* For our equation, in the case of Tanh method, we have $m = 1$.

This method admits the following expression:

$$U(\xi) = a_0 + a_1Y + b_1Y^{-1}. \tag{24}$$

Hence, by Equation (13), we have

$$U_\xi = \mu\left(1 - Y^2\right)\frac{dU}{dY} = \mu a_1 + \mu b_1 - \mu b_1Y^{-2} - \mu a_1Y^2. \tag{25}$$

Putting Equation (25) in Equation (23), we obtain

$$\begin{aligned} &k\mu a_1 + k\mu b_1 - ca_0 + k'a_0^2 + 2k'a_1b_1 + \left(2k'a_0a_1 - ca_1\right)Y + \left(k'a_1^2 - k\mu a_1\right)Y^2 \\ &+ \left(2k'a_0b_1 - cb_1\right)Y^{-1} + \left(k'b_1^2 - k\mu b_1\right)Y^{-2} = 0. \end{aligned}$$

This gives

$$\begin{aligned}&k\mu a_1 + k\mu b_1 - ca_0 + k'a_0^2 + 2k'a_1b_1 = 0\\&2k'a_0a_1 - ca_1 = 0\\&k'a_1^2 - k\mu a_1 = 0\\&2k'a_0b_1 - cb_1 = 0\\&k'b_1^2 - k\mu b_1 = 0.\end{aligned} \tag{26}$$

By solving the above system, we can obtain the following set of solutions.

Case 1:

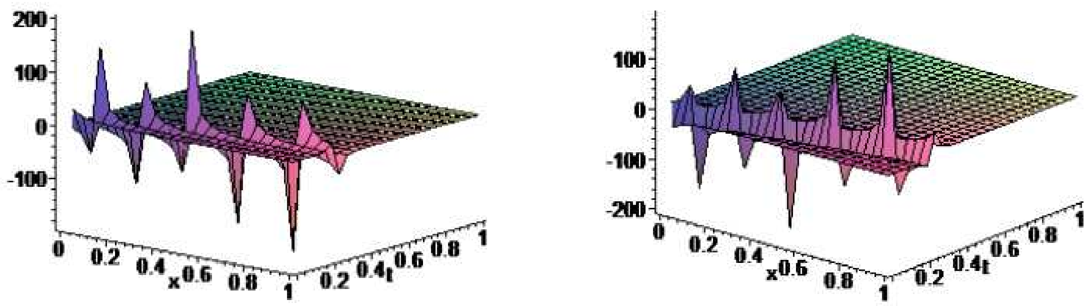
$$a_0 = \frac{2\mu k}{Ak_1}, \quad a_1 = 0, \quad b_1 = \frac{2\mu k}{Ak_1}, \quad c = 2\mu k.$$

Substituting these constants into Equation (24), we get the following exact solution:

$$u(t,x,y,z) = \frac{2\mu k}{Ak_1} + \frac{2\mu k}{Ak_1}\coth\mu\left(\frac{k_1x^{\beta_1}}{\beta_1} + \frac{k_2y^{\beta_2}}{\beta_2} + \frac{k_3z^{\beta_3}}{\beta_3} - \frac{2\mu kt^{\alpha}}{\alpha}\right). \tag{27}$$

Figure 1 shows the graph of solution (27) for some particular values of the parameters.

$$A = \mp 1, \quad \beta_1 = \frac{4}{5}, \quad \alpha = 1, \quad k_1 = 1, \quad k_2 = \frac{1}{2}, \quad k_3 = \frac{1}{4}, \quad \mu = 2.$$



(a) (b)

FIGURE 1. Graph of Equation (27) for $y = z = 0$

Case 2:

$$a_0 = -\frac{2\mu k}{Ak_1}, \quad a_1 = 0, \quad b_1 = \frac{2\mu k}{Ak_1}, \quad c = -2\mu k.$$

By substitution into Equation (24), we obtain

$$u(t,x,y,z) = -\frac{2\mu k}{Ak_1} + \frac{2\mu k}{Ak_1}\coth\mu\left(\frac{k_1x^{\beta_1}}{\beta_1} + \frac{k_2y^{\beta_2}}{\beta_2} + \frac{k_3z^{\beta_3}}{\beta_3} + \frac{2\mu kt^{\alpha}}{\alpha}\right). \tag{28}$$

Case 3:

