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ON VARIOUS CLASSES OF ONE-SIDED IDEALS IN *b*-SEMIRINGS

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Abstract. *In this paper, we introduce weak-1 right(left) ideals, weak-2 right(left) ideals, weak-1 ideals and weak-2 ideals of b-semirings. We characterize the $\uplus$ product and $\oslash$ product of various classes of weak ideals with right ideals and the $\uplus$ product and $\oslash$ product of various classes of left ideals with weak ideals. We establish the sufficient conditions for $\Upsilon \uplus \Game$ to be a weak-2 ideal and a right ideal whenever $\Game$ is a right ideal, respectively. Various sufficient conditions for $\Upsilon \oslash \Game$ to be a weak-1 right ideal are established whenever $\Game$ is a right ideal. To prove that the $\uplus$ product of left ideal and right ideal is a weak-2 ideal, also the $\oslash$ product of left ideal and right ideal is a weak-1 ideal. We establish the sufficient conditions for $\Upsilon \uplus \Game$ to be a weak-2 ideal and a left ideal whenever Υ is a left ideal, respectively. Various sufficient conditions for $\Upsilon \oslash \Game$ to be a weak-1 ideal are established whenever Υ is a left ideal. A weak-2 right ideal that need not be an ideal is exemplified by the $\uplus$ product of left ideals instance. Also, the $\oslash$ product of left ideals with a weak-2 right ideal need not be a weak-1 ideal of b-semirings.*
Keywords: *b*-semiring, Right(left) ideal, Weak-1 right(left) ideal, Weak-2 right(left) ideal, Weak-1 ideal, Weak-2 ideal

1. **Introduction.** Mathematical structures have several applications. It is important to generalize the ideals of algebraic structures and ordered algebraic structures and to make them available for further study and application. Between 1950 and 1980, mathematicians studied bi-ideals, quasi-ideals, and interior ideals. During 1950-2019, however, only mathematicians studied their applications. The notions of one-sided ideals of rings and semigroups, as well as the notions of quasi-ideals of rings and semigroups can all be considered generalizations of the notion of ideals of rings and semigroups. Semigroups are generalizations of rings and groups. Semigroup structure can be studied using certain band decompositions in semigroup theory. This research uses bi-ideals of semirings with additively reduced semilattices to open a new area of mathematics. Vandiver [1], an American mathematician, introduced the idea of semirings in 1934, while a German

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mathematician, Dedekind [2], proposed non-trivial examples of semirings in the 19th century when he studied commutative IDs for rings. In addition to their applications in the foundations of arithmetic and topological considerations, semirings occur as ideals of rings and as positive cones of partially ordered rings. As well as the allied fields of theoretical computer science, theoretical physics, and formal language theory, the modern interest in semirings comes primarily from areas of applied mathematics, including optimization theory, discrete event dynamical systems theory, automata theory, and formal language theory. In the 20th century, non-commutative rings became systematically studied. A matrix is also a non-commutative entity that occurs naturally. Cayley [3] introduced them, along with their addition and multiplication principles. Square matrices are said to follow well-known ring axioms by Peirce [4]. A fundamental contribution to the science of non-commutative rings was made by Scottish mathematician Wedderburn [5] with his Wedderburn's Theorem, which states that every finite division ring is commutative. During the 18th century, commutative and noncommutative ring theories were intertwined and impacted each other. Non-commutative rings provide a natural extension of the study of prime radicals and primary ideals for commutative rings. A few basic observations on radicals are made by Lam [6], as well as a few on non-commutative rings. The ideals of rings and semirings have been studied in many studies. Associative rings are the conceptual basis of Dedekind's ideals in algebraic numbers theory. The concept was expanded by adding algebraic numbers. By using quasi-ideals and generalized bi-ideals, Lajos [7] studied regular and intra-regular semigroups. Numerous studies have described various forms of ideals in algebraic structures like semirings [8] and rings [9]. As part of a generalization of rings, Vandiver proposed semirings [1]. In an ideal theory, there is no commutative requirement for semirings under either operation. The study of semigroups, semirings, and rings has been carried out by several authors. Describe different classes of semigroups using [10]. Associative rings are in some ways arbitrary but specified in terms of BIDs. A quasi-ideal is an extension of left ideals (LIs) and right ideals (RIs), which are examples of BIDs. Quasi-ideals were introduced by Steinfeld [15] when he introduced semigroups and rings. Commutative rings have been extensively studied using prime ideal theory. Lajos and Szász [11] defined associative rings in terms of bi-ideals. Quasi-ideals are generalizations of left ideals and right ideals and are, therefore, special cases of bi-ideals. Steinfeld [12] introduced the concept of quasi-ideals based on semigroups and rings in 1956. In semirings, prime ideals can be described in a variety of ways. In the theory of commutative rings, the prime ideal has been extensively used. In contrast to commutative rings, its application to non-commutative rings has been less extensive.

