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TWO-DIMENSIONAL FRACTIONAL WAVE EQUATION VIA A NEW NUMERICAL APPROACH

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Abstract. *The main goal of this work is to solve the fractional wave equation in two dimensions numerically with the use of some novel fractional formulas. In particular, the proposed approach used to deal with the fractional wave equation introduces two novel fractional difference formulas for approximating the Caputo differentiator of order* δ *and* 2δ*, respectively, where* $0 < \delta \leq 1$*. Such formulas, which are derived based on the Lagrange interpolating polynomial, can generate a system of linear equations that can be solved numerically to obtain, ultimately, good approximate solutions to the fractional wave equation for different fractional-order values.*

Keywords: Two-dimensional fractional wave equation, Fractional calculus, Lagrange interpolating polynomial, Fractional difference formula

1. **Introduction.** Lately, there has been a growing interest in utilizing fractional calculus as a valuable tool to represent various unusual transport phenomena [1, 2]. This heightened attention is attributed to its remarkable capacity for accurately illustrating processes characterized by memory and hereditary traits. Fractional differential equations have found extensive applications across diverse fields, including physics, chemistry, biology,

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polymer rheology, viscoelastic materials, and control theory, as evidenced by [3, 4, 5, 6, 7]. The progress in fractional calculus has spurred significant research efforts towards developing efficient numerical approaches, encompassing finite difference methods, finite element methods, finite volume methods, and spectral methods, see e.g. [8, 9, 10, 11].

Wave phenomena are a common occurrence in the natural world, appearing in diverse manifestations like electromagnetic waves, acoustic waves, and seismic waves. Traditionally, classical wave equations have been the primary means of describing these phenomena, playing a vital role in numerous applications, as referenced in sources [12, 13]. Nevertheless, as our comprehension of intricate wave behaviors has grown, it has become increasingly apparent that classical wave equations frequently lack the capability to fully represent certain crucial features. A notable breakthrough in overcoming these constraints has been the incorporation of fractional calculus into wave equations, as discussed in [14]. Fractional calculus offers a framework to include non-local and non-integer order derivatives, enabling a more intricate and comprehensive portrayal of wave dynamics, as highlighted in [15, 16]. This shift in paradigm has ushered in fresh opportunities for research and practical applications in domains where waves play a fundamental role. Numerous branches of research and engineering heavily rely on the wave equations. In order to represent wave equations, many scholars would replace regular derivatives with fractional derivatives. In particular, in [17], the fractional reduced differential-transform method was utilized to solve the fractional wave. With the use of the homotopy analysis approach, the one- and two-dimensional fractional wave equations were solved in [18] with the help of specific boundary conditions. Zhang et al. [19] also addressed the problem based on the two-dimensional fractional wave equation with supplied boundary conditions. In one such research study, the authors used four distinct fractional-order values to demonstrate the problem's graphical representations.

In this work, the two-dimensional fractional wave equation is handled with the aid of a novel fractional difference formula used to estimate the Caputo derivative operator of order 2δ, where $0 < \delta \leq 1$. As far as we know, providing a numerical solution to the two-dimensional fractional wave equation in such a way that uses a new fractional difference formula has not been dealt with by any investigator till now. This would draw attention to our provided numerical scheme to be considered for other similar problems. However, the remainder of this article is structured as follows. In the subsequent section, we review the essential fundamentals associated with fractional calculus. Moving on to Section 3, we introduce respectively two novel fractional difference formulas for approximating the Caputo differentiator of order δ and 2δ such that $0 < \delta \leq 1$. These approximations will be instrumental in our numerical solution of the two-dimensional fractional wave equation. In Section 4, we validate the effectiveness of our proposed numerical approach by presenting two illustrative examples. Finally, we conclude our work in the concluding section.

2. **Essential Preliminaries.** Within this segment, we shall remember several essential preliminaries and significant initial notions associated with fractional calculus. These pivotal ideas would establish the basis for the core findings we will subsequently unveil.

Definition 2.1. [20, 21] *The fractional Riemann-Liouville integrator of a function $g(z)$ of order $\delta > 0$ can be expressed as*

$$J^{\delta} g(z) = \frac{1}{\Gamma(\delta)} \int_0^z g(v)(z-v)^{\delta-1} dv. \tag{1}$$

In the forthcoming discussion, we examine distinct properties of the fractional Riemann-Liouville integral operator under the conditions where z and η are both greater than zero.

$$1) \quad \text{As } \delta \to 0 \Rightarrow J^{\delta} g(z) \to g(z). \tag{2}$$

$$2)\quad J^{\delta} z^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\delta+\eta+1)} z^{\delta+\eta}, \quad \eta \geq -1. \tag{3}$$

$$3)\quad J^{\delta} J^{\gamma} g(z) = J^{\gamma} J^{\delta} g(z) \quad \delta, \gamma \geq 0. \tag{4}$$

$$4)\quad J^{\delta} J^{\gamma} g(z) = J^{\delta+\gamma} g(z) \quad \delta, \gamma \geq 0. \tag{5}$$

Definition 2.2. [20, 21] *The Caputo fractional differentiator of a function $g(z)$ of order $\delta > 0$ can be expressed as*

$$D^{\delta} g(z) = \frac{1}{\Gamma(n-\delta)} \int_0^z (z-v)^{\delta-1} g^{(n)}(v) dv, \quad n-1 < \delta \leq n, \tag{6}$$

where $n \in \mathbb{N}$ and $z > 0$.