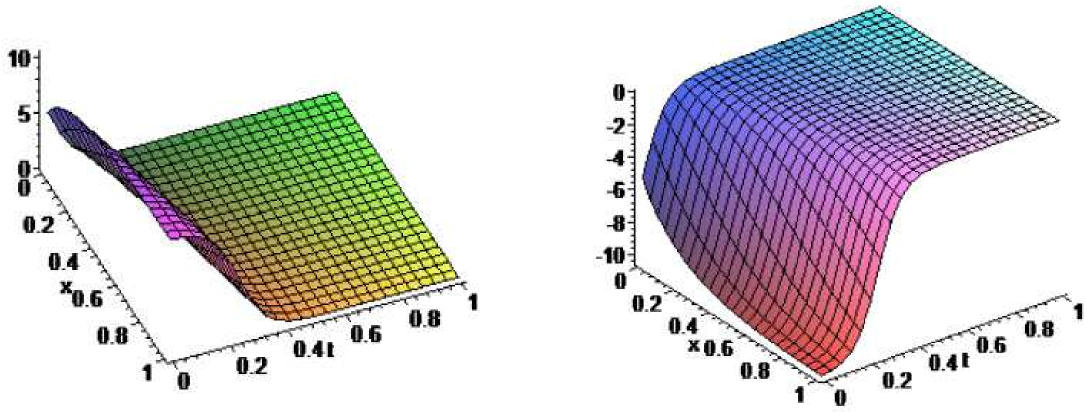
$$a_0 = \frac{2\mu k}{Ak_1}, \quad a_1 = \frac{2\mu k}{Ak_1}, \quad b_1 = 0, \quad c = 2\mu k.$$

Hence, we get

$$u(t,x,y,z) = \frac{2\mu k}{Ak_1} + \frac{2\mu k}{Ak_1}\tanh\mu\left(\frac{k_1x^{\beta_1}}{\beta_1} + \frac{k_2y^{\beta_2}}{\beta_2} + \frac{k_3z^{\beta_3}}{\beta_3} - \frac{2\mu t^{\alpha}}{\alpha}\right). \tag{29}$$

Figure 2 shows the graph of solution (29) under some particular values.

$$A = \mp 1, \quad \beta_1 = \frac{4}{5}, \quad \alpha = 1, \quad k_1 = 1, \quad k_2 = \frac{1}{2}, \quad k_3 = \frac{1}{4}, \quad \mu = 2.$$



(a) (b)

FIGURE 2. Graph of Equation (29) for $y = z = 0$

Case 4:

$$a_0 = -\frac{2\mu k}{Ak_1}, \quad a_1 = \frac{2\mu k}{Ak_1}, \quad b_1 = 0, \quad c = -2\mu k.$$

Hence, we obtain

$$u(t,x,y,z) = -\frac{2\mu k}{Ak_1} + \frac{2\mu k}{Ak_1} \tanh \mu \left(\frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} + \frac{k_3 z^{\beta_3}}{\beta_3} + \frac{2\mu k t^{\alpha}}{\alpha} \right). \tag{30}$$

Case 5:

$$a_0 = \frac{4\mu k}{Ak_1}, \quad a_1 = \frac{2\mu k}{Ak_1}, \quad b_1 = \frac{2\mu k}{Ak_1}, \quad c = 4\mu k.$$

So, we can write

$$\begin{aligned} u(t,x,y,z) = {} & \frac{4\mu k}{Ak_1} + \frac{2\mu k}{Ak_1} \tanh \mu \left(\frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} + \frac{k_3 z^{\beta_3}}{\beta_3} - \frac{4\mu k t^{\alpha}}{\alpha} \right) \\ & + \frac{2\mu k}{Ak_1} \coth \mu \left(\frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} + \frac{k_3 z^{\beta_3}}{\beta_3} - \frac{4\mu k t^{\alpha}}{\alpha} \right). \end{aligned} \tag{31}$$

Figure 3 shows the graph of Equation (31) for some values of the parameters.

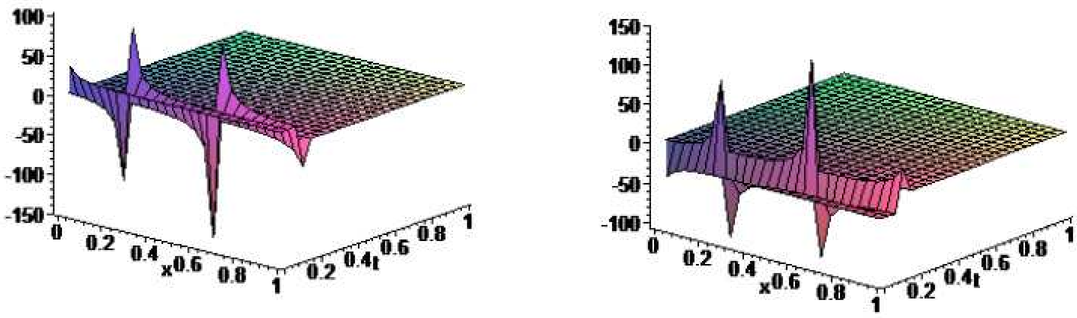
$$A = \mp 1, \quad \beta_1 = \frac{4}{5}, \quad \alpha = 1, \quad k_1 = 1, \quad k_2 = \frac{1}{2}, \quad k_3 = \frac{1}{4}, \mu = 2.$$

Case 6:

$$a_0 = -\frac{4\mu k}{Ak_1}, \quad a_1 = \frac{2\mu k}{Ak_1}, \quad b_1 = \frac{2\mu k}{Ak_1}, \quad c = -4\mu k.$$