A bi-ideal for semigroups was proposed by Good and Hughes [13]. The (m, n)-ideal introduced by Lajos and Szász [14] is also a specific instance of this concept. By combining associative rings with bi-ideals, Lajos and Szász [14] introduced bi-ideals. Both LIs and RIs can be generalized into quasi-ideals. The concept of quasi-ideals was introduced by Steinfeld [15] for semigroups and rings in 1980. Chinram [16] introduced the concept of b-semirings and its ideals. Palanikumar et al. interacted with different partial bi-ideals over non-commutative partial rings [17]. Palanikumar et al. [18] interacted new type of basis of an ordered Γ-semigroup. Recently, Palanikumar et al. discussed some algebraic structures such as semirings and ring semigroups [19, 20, 21, 22, 23]. There are several closely related structures that have been introduced in other contexts which have partially-additive semantics like in [24, 25, 26]. It is suggested that $\sum$ should be emphasized in computing science, according to the flowchart enclosed with [24]. It is possible to find partially defined infinity operations in many contexts. Various contexts can be found here, ranging from the semantics of programming languages to the integration theory of systems.

Computer scientists try to make programs more understandable by changing the computed function without changing their function. A program transformation algebraic theory is clearly required to solve this problem [27]. Palanikumar et al. addressed semigroups, semirings, rings, and ternary semirings in their recent work [20, 21, 28, 29]. Several important classical results are discussed in this paper, using weak-1 and weak-2 ideals to characterize the results in various ideals. Throughout the paper, five sections are presented. An introduction is found in Section 1. Section 2 describes b-semirings, relevant definitions, and results. In Section 3, examples of products with RIs are discussed. An example of a product with LIs is discussed in Section 4. A conclusion discussion can be found in Section 5.

2. **Preliminaries.** This section introduces the basic definitions of semirings and b-semirings and the definitions used in this article.

Definition 2.1. *An additive subgroup I of a ring R is called an LI(RI) of R if $ra \in I(ar \in I)$ for $a \in I$ and $r \in R$. I is an ideal (ID) of R if it is both an RI and an LI.*

Definition 2.2. [15] *The subset Q is a quasi-ideal of a ring R if Q is a subring of R and $SQ \cap QS \subseteq Q$.*

Definition 2.3. *The subset B is a bi-ideal of a ring R if B is a subring of R and $BAB \subseteq B$.*

Definition 2.4. [8] *A nonempty set S is said to be a semiring together with binary operations " $+$ " and " $\cdot$ " such that*
(i) $(S, +)$ is a commutative monoid,
(ii) $(S, \cdot)$ is a semigroup,
(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$, $\forall x, y, z \in S$.

Definition 2.5. [16] *The algebraic structure $(\mathscr{B}, \uplus, \oslash)$ is called a b-semiring if $(\mathscr{B}, \uplus)$ and $(\mathscr{B}, \oslash)$ are semigroups and $\zeta \uplus (\xi \oslash \tau) = (\zeta \uplus \xi) \oslash (\zeta \uplus \tau)$, $(\zeta \oslash \xi) \uplus c = (\zeta \uplus \tau) \oslash (\xi \uplus \tau)$, $\zeta \oslash (\xi \uplus \tau) = (\zeta \oslash \xi) \uplus (\zeta \oslash \tau)$, $(\zeta \uplus \xi) \oslash c = (\zeta \oslash \tau) \uplus (\xi \oslash \tau)$ for all $\zeta, \xi, \tau \in \mathscr{B}$. Let X be any set. Then $(P(X), \cap, \cup)$ and $(P(X), \cup, \cap)$ are b-semirings. A non-empty subset T of $\mathscr{B}$ is called a sub b-semiring of $\mathscr{B}$ if $(T, \uplus, \oslash)$ is a b-semiring.*

3. **Product with RIs.** Unless stated otherwise, $\mathscr{B}$ here represents a b-semiring.