Some of the Caputo differentiator properties are outlined in the following content [20, 21]:

1) $D^{\delta} c = 0,$ where c is constant. (7)

2) $D^{\delta} z^{\rho} = \dfrac{\Gamma(\rho+1)}{\Gamma(\rho-\delta+1)} z^{\rho-\delta},$ where $\rho > \delta - 1$. (8)

3) $D^{\delta}(\omega_1 g(z) + \omega_2 h(z)) = \omega_1 D^{\delta} g(z) + \omega_2 D^{\delta} h(z),$ where ω_1 and ω_2 are constants. (9)

4) $D^{\delta} J^{\delta} g(z) = g(z).$ (10)

5) $J^{\delta} D^{\delta} g(z) = g(z) - \displaystyle\sum_{i=1}^{n} g^{i}(0^+) \frac{z^i}{i!},$ where $z > 0$. (11)

6) $D^{\delta} g(z) = D^n J^{\rho} g(z),$ where $\delta > 0,$ $\rho = n - \delta,$ and n is the smallest integer number greater than δ. (12)

In the upcoming content, we recall one of the most crucial outcomes in fractional calculus that plays a pivotal role in deriving the main conclusions of this study. This outcome, attributed to Odibat and Momani in [21], extends the well-known Taylor theorem to the realm of fractional calculus.

Theorem 2.1. [21] *Suppose that $D^{k\delta} g(z) \in C^{m+1}((c, d])$ for $k = 0, 1, \ldots, m+1$, where $0 < \delta \leq 1$. We can express the expansion of the function g around the point $z = z_0$ as follows:*

$$g(z) = \sum_{i=0}^{m} \frac{(z-z_0)^{i\delta}}{\Gamma(i\delta+1)} D^{i\delta} g(z_0) + \frac{(z-z_0)^{(m+1)\delta}}{\Gamma((m+1)\delta+1)} D^{(m+1)\delta} g(\xi), \tag{13}$$

where $0 < \xi < z$ and $z \in (c, d]$.

Theorem 2.2. [22] *Suppose that there is a set of $m+1$ distinct numbers $t_0, t_1, \ldots, t_m$ in which the function g has values at these numbers. Then, there is a unique m^{th}-Lagrange interpolating polynomial $P(t)$ with*

$$g(t_j) = P(t_j),$$

for each $j = 0, 1, \ldots, m$. This polynomial has the form

$$P(t) = g(t_0) L_{m,0}(t) + \cdots + g(t_m) L_{m,m}(t) = \sum_{j=0}^{m} g(t_j) L_{m,j}(t), \tag{14}$$

for $j = 0, 1, \ldots, m$, where

$$L_{m,j}(t) = \frac{(t-t_0)(t-t_1)\cdots(t-t_{j-1})(t-t_{j+1})\cdots(t-t_m)}{(t_j-t_0)(t_j-t_1)\cdots(t_j-t_{j-1})(t_j-t_{j+1})\cdots(t_j-t_m)} = \prod_{\substack{i=0\\ i\neq j}}^{m} \frac{t-t_i}{t_j-t_i}. \tag{15}$$

For simplification, we denote $L_{m,j}(t)$ as $L_j(t)$, for $j = 0, 1, \ldots, m$.

Theorem 2.3. [22] *Suppose that $g \in C^{m+1}([c,d])$ in which the interval $[c,d]$ has distinct numbers $t_0, t_1, \ldots, t_m$. Then, there is an unknown number $\xi(t)$ between these numbers, and so in (c,d) such that*

$$g(t) = P(t) + \frac{g^{(m+1)}(\xi(t))}{(m+1)!}(t-t_0)(t-t_1)\cdots(t-t_m), \tag{16}$$

where $P(t)$ is previously defined in Equation (14).

3. **Main Results.** This section primarily aims to highlight the main findings of this research. It is divided into two subsections, the first one aims to respectively establish two novel fractional difference formulas for approximating the Caputo differentiator of order δ and 2δ in which $0 < \delta \leq 1$, while the second one aims to address the two-dimensional fractional wave equation with the use of the aforementioned formulas.

3.1. **Fractional difference formulas.** We begin this part by introducing a significant result related to the fractional difference formula that can be employed for approximating the Caputo fractional differentiator of order δ, where $0 < \delta \leq 1$.

Theorem 3.1. *Given that t_0, t_1, t_2 are three distinct points within the interval $[c,d]$ such that $c = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h = d$ with $h > 0$. Assume that the function g is a member of the class $C^3([c,d])$. The fractional difference formula to approximate Caputo differentiator of order δ is of the form*

$$\begin{aligned} D^\delta g(t) = {} & \frac{t^{2-\delta}}{h^2\Gamma(3-\delta)}\left(g(t_0) - 2g(t_1) + g(t_2)\right) \\ & - \frac{t^{1-\delta}}{2h^2\Gamma(2-\delta)}\left(g(t_0)(t_1+t_2) - 2g(t_1)(t_0+t_2) + g(t_2)(t_0+t_1)\right) \\ & + \frac{g^{(3)}(\xi)}{6}\left(\frac{6}{\Gamma(4-\delta)}t^{3-\delta} - \frac{2(t_0+t_1+t_2)}{\Gamma(3-\delta)}t^{2-\delta} + \frac{(t_0t_1+t_0t_2+t_1t_2)}{\Gamma(2-\delta)}t^{1-\delta}\right), \end{aligned} \tag{17}$$

where $0 < \delta \leq 1$ and $\xi \in (c,d)$.