Substituting the results into Equation (24), we obtain

$$\begin{aligned} u(t,x,y,z) = {} & -\frac{4\mu k}{Ak_1} + \frac{2\mu k}{Ak_1} \tanh \mu \left(\frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} + \frac{k_3 z^{\beta_3}}{\beta_3} + \frac{4\mu k t^{\alpha}}{\alpha} \right) \\ & + \frac{2\mu k}{Ak_1} \coth \mu \left(\frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} + \frac{k_3 z^{\beta_3}}{\beta_3} + \frac{4\mu k t^{\alpha}}{\alpha} \right). \end{aligned} \tag{32}$$



(a) (b)

FIGURE 3. Graph of Equation (31) for $y = z = 0$

4.1.2. *Traveling waves using* $\left(\frac{G'}{G}\right)$*-expansion method.* Suppose that the solutions of Equation (11) can be expressed using a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right), \tag{33}$$

such that a_0 and a_1 are constants to be determined later.

We have

$$U'(\xi) = -a_1\mu - a_1\lambda \left(\frac{G'}{G}\right) - a_1 \left(\frac{G'}{G}\right)^2, \tag{34}$$

and

$$U^2(\xi) = a_0^2 + 2a_0a_1 \left(\frac{G'}{G}\right) + a_1^2 \left(\frac{G'}{G}\right)^2. \tag{35}$$

Putting Equations (33), (34) and (35) in Equation (23), we have

$$\begin{aligned} &-ka_1\mu - ka_1\lambda \left(\frac{G'}{G}\right) - ka_1 \left(\frac{G'}{G}\right)^2 - ca_0 - ca_1 \left(\frac{G'}{G}\right) + k'a_0^2 \\ &+ 2k'a_0a_1 \left(\frac{G'}{G}\right) + k'a_1^2 \left(\frac{G'}{G}\right)^2 = 0. \end{aligned} \tag{36}$$

This allows us to get

$$\begin{aligned} &-ka_1\mu - ca_0 + k'a_0^2 = 0 \\ &-ka_1\lambda - ca_1 + 2k'a_0a_1 = 0 \\ &k'a_1^2 - ka_1 = 0. \end{aligned}$$

Hence, it yields that

$$a_0 = \frac{k}{3Ak_1}\left(\sqrt{\lambda^2 + 12\mu} - \lambda\right), \quad a_1 = \frac{2k}{Ak_1}, \quad c = -\frac{2}{3}\lambda k - \frac{1}{3}k\sqrt{\lambda^2 + 12\mu}, \tag{37}$$

where, λ and μ are arbitrary constants.

Substituting Equation (37) into Equation (33), we obtain

$$U(\xi) = \frac{k}{3Ak_1}\left(\sqrt{\lambda^2 + 12\mu} - \lambda\right) - \left(\frac{2}{3}\lambda k + \frac{1}{3}k\sqrt{\lambda^2 + 12\mu}\right)\left(\frac{G'}{G}\right). \tag{38}$$

By substituting the general solution of Equation (19) into Equation (38), we have three types of traveling wave solutions for the above conformable FZK equation as follows.

When $\lambda^2 - 4\mu > 0$, we obtain the following traveling wave solution

$$U(\xi) = \frac{k}{3Ak_1}\left(\sqrt{\lambda^2+12\mu}-\lambda\right) + \left(\frac{1}{3}\lambda^2 k + \frac{1}{6}\lambda k\sqrt{\lambda^2+12\mu}\right) - \frac{k\sqrt{\lambda^2-4\mu}}{6}\left(2\lambda+\sqrt{\lambda^2+12\mu}\right)\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}, \tag{39}$$

where, $\xi = \frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} + \frac{k_3 z^{\beta_3}}{\beta_3} + \left(2\lambda + \sqrt{\lambda^2+12\mu}\right)\frac{kt^{\alpha}}{3\alpha}$ and C_1, C_2 are arbitrary constants.

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of Equation (39) can be written as

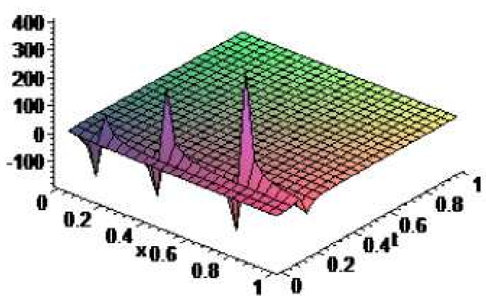
$$U(\xi) = \frac{1}{2}\lambda^2 k\left(1 - \tanh\left(\frac{\lambda k_1 x^{\beta_1}}{2\beta_1} + \frac{\lambda k_2 y^{\beta_2}}{2\beta_2} + \frac{\lambda k_3 z^{\beta_3}}{2\beta_3} + \frac{\lambda^2 k t^{\alpha}}{2\alpha}\right)\right). \tag{40a}$$