Definition 3.1. *Let Υ and $\eth$ be the subsets of $(\mathscr{B}, \uplus, \oslash)$. Then the $\uplus$ product and $\oslash$ product of Υ and $\eth$, denoted by $\Upsilon \uplus \eth$ and $\Upsilon \oslash \eth$, respectively, are defined as $\Upsilon \uplus \eth = \{\zeta \uplus \xi \mid \zeta \in \Upsilon \text{ and } \xi \in \eth\}$ and $\Upsilon \oslash \eth = \{\zeta \oslash \xi \mid \zeta \in \Upsilon \text{ and } \xi \in \eth\}$.*

Definition 3.2. *The subset Υ of $\mathscr{B}$ is called a weak-1 right ideal(W1RI) (weak-1 left ideal(W1LI)) of $\mathscr{B}$ if $\zeta_1 \uplus \zeta_2 \in \Upsilon$ and $\zeta_1 \oslash s \in \Upsilon$ $(s \oslash \zeta_1 \in \Upsilon)$ for all $\zeta_1, \zeta_2 \in \Upsilon$ and $s \in \mathscr{B}$.*

Definition 3.3. *The subset Υ of $\mathscr{B}$ is called a weak-1 ideal(W1ID) of $\mathscr{B}$ if it is a W1RI and a W1LI.*

Definition 3.4. *The subset Υ of $\mathscr{B}$ is called a weak-2 right ideal(W2RI) (weak-2 left ideal(W2LI)) of $\mathscr{B}$ if $\zeta_1 \oslash \zeta_2 \in \Upsilon$ and $\zeta_1 \uplus s \in \Upsilon$ $(s \uplus \zeta_1 \in \Upsilon)$ for all $\zeta_1, \zeta_2 \in \Upsilon$ and $s \in \mathscr{B}$.*

Definition 3.5. *The subset Υ of $\mathscr{B}$ is called a weak-2 ideal(W2ID) of $\mathscr{B}$ if it is a W2RI and a W2LI.*

Definition 3.6. *The subset Υ of $\mathscr{B}$ is called an RI(LI) of $\mathscr{B}$ if it is a W1RI(W1LI) and a W2RI(W2LI).*

Definition 3.7. *The subset Υ of $\mathscr{B}$ is called an ID of $\mathscr{B}$ if it is an RI and an LI.*

Theorem 3.1. *If Υ is a W1LI and $\eth$ is an RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ is a W2RI of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ is a W1ID of $\mathscr{B}$.

Proof: Let Υ be a W1LI and $\eth$ be an RI of $\mathscr{B}$.
(i) For $x, y \in \Upsilon \uplus \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \oslash y &= (\zeta \uplus \xi) \oslash (\zeta_1 \uplus \xi_1) \\ &= [(\zeta \uplus \xi) \oslash \zeta_1] \uplus [(\zeta \uplus \xi) \oslash \xi_1] \\ &= [(\zeta \uplus \xi) \oslash \zeta_1] \uplus s \\ &= [(\zeta \oslash \zeta_1) \uplus (\xi \oslash \zeta_1)] \uplus s \\ &= (\zeta_2 \uplus \xi_2) \uplus s \\ &= \zeta_2 \uplus (\xi_2 \uplus s) \\ &= \zeta_2 \uplus \xi_3 \in \Upsilon \uplus \eth, \qquad (1) \\ x \uplus s &= (\zeta \uplus \xi) \uplus s \\ &= \zeta \uplus (\xi \uplus s) \\ &= \zeta \uplus \xi' \in \Upsilon \uplus \eth. \qquad (2) \end{aligned}$$