Proof: With the use of Equation (14) coupled with Equation (16), we get

$$g(t) = \sum_{j=0}^{m} g(t_j)L_j(t) + \frac{(t-t_0)\cdots(t-t_m)}{(m+1)!}g^{(m+1)}(\xi). \tag{18}$$

For $m = 2$, we can have

$$g(t) = \sum_{j=0}^{2} g(t_j)L_j(t) + \frac{(t-t_0)(t-t_1)(t-t_2)}{3!}g^{(3)}(\xi), \tag{19}$$

or

$$g(t) = g(t_0)L_0(t) + g(t_1)L_1(t) + g(t_2)L_2(t) + \frac{1}{6}g^{(3)}(\xi)(t-t_0)(t-t_1)(t-t_2).$$

By using Equation (15), we obtain

$$\begin{aligned} g(t) = {} & g(t_0)\frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)} + g(t_1)\frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)} + g(t_2)\frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)} \\ & + \frac{1}{6}g^{(3)}(\xi)(t-t_0)(t-t_1)(t-t_2). \end{aligned} \tag{20}$$

Simplifying the above assertion yields

$$
\begin{aligned}
g(t) = & \frac{g(t_0)}{(t_0-t_1)(t_0-t_2)}\left(t^2-(t_1+t_2)t+t_1t_2\right)+\frac{g(t_1)}{(t_1-t_0)(t_1-t_2)}\left(t^2-(t_0+t_2)t+t_0t_2\right) \\
& +\frac{g(t_2)}{(t_2-t_0)(t_2-t_1)}\left(t^2-(t_0+t_1)t+t_0t_1\right)+\frac{g^{(3)}(\xi)}{6}(t-t_0)(t-t_1)(t-t_2).
\end{aligned}
$$

By using the assumptions $t_1 = t_0 + h$ and $t_2 = t_0 + 2h$, we can then rewrite $g(t)$ in terms of h in the form

$$
\begin{aligned}
g(t) = & \frac{g(t_0)}{2h^2}\left(t^2-(t_1+t_2)t+t_1t_2\right)-\frac{g(t_1)}{h^2}\left(t^2-(t_0+t_2)t+t_0t_2\right) \\
& +\frac{g(t_2)}{2h^2}\left(t^2-(t_0+t_1)t+t_0t_1\right) \\
& +\frac{g^{(3)}(\xi)}{6}\left(t^3-(t_0+t_1+t_2)t^2+(t_0t_1+t_0t_2+t_1t_2)t-t_0t_1t_2\right).
\end{aligned}
$$

By applying Caputo differentiator to the both sides of the above assertion, we get

$$
\begin{aligned}
D^{\delta}g(t) = & \frac{g(t_0)}{2h^2}\left[\frac{2t^{2-\delta}}{\Gamma(3-\delta)}-\frac{(t_1+t_2)t^{1-\delta}}{\Gamma(2-\delta)}\right] \\
& +\frac{g(t_1)}{h^2}\left[\frac{2t^{2-\delta}}{\Gamma(3-\delta)}-\frac{(t_0+t_2)t^{1-\delta}}{\Gamma(2-\delta)}\right]+\frac{g(t_2)}{2h^2}\left[\frac{2t^{2-\delta}}{\Gamma(3-\delta)}-\frac{(t_0+t_1)t^{1-\delta}}{\Gamma(2-\delta)}\right] \\
& +\frac{g^{(3)}(\xi)}{6}\left[\frac{6t^{3-\delta}}{\Gamma(4-\delta)}-\frac{2(t_0+t_1+t_2)t^{2-\delta}}{\Gamma(3-\delta)}+\frac{(t_0t_1+t_2t_1+t_0t_2)t^{1-\delta}}{\Gamma(2-\delta)}\right].
\end{aligned}
$$

This immediately implies

$$
\begin{aligned}
D^{\delta}g(t) = & \frac{g(t_0)}{2h^2}\frac{2t^{2-\delta}}{\Gamma(3-\delta)}-\frac{g(t_1)}{h^2}\frac{2t^{2-\delta}}{\Gamma(3-\delta)}+\frac{g(t_2)}{2h^2}\frac{2t^{2-\delta}}{\Gamma(3-\delta)}-\frac{g(t_0)}{2h^2}\frac{\left((t_1+t_2)t^{1-\delta}\right)}{\Gamma(2-\delta)} \\
& +\frac{g(t_1)}{h^2}\frac{\left((t_0+t_2)t^{1-\delta}\right)}{\Gamma(2-\delta)}-\frac{g(t_2)}{2h^2}\frac{\left((t_0+t_1)t^{1-\delta}\right)}{\Gamma(2-\delta)} \\
& -\frac{g^3(\xi)}{6}\left[\frac{6t^{3-\delta}}{\Gamma(4-\delta)}-\frac{(t_0+t_1+t_2)\left(2t^{2-\delta}\right)}{\Gamma(3-\delta)}+\frac{(t_0t_1+t_2t_1+t_0t_2)t^{1-\delta}}{\Gamma(2-\delta)}\right],
\end{aligned}
$$

which leads consequently to the desired result. □

Corollary 3.1. *Given that t_0, t_1, t_2 are three distinct points within the interval $[c, d]$ such that $c = t_0 < t_1 = t_0 + h < t_2 = t_0 + 2h = d$ with $h > 0$. Assume that the function g is a member of the class $C^3([c, d])$. The fractional difference formula to approximate Caputo differentiator of order 2δ is of the form*

$$
\begin{aligned}
D^{2\delta}g(t) = & \frac{t^{2-2\delta}}{h^2\Gamma(3-2\delta)}\left(g(t_0)-2g(t_1)+g(t_2)\right) \\
& +\frac{g^3(\xi)}{6}\left(\frac{6t^{3-\delta}}{\Gamma(4-\delta)}-\frac{2(t_0+t_1+t_2)t^{2-\delta}}{\Gamma(3-\delta)}+\frac{(t_0t_1+t_0t_2+t_1t_2)t^{1-\delta}}{\Gamma(2-\delta)}\right), \qquad (21)
\end{aligned}
$$

for an unknown $\xi \in (c, d)$ and $t \in [c, d]$.