If $C_1 = 0$, $C_2 \neq 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of Equation (39) can be written as

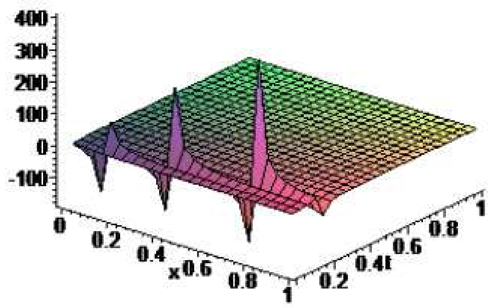
$$U(\xi) = \frac{1}{2}\lambda^2 k\left(1 - \coth\left(\frac{\lambda k_1 x^{\beta_1}}{2\beta_1} + \frac{\lambda k_2 y^{\beta_2}}{2\beta_2} + \frac{\lambda k_3 z^{\beta_3}}{2\beta_3} + \frac{\lambda^2 k t^{\alpha}}{2\alpha}\right)\right). \tag{40b}$$

Figure 4 shows the graph of solution (40a) for some particular values.

$$A = \mp 2, \quad \beta_1 = \frac{1}{2}, \quad \alpha = \frac{3}{4}, \quad k_1 = -\frac{1}{2}, \quad k_2 = \frac{3}{2}, \quad k_3 = \frac{3}{4}, \quad \mu = 1.$$



(a)



(b)

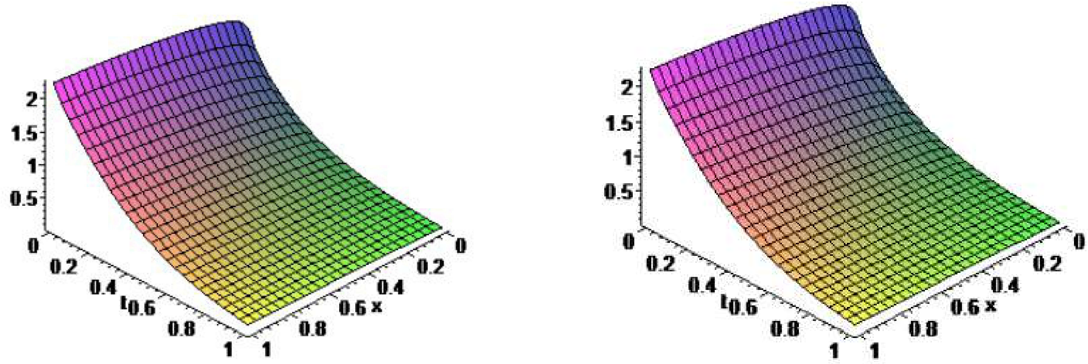
FIGURE 4. Graph of Equation (40a) for $y = z = 0$

Figure 5 shows the graph of solution Equation (40b) for some values of arbitrary constants.

$$A = \mp 2, \quad \beta_1 = \frac{1}{2}, \quad \alpha = \frac{3}{4}, \quad k_1 = -\frac{1}{2}, \quad k_2 = \frac{3}{2}, \quad k_3 = \frac{3}{4}, \quad \mu = 1.$$

When $\lambda^2 - 4\mu = 0$, we obtain

$$U(\xi) = \frac{k}{3Ak_1}\left(\sqrt{\lambda^2+12\mu}-\lambda\right) + \frac{\lambda k}{6}\left(2\lambda+\sqrt{\lambda^2+12\mu}\right) - \frac{k}{3}\left(2\lambda+\sqrt{\lambda^2+12\mu}\right)\frac{C_2}{C_1+C_2\xi}. \tag{41}$$



(a) (b)

FIGURE 5. Graph of solution (40b) for $y = z = 0$

When $\lambda^2 - 4\mu < 0$, we obtain

$$\begin{aligned} U(\xi) \;=\; & \frac{k}{3Ak_1}\left(\sqrt{\lambda^2+12\mu}-\lambda\right)+\frac{\lambda k}{6}\left(2\lambda+\sqrt{\lambda^2+12\mu}\right) \\ & -\frac{k}{6}\sqrt{4\mu-\lambda^2}\left(2\lambda+\sqrt{\lambda^2+12\mu}\right)\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}. \end{aligned} \tag{42}$$

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of Equation (42) can be written as

$$U(\xi) = \frac{1}{2}\lambda^2 k - \frac{k\lambda^2}{2}\left(\tan\frac{\lambda}{2}\left(\frac{k_1 x^{\beta_1}}{\beta_1}+\frac{k_2 y^{\beta_2}}{\beta_2}+\frac{k_3 z^{\beta_3}}{\beta_3}+\frac{\lambda k t^{\alpha}}{\alpha}\right)\right). \tag{43a}$$