Therefore, $\Upsilon \uplus \eth$ is a W2RI of $\mathscr{B}$.
(ii) For $x, y \in \Upsilon \oslash \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \uplus y &= (\zeta \oslash \xi) \uplus (\zeta_1 \oslash \xi_1) \\ &= [(\zeta \oslash \xi) \uplus \zeta_1] \oslash [(\zeta \oslash \xi) \uplus \xi_1] \\ &= [(\zeta \oslash \xi) \uplus \zeta_1] \oslash s \\ &= [(\zeta \uplus \zeta_1) \oslash (\xi \uplus \zeta_1)] \oslash s \\ &= (\zeta' \oslash \xi'') \oslash s \\ &= \zeta' \oslash (\xi'' \oslash s) \\ &= \zeta' \oslash \xi''' \in \Upsilon \oslash \eth, \qquad (3) \\ x \oslash s &= (\zeta \oslash \xi) \oslash s \\ &= \zeta \oslash (\xi_2 \oslash s) \\ &= \zeta \oslash \xi_4 \in \Upsilon \oslash \eth, \qquad (4) \\ s \oslash x &= s \oslash (\zeta \oslash \xi) \\ &= (s \oslash \zeta) \oslash \xi \\ &= \zeta_3 \oslash \xi \in \Upsilon \oslash \eth. \end{aligned}$$

Therefore, $\Upsilon \oslash \eth$ is a W1ID of $\mathscr{B}$. □

Remark 3.1. *If Υ is a W1LI and $\eth$ is an RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ need not be a W2ID of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ need not be an ID of $\mathscr{B}$.

Example 3.1. *Let $(\mathscr{B}, \uplus, \oslash)$ be a b-semiring by the following table.*

$\uplus$	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1
ζ_2	ζ_2	ζ_2	ζ_2	ζ_2	ζ_2	ζ_2
ζ_3	ζ_3	ζ_3	ζ_3	ζ_3	ζ_3	ζ_3
ζ_4	ζ_4	ζ_4	ζ_4	ζ_4	ζ_4	ζ_4
ζ_5	ζ_5	ζ_5	ζ_5	ζ_5	ζ_5	ζ_5
ζ_6	ζ_6	ζ_6	ζ_6	ζ_6	ζ_6	ζ_6

$\oslash$	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1
ζ_2	ζ_1	ζ_2	ζ_1	ζ_2	ζ_2	ζ_2
ζ_3	ζ_1	ζ_1	ζ_3	ζ_1	ζ_3	ζ_3
ζ_4	ζ_1	ζ_2	ζ_1	ζ_4	ζ_2	ζ_4
ζ_5	ζ_1	ζ_2	ζ_3	ζ_2	ζ_5	ζ_5
ζ_6	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6

Clearly, $\Upsilon = \{\zeta_1, \zeta_3\}$ *is a W1LI and* $\eth = \{\zeta_1, \zeta_2, \zeta_4\}$ *is an RI of* $\mathscr{B}$*. Now,* $\zeta_5 \uplus \zeta_3 = \zeta_5 \notin \Upsilon \uplus \eth$ *implies that* $\Upsilon \uplus \eth$ *is not a W2LI of* $\mathscr{B}$*. By taking* $C = \{\zeta_1, \zeta_2, \zeta_3\}$ *as a W1LI and* $D = \{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$ *as an RI of* $\mathscr{B}$*,* $C \oslash D$ *fails to be a W2LI of* $\mathscr{B}$ *by* $\zeta_6 \uplus \zeta_2 = \zeta_6 \notin C \oslash D$*.*

Theorem 3.2. *If* Υ *is a W1RI and* $\eth$ *is an RI of* $\mathscr{B}$*, then*
(i) $\Upsilon \uplus \eth$ *is an RI of* $\mathscr{B}$*,*
(ii) $\Upsilon \oslash \eth$ *is a W1RI of* $\mathscr{B}$*.*