Proof: This result can be easily held by just operating the Caputo differentiator to Formula (17) once again, ultimately yielding the desired outcome. □

3.2. **Two-dimensional fractional wave equation.** The conventional form of the two-dimensional wave equation can be generally expressed as

$$u_{tt}u(x,y,t) = c^2(u_{xx}(x,y,t) + u_{yy}(x,y,t)),$$

with initial conditions

$$u(x,y,0) = f(x,y), \quad u_t(x,y,0) = 0, \quad 0 < x < l,\ 0 < y < l,$$

and with boundary conditions

$$u(0,y,t) = a_1, \quad u(l,y,t) = b_1, \quad u(x,0,t) = a_2, \quad u(x,l,t) = b_2,$$

where $t > 0$. With the use of the Caputo fractional differentiator (6), the above equation can be expressed in its fractional-order case as follows:

$$\frac{\partial^{2\delta} u(x,y,t)}{\partial t^{2\delta}} = c^2 \left(\frac{\partial^{2\delta} u(x,y,t)}{\partial x^{2\delta}} + \frac{\partial^{2\delta} u(x,y,t)}{\partial y^{2\delta}} \right) \tag{22}$$

with initial conditions

$$u(x,y,0) = f(x,y), \quad \frac{\partial^{\delta}}{\partial t^{\delta}}(x,y,0) = 0, \quad 0 < x < l,\ 0 < y < l,$$

and with boundary conditions

$$u(0,y,t) = a_1, \quad u(l,y,t) = b_1, \quad u(x,0,t) = a_2, \quad u(x,l,t) = b_2,$$

where $t > 0$. Herein, when we partition the intervals $[a_1, b_1]$, $[a_2, b_2]$ and $[a_3, b_3]$ into multiple nodes, these points of intersection are commonly referred to as mesh points, nodal points or grid points. These points can be expressed as follows: $x_i = ih_x$, $y_j = jh_y$ and $t_k = kh_t$, where $i = 1, 2, \ldots, n_x$, $j = 1, 2, \ldots, n_y$ and $k = 1, 2, \ldots, n_t$. Now, to tackle problem (22), we employ the fractional difference formula (21) reported in Corollary 3.1. To accomplish this, we need to recognize that Equation (22) can be reformulated numerically using the aforementioned formula as follows:

$$\begin{aligned} &\frac{t_1^{2-2\delta}}{h_t^2 \Gamma(3-2\delta)} \left(u_{i,j}^{k+1} - 2u_{i,j}^{k} + u_{i,j}^{k-1} \right) \\ &= \frac{c^2 x_1^{2-2\delta}}{h_x^2 \Gamma(3-2\delta)} \left(u_{i+1,j}^{k} - 2u_{i,j}^{k} + u_{i-1,j}^{k} \right) + \frac{c^2 y_1^{2-2\delta}}{h_y^2 \Gamma(3-2\delta)} \left(u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k} \right), \end{aligned} \tag{23}$$

for $i = 1, 2, \ldots, n_x$, $j = 1, 2, \ldots, n_y$ and $k = 1, 2, \ldots, n_t$ in which $n_x = \frac{b_1 - a_1}{h_x}$, $n_y = \frac{b_2 - a_2}{h_y}$ and $n_t = \frac{b_3 - a_3}{h_t}$ with step sizes $h_x, h_y, h_t > 0$ such that $x_1 = \frac{a_1 + b_1}{2}$, $y_1 = \frac{a_2 + b_2}{2}$ and $t_1 = \frac{b_3 + a_3}{2}$, where $x \in [a_1, b_1]$, $y \in [a_2, b_2]$ and $t \in [a_3, b_3]$. In fact, Equation (23) can be written in the form

$$\begin{aligned} &u_{i,j}^{k+1} - 2u_{i,j}^{k} + u_{i,j}^{k-1} \\ &= c^2 \left(\frac{h_t}{h_x} \right)^2 \left(\frac{x_1}{t_1} \right)^{2-2\delta} \left(u_{i+1,j}^{k} - 2u_{i,j}^{k} + u_{i-1,j}^{k} \right) + c^2 \left(\frac{h_t}{h_y} \right)^2 \left(\frac{y_1}{t_1} \right)^2 \left(u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k} \right). \end{aligned}$$

This consequently yields

$$u_{i,j}^{k+1} = 2u_{i,j}^{k} + u_{i,j}^{k-1} + s_1 \left(u_{i+1,j}^{k} - 2u_{i,j}^{k} + u_{i-1,j}^{k} \right) + s_2 \left(u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k} \right), \tag{24}$$

where

$$s_1 = c^2 \left(\frac{h_t}{h_x} \right)^2 \left(\frac{x_1}{t_1} \right)^{2-2\delta} \text{ and } s_2 = c^2 \left(\frac{h_t}{h_y} \right)^2 \left(\frac{y_1}{t_1} \right)^2.$$

As a consequence, the iterative equation presented in Equation (24) can lead to the formation of a linear system. This system can be algebraically solved using a prepared MATLAB code to obtain the desired approximate solution for the given problem (22).

4. **Illustrative Examples.** In this section, our objective is to validate the efficacy of our proposed approach. We utilize figures to visually present and compare the results we have obtained.