Taking $C_1 = 0$, $C_2 \neq 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of Equation (42) can be written as

$$U(\xi) = \frac{1}{2}\lambda^2 k - \frac{k\lambda^2}{2}\left(\cot\frac{\lambda}{2}\left(\frac{k_1 x^{\beta_1}}{\beta_1}+\frac{k_2 y^{\beta_2}}{\beta_2}+\frac{k_3 z^{\beta_3}}{\beta_3}+\frac{\lambda k t^{\alpha}}{\alpha}\right)\right). \tag{43b}$$

Figure 6 shows the graph of solution (43a) for some values.

$$A = \mp 2, \quad \beta_1 = \frac{1}{2}, \quad \alpha = \frac{3}{4}, \quad k_1 = -\frac{1}{2}, \quad k_2 = \frac{3}{2}, \quad k_3 = \frac{3}{4}, \quad \mu = 1.$$

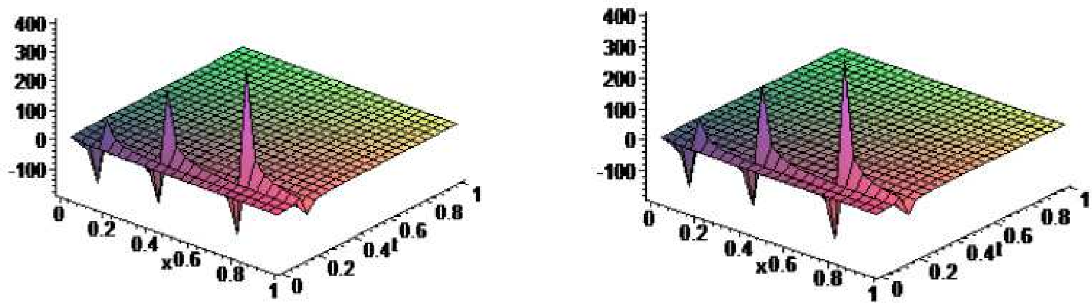
Figure 7 shows the graph of solution Equation (43b) for some values.

$$A = \mp 2, \quad \beta_1 = \frac{1}{2}, \quad \alpha = \frac{3}{4}, \quad k_1 = -\frac{1}{2}, \quad k_2 = \frac{3}{2}, \quad k_3 = \frac{3}{4}, \quad \mu = 1.$$

4.2. **Space-time conformable FZZ equation.** We consider the space-time conformable FZZ equation:

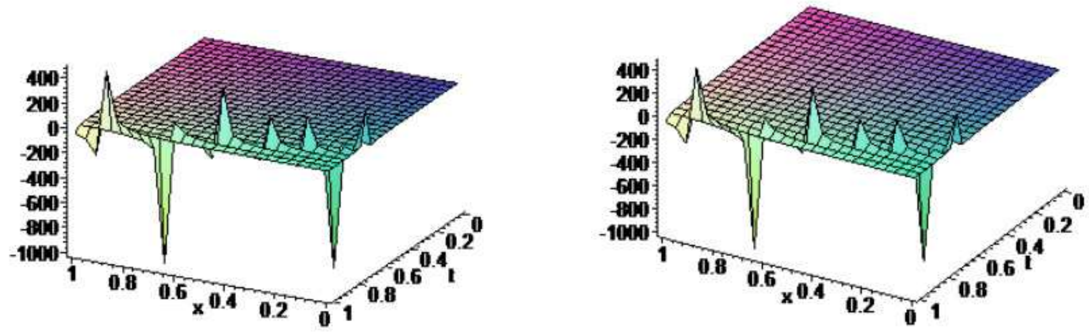
$$\begin{aligned} & T_t^{2\alpha}u(t,x,y)\left(\frac{T_x^{\beta_1}T_y^{\beta_2}u(t,x,y)}{u(t,x,y)}\right) - T_x^{2\beta_1}u(t,x,y)\left(\frac{T_x^{\beta_1}T_y^{\beta_2}u(t,x,y)}{u(t,x,y)}\right) \\ & + 2T_t^{\alpha}u(t,x,y)\left(T_x^{\beta_1}u^2(t,x,y)\right) = 0, \end{aligned} \tag{44}$$

where, $T_x^{\beta_1}$, $T_y^{\beta_2}$, T_t^{α} are the conformable fractional derivative, with $0 < \alpha, \beta_1, \beta_2 \leq 1$.



(a) (b)

FIGURE 6. Graphic of Equation (43a) for $y = z = 0$



(a) (b)

FIGURE 7. Graph of solution (43b) for $y = z = 0$

We introduce the following transformation:

$$u(t, x, y) = U(\xi), \quad \xi = \frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} - \frac{ct^\alpha}{\alpha}, \tag{45}$$

where k_1, k_2, c are constants.