Proof: Let Υ be a W1RI and $\eth$ be an RI of $\mathscr{B}$.
(i) By (1) and (2), $\Upsilon \uplus \eth$ is a W2RI of $\mathscr{B}$. For $x \in \Upsilon \uplus \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \oslash s &= (\zeta \uplus \xi) \oslash s \\ &= (\zeta \oslash s) \uplus (\xi \oslash s) \\ &= \zeta_1 \uplus \xi_1 \in \Upsilon \uplus \eth. \end{aligned}$$

Then, $\Upsilon \uplus \eth$ is an RI of $\mathscr{B}$.
(ii) By (3) and (4), $\Upsilon \oslash \eth$ is a W1RI of $\mathscr{B}$. □

Remark 3.2. *If* Υ *is a W1RI and* $\eth$ *is an RI of* $\mathscr{B}$*, then* $\Upsilon \oslash \eth$ *need not be an RI of* $\mathscr{B}$*.*

Example 3.2. *Consider* $(\mathscr{B}, \uplus, \oslash)$ *as a b-semiring by the following table.*

$\uplus$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

$\oslash$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

By taking $\Upsilon = \{b, d\}$ *and* $\eth = \{a, c\}$ *as a W1RI and an RI of* $\mathscr{B}$*, respectively,* $\Upsilon \oslash \eth$ *fails to be a W2RI of* $\mathscr{B}$ *by* $d \uplus a = a \notin \Upsilon \oslash \eth$*.*

Theorem 3.3. *If* Υ *is a W2LI and* $\eth$ *is an RI of* $\mathscr{B}$*, then*
(i) $\Upsilon \uplus \eth$ *is a W2ID of* $\mathscr{B}$*,*
(ii) $\Upsilon \oslash \eth$ *is a W1RI of* $\mathscr{B}$*.*

Proof: Let Υ be a W2LI and $\eth$ be an RI of $\mathscr{B}$.
(i) For $x, y \in \Upsilon \uplus \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \oslash y &= (\zeta \uplus \xi) \oslash (\zeta_1 \uplus \xi_1) \\ &= [(\zeta \uplus \xi) \oslash \zeta_1] \uplus [(\zeta \uplus \xi) \oslash \xi_1] \\ &= [(\zeta \uplus \xi) \oslash \zeta_1] \uplus s \\ &= [(\zeta \oslash \zeta_1) \uplus (\xi \oslash \zeta_1)] \uplus s \\ &= (\zeta_2 \uplus \xi_2) \uplus s \\ &= \zeta_2 \uplus (\xi_2 \uplus s) \\ &= \zeta_2 \uplus \xi_3 \in \Upsilon \uplus \eth, \end{aligned} \tag{5}$$

$$x \uplus s = (\zeta \uplus \xi) \uplus s$$

$$\begin{aligned} &= \zeta \uplus (\xi \uplus s) \\ &= \zeta \uplus \xi' \in \Upsilon \uplus \eth, \qquad (6) \\ s \uplus x &= s \uplus (\zeta \uplus \xi) \\ &= (s \uplus \zeta) \uplus \xi \\ &= \zeta' \uplus \xi \in \Upsilon \uplus \eth. \end{aligned}$$

Hence, $\Upsilon \uplus \eth$ is a W2ID of $\mathscr{B}$.
(ii) For $x, y \in \Upsilon \oslash \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \uplus y &= (\zeta \oslash \xi) \uplus (\zeta_1 \oslash \xi_1) \\ &= [(\zeta \oslash \xi) \uplus \zeta_1] \oslash [(\zeta \oslash \xi) \uplus \xi_1] \\ &= [(\zeta \oslash \xi) \uplus \zeta_1] \oslash s \\ &= [(\zeta \uplus \zeta_1) \oslash (\xi \uplus \zeta_1)] \oslash s \\ &= (\zeta'' \oslash \xi'') \oslash s \\ &= \zeta'' \oslash (\xi'' \oslash s) \\ &= \zeta'' \oslash \xi''' \in \Upsilon \oslash \eth, \qquad (7) \\ x \oslash s &= (\zeta \oslash \xi) \oslash s \\ &= \zeta \oslash (\xi \oslash s) \\ &= \zeta \oslash \xi_4 \in \Upsilon \oslash \eth. \qquad (8) \end{aligned}$$

Therefore, $\Upsilon \oslash \eth$ is a W1RI of $\mathscr{B}$. □

Remark 3.3. *If Υ is a W2LI and $\eth$ is an RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ need not be an ID of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ need not be a W1ID of $\mathscr{B}$.