Example 4.1. *Consider the following 2-dimensional fractional wave equation:*

$$\frac{\partial^{2\delta} u}{\partial t^{2\delta}} = c^2 \left(\frac{\partial^{2\delta} u}{\partial x^{2\delta}} + \frac{\partial^{2\delta} u}{\partial y^{2\delta}} \right), \tag{25}$$

with initial conditions

$$u(x, y, 0) = \sin(\pi x) \sin(\pi y), \quad \frac{\partial^{\delta} u}{\partial t^{\delta}}(x, y, 0) = 0, \quad 0 < x < 1, \ 0 < y < 1,$$

and with boundary conditions

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0,$$

where $t > 0$. Herein, we assume that $c = 1$, and consequently the analytical solution of problem (25), which can be seen in Figure 1, has the form

$$u(x, y, t) = \sin(\pi x) \sin(\pi y) \cos\left(\sqrt{2}\pi t\right).$$

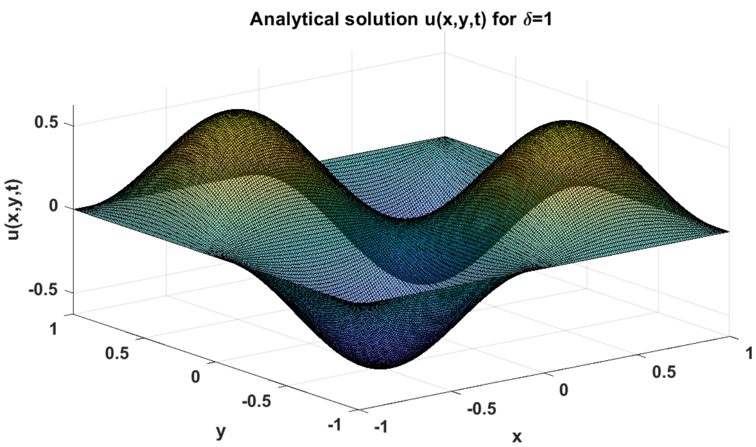

FIGURE 1. Analytical solution $u(x, y, t)$ of problem (25) when $\delta = 1$

In the following content, we aim to test our numerical approach in dealing with problem (25). To do so, we plot in Figure 2 the numerical solution $u(x, y, t)$ of such a problem when $\delta = 1$ by applying the proposed approach discussed in the previous section. In addition, in order to further illustrate the validity of our proposed technique, we plot Figure 3, which presents a graphical comparison between the numerical and analytical solutions of problem (25) when $\delta = 1$, and Figure 4, which displays the absolute value of errors between these two solutions when $\delta = 1$. The subsequent two figures amply demonstrate that the suggested numerical solution is a highly accurate approximation of the analytical answer, hence validating the previously mentioned numerical technique.

In what follows, we address problem (25) once again, but this time at a few fractional-order values. In this connection, we plot Figure 5 that represents another graphical comparison between the numerical and analytical solutions of problem (25) when $\delta = 0.75$ and $\delta = 1$, respectively. In fact, there is a clear distinction between these two solutions since

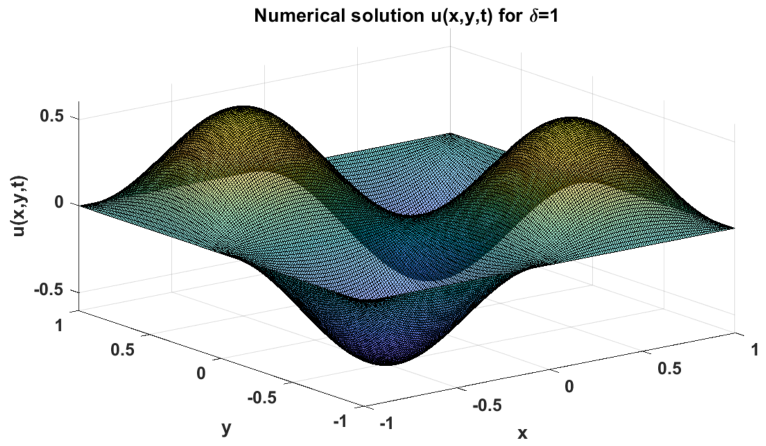

FIGURE 2. Numerical solution $u(x,y,t)$ of problem (25) when $\delta = 1$

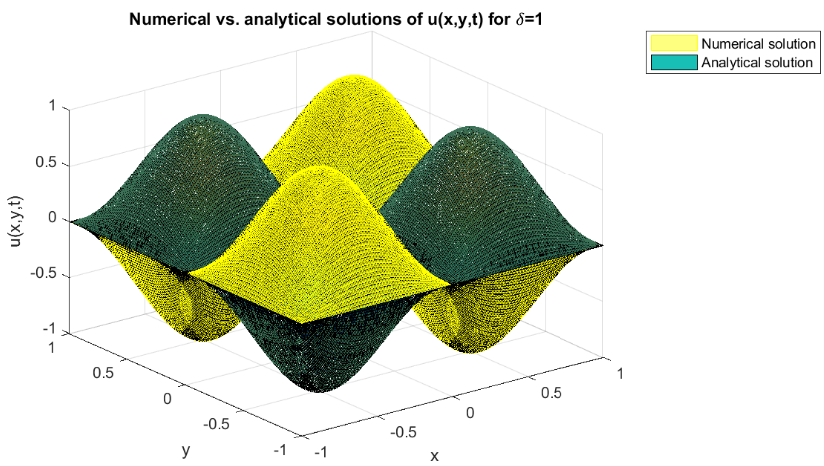

FIGURE 3. Comparison between numerical and analytical solutions of problem (25) when $\delta = 1$