We have

$$\begin{aligned} &T_t^\alpha u = -cU_\xi, \quad T_t^{2\alpha} u = c^2 U_{\xi\xi}, \quad T_x^{2\beta_1} u = k_1^2 U_{\xi\xi}, \quad T_x^{\beta_1} T_y^{\beta_2} u = k_1 k_2 U_{\xi\xi}, \\ &T_x^{\beta_1} u^2 = 2k_1 U U_\xi. \end{aligned} \tag{46}$$

Substituting Equations (45) and (46) in Equation (44), we have

$$k \left(U_{\xi\xi}\right)^2 - k' \left(UU_\xi\right)^2 = 0, \tag{47}$$

where $k = \mu^4 k_1 k_2 \left(c^2 - k_1^2\right)$ and $k' = \mu c k_1$.

4.2.1. *Traveling waves using Tanh method.* We note that in this case, we have $m = 1$.

The Tanh method admits the use of the finite expansion

$$U(\xi) = a_0 + a_1 Y + b_1 Y^{-1}. \tag{48}$$

Thanks to Equations (13) and (14), we have

$$\begin{aligned} U_\xi &= \mu a_1 + \mu b_1 - \mu a_1 Y^2 - \mu b_1 Y^{-2}, \\ U_{\xi\xi} &= -2\mu^2 a_1 Y + 2\mu^2 b_1 Y^{-3} + 2\mu^2 a_1 Y^3 - 2\mu^2 b_1 Y^{-1}. \end{aligned} \tag{49}$$

Putting Equations (48) and (49) in Equation (47), we obtain

$$
\begin{aligned}
&4ka_1b_1 - k'\left(a_0a_1 + a_1b_1\right)^2 - 4k'a_1^2b_1^2 - 2k'a_0^2a_1b_1 \\
&+ \left(2k'a_1^3\left(a_0 + b_1\right) - 2k'a_0a_1^2b_1 + 2k'a_0a_1b_1^2\right)Y \\
&+ \left(2k'a_1b_1^2\left(a_0 + b_1\right) - 2k'a_0a_1b_1^2 + 2k'a_0a_1^2b_1\right)Y^{-1} \\
&+ \left(k\left(a_1^2 - 2a_1b_1\right) - k'a_1^2\left(a_1^2 - 2a_0^2 - 2a_0b_1 - 2b_1^2\right)\right)Y^2 \\
&+ \left(k\left(b_1^2 - 2a_1b_1\right) - k'\left(b_1^4 - 2a_1^2b_1^2 - 2a_0^2a_1b_1 - 2a_0a_1b_1^2\right)\right)Y^{-2} \\
&+ 2k'a_1^3\left(2a_0 + b_1\right)Y^3 + 2k'b_1^2\left(a_0a_1 + a_0b_1 + a_1b_1\right)Y^{-3} \\
&+ \left(2k'a_1^4 - k'a_0^2a_1^2 - 2ka_1^2\right)Y^4 + \left(2k'b_1^4 - k'a_0^2b_1^2 - 2kb_1^2\right)Y^{-4} \\
&- 2k'a_0a_1^3Y^5 - 2k'a_0b_1^3Y^{-5} + \left(ka_1^2 - k'a_1^4\right)Y^6 + \left(kb_1^2 - k'b_1^4\right)Y^{-6} = 0.
\end{aligned}
$$

This allows us to obtain

$$
\begin{aligned}
&4ka_1b_1 - k'\left(a_0a_1 + a_1b_1\right)^2 - 4k'a_1^2b_1^2 - 2k'a_0^2a_1b_1 = 0 \\
&k'a_1^3\left(a_0 + b_1\right) - k'a_0a_1^2b_1 + k'a_0a_1b_1^2 = 0 \\
&k'a_1b_1^2\left(a_0 + b_1\right) - k'a_0a_1b_1^2 + k'a_0a_1^2b_1 = 0 \\
&k\left(a_1^2 - 2a_1b_1\right) - k'a_1^2\left(a_1^2 - 2a_0^2 - 2a_0b_1 - 2b_1^2\right) = 0 \\
&k\left(b_1^2 - 2a_1b_1\right) - k'\left(b_1^4 - 2a_1^2b_1^2 - 2a_0^2a_1b_1 - 2a_0a_1b_1^2\right) = 0 \\
&2k'a_1^3\left(2a_0 + b_1\right) = 0 \\
&2k'b_1^2\left(a_0a_1 + a_0b_1 + a_1b_1\right) = 0 \\
&2k'a_1^4 - k'a_0^2a_1^2 - 2ka_1^2 = 0 \\
&2k'b_1^4 - k'a_0^2b_1^2 - 2kb_1^2 = 0 \\
&k'a_0a_1^3 = 0 \\
&k'a_0b_1^3 = 0 \\
&ka_1^2 - k'a_1^4 = 0 \\
&kb_1^2 - k'b_1^4 = 0.
\end{aligned} \tag{50}
$$

Hence, we have

$$
a_0 = 0, \quad a_1 = a_1 \quad b_1 = 0, \quad k_2 = \frac{ca_1^2}{\mu^3\left(c^2 - k_1^2\right)}.
$$

Substituting these results into Equation (48), we get the following exact solution:

$$
u(t,x,y) = a_1 \tanh \mu\left(\frac{k_1x^{\beta_1}}{\beta_1} + \frac{ca_1^2}{\mu^3\left(c^2 - k_1^2\right)}\frac{y^{\beta_2}}{\beta_2} - \frac{ct^\alpha}{\alpha}\right). \tag{51}
$$

4.2.2. *Traveling waves using* $\left(\frac{G'}{G}\right)$*-expansion method.* Suppose that the solutions of (11) can be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows:

$$
U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \tag{52}
$$

such that a_0 and a_1 are constants to be determined later.