Example 3.3. *Consider the b-semiring $\mathscr{B}$ as in Example 3.2. Clearly, $\Upsilon = \{a, b\}$ is a W2LI and $\eth = \{a, c\}$ is an RI of $\mathscr{B}$. Now, $b \oslash a = b \notin \Upsilon \uplus \eth$ implies that $\Upsilon \uplus \eth$ is not a W1LI of $\mathscr{B}$. By taking $C = \{a, b, c\}$ as an RI of $\mathscr{B}$, $\Upsilon \oslash C$ fails to be a W1LI of $\mathscr{B}$ by $d \oslash \xi = d \notin \Upsilon \oslash C$.*

Corollary 3.1. *If Υ is an LI and $\eth$ is an RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ is a W2ID of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ is a W1ID of $\mathscr{B}$.

Proof: Let Υ be an LI and $\eth$ be an RI of $\mathscr{B}$.
(i) By Theorem 3.3, $\Upsilon \uplus \eth$ is a W2ID of $\mathscr{B}$.
(ii) By Theorem 3.3, $\Upsilon \oslash \eth$ is a W1RI of $\mathscr{B}$. For $x \in \Upsilon \oslash \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} s \oslash x &= s \oslash (\zeta \oslash \xi) \\ &= (s \oslash \zeta) \oslash \xi \\ &= \zeta' \oslash \xi \in \Upsilon \oslash \eth. \end{aligned}$$

Therefore, $\Upsilon \oslash \eth$ is a W1ID of $\mathscr{B}$. □

Remark 3.4. *When Υ is an LI and $\eth$ is an RI of $\mathscr{B}$, $\Upsilon \uplus \eth$ and $\Upsilon \oslash \eth$ need not be an ID of $\mathscr{B}$.*

Example 3.4. *Let $(\mathscr{B}_1, \uplus, \oslash)$ be a b-semiring by the following table.*

$\uplus$	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1	ζ_1
ζ_2	ζ_1	ζ_2	ζ_1	ζ_2	ζ_2	ζ_2
ζ_3	ζ_1	ζ_1	ζ_3	ζ_1	ζ_3	ζ_3
ζ_4	ζ_1	ζ_2	ζ_1	ζ_4	ζ_2	ζ_4
ζ_5	ζ_1	ζ_2	ζ_3	ζ_2	ζ_5	ζ_5
ζ_6	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6

$\oslash$	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_1	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_2	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_3	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_4	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_5	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
ζ_6	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6

Taking $\Upsilon = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ is an LI and $\eth = \mathscr{B}$ is an RI of $\mathscr{B}$, $\zeta_4 \oslash \zeta_5 = \zeta_5 \notin \Upsilon \uplus \eth$. Thus, $\Upsilon \uplus \eth$ is not a W1RI of $\mathscr{B}$. Let $(\mathscr{B}_1, \uplus, \oslash)$ be a b-semiring by the following table.

$\uplus$	a	b	c	d
a	a	b	c	d
b	a	b	c	d
c	a	b	c	d
d	a	b	c	d

$\oslash$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

Let $\Upsilon = \{a, b\}$ be an LI and $\eth = \mathscr{B}$ be an RI of b-semirings $\mathscr{B}$, $b \uplus c = c \notin \Upsilon \oslash \eth$. Thus, $\Upsilon \oslash \eth$ is not a W2RI of $\mathscr{B}$.

Corollary 3.2. *If Υ is a W2RI and $\eth$ is an RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ is a W2RI of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ is an RI of $\mathscr{B}$.