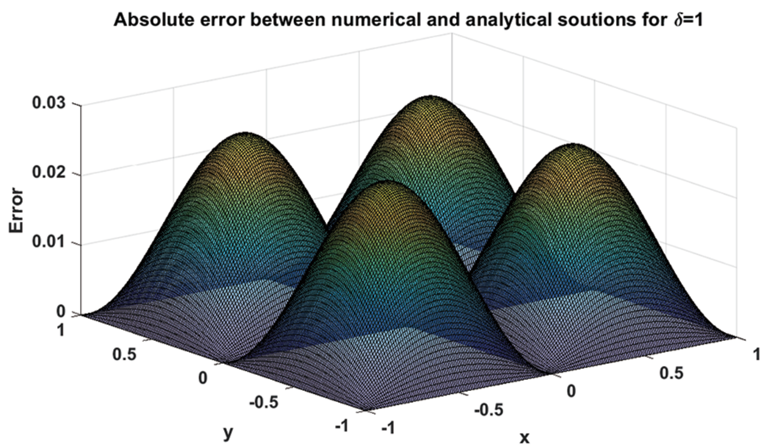

FIGURE 4. Absolute value of errors between numerical and analytical solutions of problem (25) when $\delta = 1$

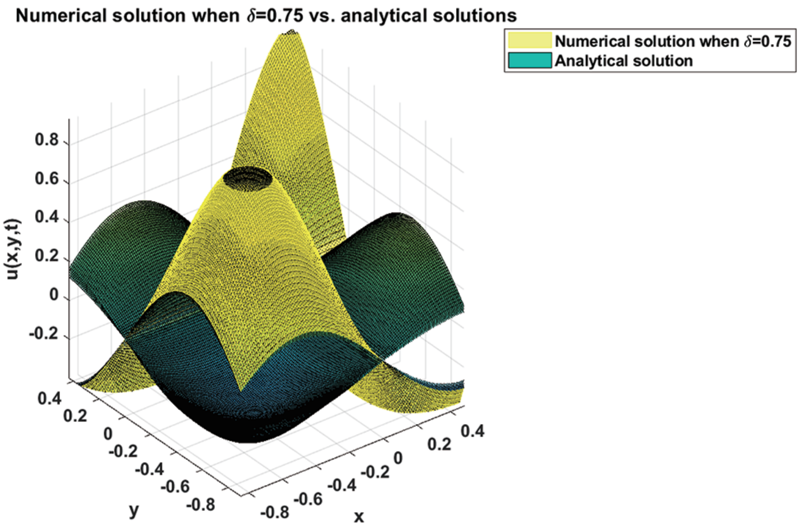

FIGURE 5. Comparison between numerical and analytical solutions of problem (25) when $\delta = 0.75$ and $\delta = 1$, respectively

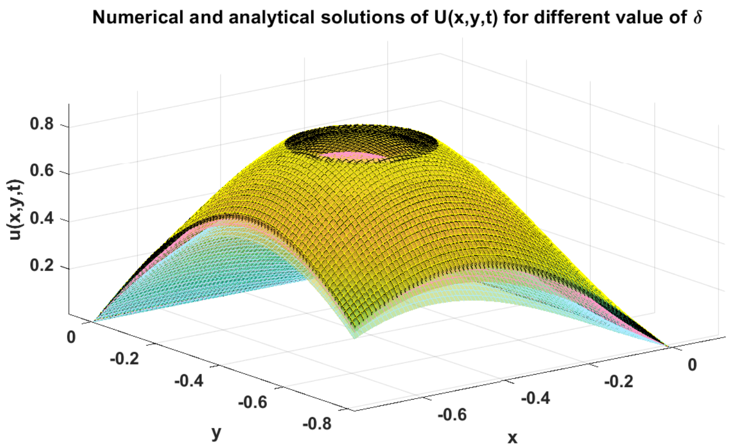

FIGURE 6. (color online) Comparison between numerical and analytical solutions of problem (25) for different values of δ: Blue for $\delta = 0.5$, Red for $\delta = 0.75$, and Yellow for $\delta = 1$

in the numerical approach, δ is taken to be 0.75, whereas in the analytical one, it is taken to be 1. This fact is further supported by Figure 6, which provides an additional graphical comparison of the analytical and numerical solutions to problem (25) for various δ values.

Example 4.2. *Consider the following 2-dimensional fractional wave equation:*

$$\frac{\partial^{2\delta} u}{\partial t^{2\delta}} = c^2 \left(\frac{\partial^{2\delta} u}{\partial x^{2\delta}} + \frac{\partial^{2\delta} u}{\partial y^{2\delta}} \right), \tag{26}$$

with initial conditions

$$u(x, y, 0) = \sin(2\pi x) \sin(\pi y), \quad \frac{\partial^{\delta} u}{\partial t^{\delta}}(x, y, 0) = 0, \quad 0 < x < 1, \ 0 < y < 1,$$

and with the boundary conditions

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0,$$

where $t > 0$. Herein, we assume that $c = 2$, and consequently the analytical solution of problem (26), which can be seen in Figure 7, has the form