We have

$$U'(\xi) = -a_1\mu - a_1\lambda\left(\frac{G'}{G}\right) - a_1\left(\frac{G'}{G}\right)^2,$$
$$U''(\xi) = a_1\lambda\mu + \left(a_1\lambda^2 + 2\mu a_1\right)\left(\frac{G'}{G}\right) + 3a_1\lambda\left(\frac{G'}{G}\right)^2 + 2a_1\left(\frac{G'}{G}\right)^3, \tag{53}$$

and

$$U^2(\xi) = a_0^2 + 2a_0a_1\left(\frac{G'}{G}\right) + a_1^2\left(\frac{G'}{G}\right)^2,$$
$$\left(U'(\xi)\right)^2 = a_1^2\mu^2 + 2a_1^2\mu\lambda\left(\frac{G'}{G}\right) + a_1^2\left(\lambda^2 + 2\mu\right)\left(\frac{G'}{G}\right)^2 + 2a_1^2\lambda\left(\frac{G'}{G}\right)^3 + a_1^2\left(\frac{G'}{G}\right)^4. \tag{54}$$

Putting Equations (52), (53) and (54) in Equation (44), we have

$$-k'a_0^2a_1^2\mu^2 + \left(ka_1^2\lambda^2\mu^2 + 2k\lambda\mu a_1^2\left(\lambda^2 + 2\mu\right) - 2k'a_0^2a_1^2\mu\lambda - 2k'a_0a_1^3\mu^2\right)\left(\frac{G'}{G}\right)$$
$$+ \left(ka_1^2\left(\lambda^2 + 2\mu^2 + 6\lambda^2\mu + 6\lambda^3 + 12\lambda\mu\right) - k'a_0^2a_1^2\left(\lambda^2 + 2\mu\right) - 4k'a_0a_1^3\mu\lambda - k'a_1^4\mu^2\right)\left(\frac{G'}{G}\right)^2$$
$$+ \left(4ka_1^2\left(\lambda\mu + \lambda^2 + 2\mu\right) - 2k'a_0^2a_1^2\lambda - 2k'a_0a_1^3\left(\lambda^2 + 2\mu\right) - 2k'a_1^4\mu\lambda\right)\left(\frac{G'}{G}\right)^3$$
$$+ \left(9ka_1^2\lambda^2 - k'a_0^2a_1^2 - 4k'a_0a_1^3\lambda - k'a_1^4\left(\lambda^2 + 2\mu\right)\right)\left(\frac{G'}{G}\right)^4$$
$$+ \left(12ka_1^2\lambda - k'\left(a_0a_1^3 + 2a_1^4\lambda\right)\right)\left(\frac{G'}{G}\right)^5 + \left(4ka_1^2 - k'a_1^4\right)\left(\frac{G'}{G}\right)^6 = 0.$$

Consequently,

$$\begin{aligned}
&k'a_0^2a_1^2\mu^2 = 0\\
&ka_1^2\lambda^2\mu^2 + 2k\lambda\mu a_1^2\left(\lambda^2 + 2\mu\right) - 2k'a_0^2a_1^2\mu\lambda - 2k'a_0a_1^3\mu^2 = 0\\
&ka_1^2\left(\lambda^2 + 2\mu^2 + 6\lambda^2\mu + 6\lambda^3 + 12\lambda\mu\right) - k'a_0^2a_1^2\left(\lambda^2 + 2\mu\right) - 4k'a_0a_1^3\mu\lambda - k'a_1^4\mu^2 = 0\\
&4ka_1^2\left(\lambda\mu + \lambda^2 + 2\mu\right) - 2k'a_0^2a_1^2\lambda - 2k'a_0a_1^3\left(\lambda^2 + 2\mu\right) - 2k'a_1^4\mu\lambda = 0\\
&9ka_1^2\lambda^2 - k'a_0^2a_1^2 - 4k'a_0a_1^3\lambda - k'a_1^4\left(\lambda^2 + 2\mu\right) = 0\\
&12ka_1^2\lambda - k'\left(a_0a_1^3 + 2a_1^4\lambda\right) = 0\\
&4ka_1^2 - k'a_1^4 = 0.
\end{aligned} \tag{55}$$

Therefore,

$$a_0 = a_0, \quad a_1 = a_1, \quad k_1 = 0.$$

By substituting the general solution of Equation (19) into Equation (52), we obtain three types of traveling wave solutions of the above space time conformable FZZ equation as follows.