Proof: Let Υ be a W2RI and $\eth$ be an RI of $\mathscr{B}$.
(i) By (5) and (6), $\Upsilon \uplus \eth$ is a W2RI of $\mathscr{B}$.
(ii) By (7) and (8), $\Upsilon \oslash \eth$ is a W1RI of $\mathscr{B}$. For $x \in \Upsilon \oslash \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \uplus s &= (\zeta \oslash \xi) \uplus s \\ &= (\zeta \uplus s) \oslash (\xi \uplus s) \\ &= \zeta_1 \oslash \xi_1 \in \Upsilon \oslash \eth. \end{aligned}$$

Hence, $\Upsilon \oslash \eth$ is an RI of $\mathscr{B}$. □

Remark 3.5. *If Υ is a W2RI and $\eth$ is an RI of $\mathscr{B}$, then $\Upsilon \uplus \eth$ need not be an RI of $\mathscr{B}$.*

Example 3.5. *$\Upsilon = \{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$ is a W2RI and $\eth = \mathscr{B}$ is an RI of $\mathscr{B}$ as in Example 3.4. Now, $\Upsilon \uplus \eth$ fails to be a W1RI of $\mathscr{B}$ by $\zeta_5 \oslash \zeta_6 = \zeta_6 \notin \Upsilon \uplus \eth$.*

4. **Product with LIs.** In this section, various interesting properties of LIs, along with examples, are provided.

Theorem 4.1. *If Υ is an LI and $\eth$ is a W1RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ is a W2LI of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ is a W1ID of $\mathscr{B}$.

Proof: Let Υ be an LI and $\eth$ be a W1RI of $\mathscr{B}$.
(i) For $x, y \in \Upsilon \uplus \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \oslash y &= (\zeta \uplus \xi) \oslash (\zeta_1 \uplus \xi_1) \\ &= [\zeta \oslash (\zeta_1 \uplus \xi_1)] \uplus [\xi \oslash (\zeta_1 \uplus \xi_1)] \\ &= s \uplus [(\xi \oslash \zeta_1) \uplus (\xi \oslash \xi_1)] \\ &= s \uplus (\zeta_2 \uplus \xi_2) \\ &= (s \uplus \zeta_2) \uplus \xi_2 \\ &= \zeta_3 \uplus \xi \in \Upsilon \uplus \eth, \end{aligned}$$

$$\begin{aligned} s \uplus x &= s \uplus (\zeta \uplus \xi) \\ &= (s \uplus \zeta) \uplus \xi \\ &= \zeta^{'} \uplus \xi \in \Upsilon \uplus \partial. \end{aligned}$$

Therefore, $\Upsilon \uplus \partial$ is a W2LI of $\mathscr{B}$.
(ii) For $x, y \in \Upsilon \oslash \partial$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \uplus y &= (\zeta \oslash \xi) \uplus (\zeta_1 \oslash \xi_1) \\ &= [\zeta \uplus (\zeta_1 \oslash \xi_1)] \oslash [\xi \uplus (\zeta_1 \oslash \xi_1)] \\ &= s \oslash [(\xi \uplus \zeta_1) \oslash (\xi \uplus \xi_1)] \\ &= s \oslash (\zeta^{''} \oslash \xi^{'}) \\ &= (s \oslash \zeta^{''}) \oslash \xi^{'} \\ &= \zeta_4 \oslash \xi^{'} \in \Upsilon \oslash \partial, \\ s \oslash x &= s \oslash (\zeta \oslash \xi) \\ &= (s \oslash \zeta) \oslash b \\ &= \zeta^{'''} \oslash b \in \Upsilon \oslash \partial, \\ x \oslash s &= (\zeta \oslash \xi) \oslash s \\ &= \zeta \oslash (\xi \oslash s) \\ &= \zeta \oslash \xi^{''} \in \Upsilon \oslash \partial. \end{aligned}$$

Therefore, $\Upsilon \oslash \partial$ is a W1ID of $\mathscr{B}$. □

Remark 4.1. *If Υ is an LI and ∂ is a W1RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \partial$ need not be a W2ID of $\mathscr{B}$,
(ii) $\Upsilon \oslash \partial$ need not be an ID of $\mathscr{B}$.