$$u(x, y, t) = \sin(2\pi x)\sin(\pi y)\cos\left(\sqrt{5}\pi t\right).$$

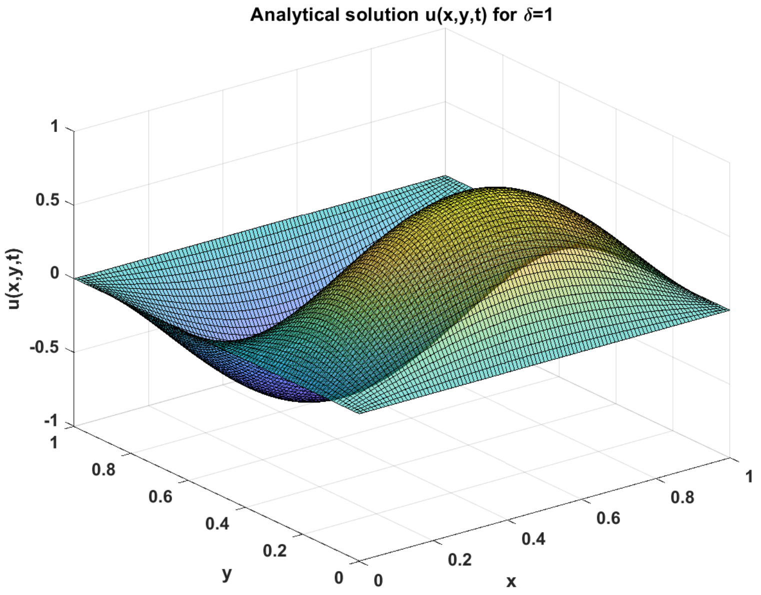

FIGURE 7. Analytical solution $u(x, y, t)$ of problem (26) when $\delta = 1$

In the following content, we aim to test our numerical approach in dealing with problem (26). To do so, we plot in Figure 8 the numerical solution $u(x, y, t)$ of such a problem when $\delta = 1$ by applying the proposed approach discussed in the previous section. In addition, in order to further illustrate the validity of our proposed technique, we plot Figure 9,

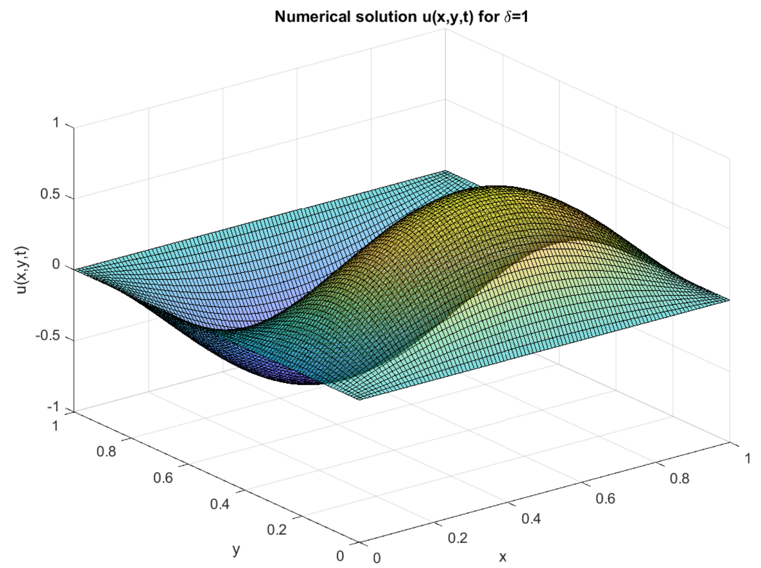

FIGURE 8. Numerical solution $u(x, y, t)$ of problem (26) when $\delta = 1$

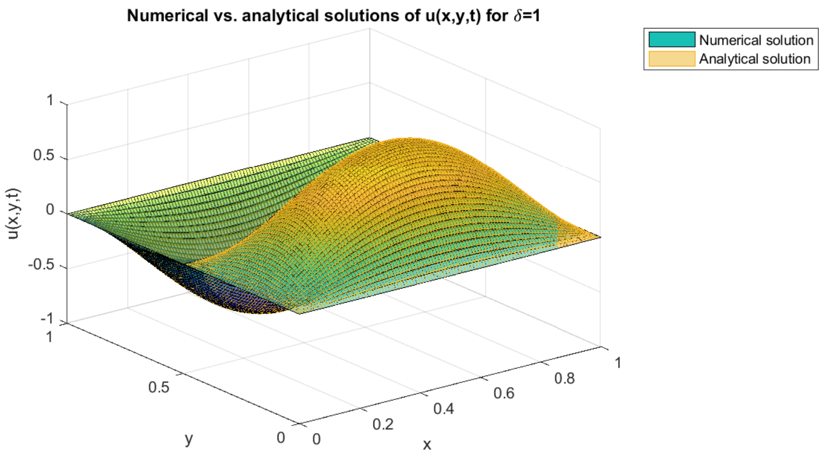

FIGURE 9. Comparison between numerical and analytical solutions of problem (26) when $\delta = 1$

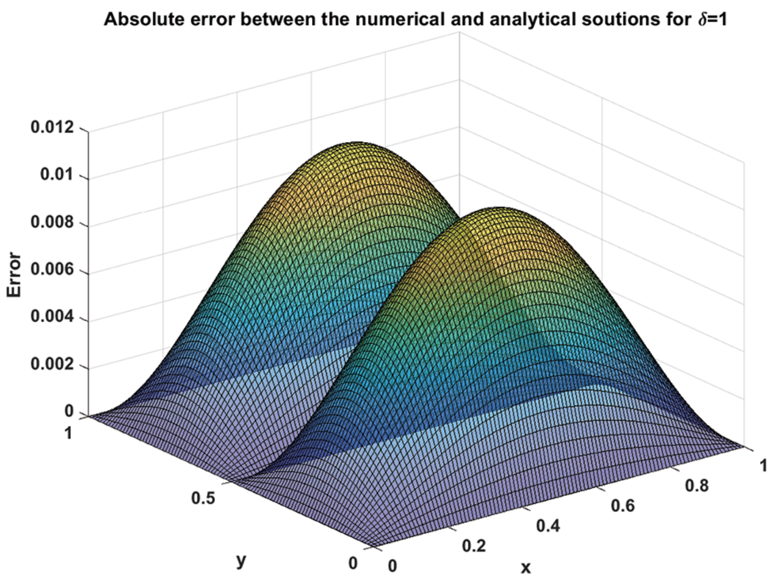

FIGURE 10. Absolute value of errors between numerical and analytical solutions of problem (26) when $\delta = 1$