When $\lambda^2 - 4\mu > 0$, we obtain the following traveling wave solution:

$$U(\xi) = a_0 - \frac{\lambda a_1}{2} + \frac{a_1\sqrt{\lambda^2 - 4\mu}}{2} \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}, \tag{56}$$

where, $\xi = \frac{k_1 x^{\beta_1}}{\beta_1} + \frac{k_2 y^{\beta_2}}{\beta_2} - \frac{ct^\alpha}{\alpha}$ and C_1, C_2 are arbitrary constants.

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of Equation (56) can be written as

$$U(\xi) = a_0 - \frac{\lambda a_1}{2} + \frac{a_1\sqrt{\lambda^2 - 4\mu}}{2} \tanh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right). \tag{57a}$$

If $C_1 = 0$, $C_2 \neq 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of (56) can be written as

$$U(\xi) = a_0 - \frac{\lambda a_1}{2} + \frac{a_1\sqrt{\lambda^2 - 4\mu}}{2} \coth\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2}\xi\right). \tag{57b}$$

When $\lambda^2 - 4\mu = 0$, we obtain

$$U(\xi) = a_0 - \frac{\lambda a_1}{2} + \frac{a_1 C_2}{C_1 + C_2\xi}. \tag{58}$$

When $\lambda^2 - 4\mu < 0$, we obtain

$$U(\xi) = a_0 - \frac{\lambda a_1}{2} + \frac{a_1\sqrt{4\mu - \lambda^2}}{2} \frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}. \tag{59}$$

In particular, if $C_1 \neq 0$, $C_2 = 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of Equation (59) can be written as

$$U(\xi) = a_0 - \frac{\lambda a_1}{2} - \frac{a_1\sqrt{4\mu - \lambda^2}}{2} \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right). \tag{60a}$$

If $C_1 = 0$, $C_2 \neq 0$, $\lambda > 0$, $\mu = 0$, then the traveling wave solution of Equation (59) can be written as

$$U(\xi) = a_0 - \frac{\lambda a_1}{2} + \frac{a_1\sqrt{4\mu - \lambda^2}}{2} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\xi\right). \tag{60b}$$

5. **Conclusion.** In conclusion, our study has focused on the space-time FZK equation and the space-time FZZ equation using conformable fractional derivatives. We have applied the Tanh and the $\left(\frac{G'}{G}\right)$-expansion methods to finding new sets of traveling wave solutions for these two fractional evolution problems. In addition, we have plotted graphs for some of the obtained new traveling waves. The obtained solutions are significant in the study of the dynamics and behavior of these fractional equations. Our research contributes to the development of conformable fractional calculus and also to the above two space-time fractional evolution equations.

Further investigations could focus on exploring the properties of the obtained solutions and their potential applications. Our results may have practical applications in the fields of material science, optics, and acoustics.

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Author Biography

Ahmed Anber is an Associate Professor of Mathematics at the University of Sciences and the Technology of Oran (USTO), Department of Mathematics, Faculty of Mathematics and Informatic. His fields of interests include fractional differential equations, integral inequalities and numerical methods for differential equations. He has obtained his Ph.D. degree at UMAB University. He has taught mathematical analysis, linear algebra and complex analysis.

Iqbal Jebril is a Professor at the Department of Mathematics, Al Zaytoonah University of Jordan, Amman, Jordan. He obtained his Ph.D. degree in 2005 from National University of Malaysia (UKM). His fields of interest include functional analysis, operator theory and fuzzy logic. He had several prestigious Journal/Conference publications and was in various journals and conferences committees.

Zoubir Dahmani is a Professor of Mathematics at Saad Dahleb Blida 1 University and a member of Laboratory of LPAM of the University of Mostaganem. His fields of interests include differential equations and dynamical systems, inequality theory, fractional calculus, fixed point theory, numerical methods for fractional PDEs, probability and statistics. He has obtained his Ph.D. degree at USTHB University of Algiers and La Rochelle (France), 2009. He has published more than 250 research papers on Pure and Applied Maths. In 2021, 2022, and 2023, he has been selected as the best cited researcher at UMAB University. He has taught differential equations, operator theory, numerical analysis and applied mathematics.

Nabil Bedjaoui is a Professor of Mathematics at the University of Picardie Jules Vernes, France and a member of Laboratory of LAMFA. His fields of research are differential equations and applied analysis. He has obtained his Ph.D. degree at Ecole Polytechnique, Palaiseau (France), 1996. He is responsible of Applied Mathematics at his university. Also, he has taught differential equations and applied analysis.

Abdelkader Lamamri is a Professor of Mathematics at the Saad Dahlab Blida 1 University. His research interests include discrete optimization, graph theory and fractional calculus. He has obtained his Ph.D. degree at Saad Dahlab Blida 1 University. He has taught operational research, combinatory optimization and algorithmic for optimization.