which presents a graphical comparison between the numerical and analytical solutions of problem (26) when $\delta = 1$*, and Figure 10, which displays the absolute value of errors between these two solutions when* $\delta = 1$*. The subsequent two figures amply demonstrate that the suggested numerical solution is a highly accurate approximation of the analytical answer, hence validating the previously mentioned numerical technique.*

In what follows, we address problem (26) once again, but this time at a few fractional-order values. In this connection, we plot Figure 11 that represents another graphical comparison between the numerical and analytical solutions of problem (26) when $\delta = 0.75$ *and* $\delta = 1$*, respectively.*

In fact, there is a clear distinction between these two solutions since in the numerical approach, δ *is taken to be* 0.75*, whereas in the analytical one, it is taken to be* 1*. This is*

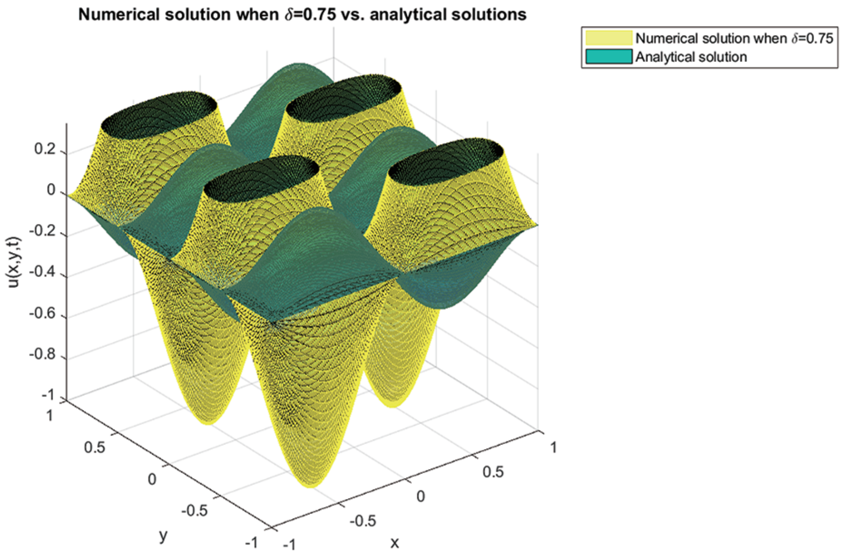

FIGURE 11. Comparison between numerical and analytical solutions of problem (26) when $\delta = 0.75$ and $\delta = 1$, respectively

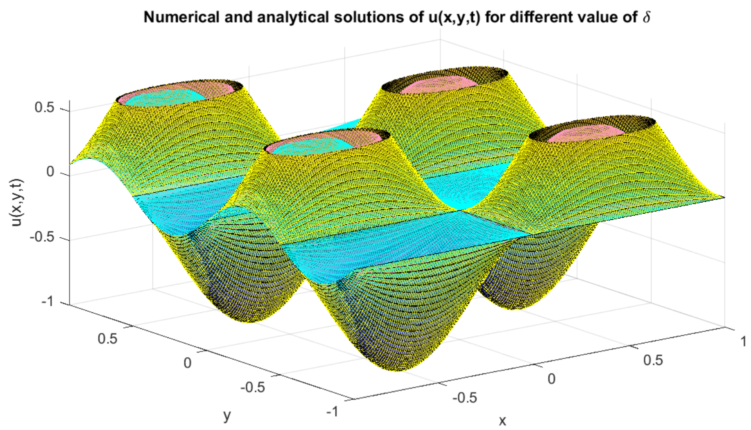

FIGURE 12. (color online) Comparison between numerical and analytical solutions of problem (26) for different values of δ: Blue for $\delta = 0.5$, Red for $\delta = 0.75$, and Yellow for $\delta = 1$

further supported by Figure 12, which provides an additional graphical comparison of the analytical and numerical solutions to problem (26) for various δ values.

Drawing from the examples and figures showcased earlier, it becomes clear that the proposed formula provides a dependable estimation for the two-dimensional wave equation in the fractional-order domain, exhibiting strong agreement with the analytical values provided. This observation indicates that the formula holds significant potential for a wide range of applications, particularly in the context of ordinary and partial differential equations. This promising outcome opens the door to utilizing the formula in various real-world scenarios, where it can contribute to more efficient and precise solutions for similar problems.

5. **Conclusion.** The two-dimensional fractional wave equation has been successfully solved with the aid of two novel fractional difference formulas that have been derived for

approximating the Caputo differentiator of order δ and 2δ, respectively, where $0 < \delta \leq 1$. As a result, several numerical solutions to the fractional wave equation have been generated according to different fractional-order values, for which its analytical solution is completely coincident with the case where the fractional-order value is 1. From this point of view, one can notice that the proposed numerical approach can have a great deal of potential for a variety of applications, especially when it comes to ordinary and partial differential equations, such as wave equations, heat equations, beam equations, Poisson's equations, and Schrödinger equations.

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