Example 4.1. *By taking $\Upsilon = \{a, c\}$ as an LI and $\partial = \{a, b\}$ as a W1RI of $\mathscr{B}$ as in Example 3.4, $\Upsilon \uplus \partial$ fails to be a W2RI of $\mathscr{B}$ by $b \uplus c = c \notin \Upsilon \uplus \partial$. Now, $C = \{a, b, c\}$ as an LI of $\mathscr{B}$, because of $b \uplus d = d \notin C \oslash \partial$, $C \oslash \partial$ is not a W2RI of $\mathscr{B}$.*

Theorem 4.2. *If Υ is an LI and ∂ is a W1LI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \partial$ is an LI of $\mathscr{B}$,
(ii) $\Upsilon \oslash \partial$ is a W1LI of $\mathscr{B}$.

Proof: Let Υ be an LI and ∂ be a W1LI of $\mathscr{B}$.
(i) For $x \in \Upsilon \uplus \partial$ and $s \in \mathscr{B}$,

$$\begin{aligned} s \oslash x &= s \oslash (\zeta \uplus \xi) \\ &= (s \oslash \zeta) \uplus (s \oslash \xi) \\ &= \zeta_1 \uplus \xi_1 \in \Upsilon \uplus \partial, \\ s \uplus x &= s \uplus (\zeta \uplus \xi) \\ &= (s \uplus \zeta) \uplus \xi \\ &= \zeta^{'} \uplus \xi \in \Upsilon \uplus \partial. \end{aligned}$$

Therefore, $\Upsilon \uplus \partial$ is an LI of $\mathscr{B}$.
(ii) For $x, y \in \Upsilon \oslash \partial$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \uplus y &= (\zeta \oslash \xi) \uplus (\zeta_1 \oslash \xi_1) \\ &= \xi^{'} \uplus (\zeta_1 \oslash \xi_1) \\ &= (\xi^{'} \uplus \zeta_1) \oslash (\xi^{'} \uplus \xi_1) \end{aligned}$$

$$
\begin{aligned}
&= \zeta_2 \oslash \xi_2 \in \Upsilon \oslash \eth, \\
s \oslash x &= s \oslash (\zeta \oslash \xi) \\
&= (s \oslash \zeta) \oslash \xi \\
&= \zeta_3 \oslash \xi \in \Upsilon \oslash \eth.
\end{aligned}
$$

Therefore, $\Upsilon \oslash \eth$ is a W1LI of $\mathscr{B}$. □

Remark 4.2. *When Υ is an LI and $\eth$ is a W1LI of $\mathscr{B}$, $\Upsilon \oslash \eth$ fails to be a W1ID of $\mathscr{B}$.*

Example 4.2. *Consider the b-semiring $\mathscr{B}$ as in Example 3.4. $\Upsilon = \{\zeta_1, \zeta_2\}$ is an LI and $\eth = \{\zeta_3, \zeta_5\}$ is a W1LI of $\mathscr{B}$, but $\Upsilon \oslash \eth$ fails to be a W1RI of $\mathscr{B}$ by $\zeta_5 \oslash \zeta_6 = \zeta_6 \notin \Upsilon \oslash \eth$.*

Corollary 4.1. *If Υ is an LI and $\eth$ is a W1ID of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ is an LI of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ is a W1ID of $\mathscr{B}$.

Proof: Let Υ be an LI and $\eth$ be a W1ID of $\mathscr{B}$.
(i) By Theorem 4.2, $\Upsilon \uplus \eth$ is an LI of $\mathscr{B}$.
(ii) By Theorem 4.2, $\Upsilon \oslash \eth$ is a W1LI of $\mathscr{B}$. For $x \in \Upsilon \oslash \eth$ and $s \in \mathscr{B}$,

$$
\begin{aligned}
x \oslash s &= (\zeta \oslash \xi) \oslash s \\
&= \zeta \oslash (\xi \oslash s) \\
&= \zeta \oslash \xi_1 \in \Upsilon \oslash \eth.
\end{aligned}
$$

Therefore, $\Upsilon \oslash \eth$ is a W1ID of $\mathscr{B}$. □

Remark 4.3. *When Υ is an LI and $\eth$ is a W1ID of $\mathscr{B}$, $\Upsilon \oslash \eth$ need not be an ID of $\mathscr{B}$ by the given example.*

Example 4.3. *$\Upsilon = \{a, b, c\}$ is an LI and $\eth = \{a, b\}$ is a W1ID of a b-semiring $\mathscr{B}$ as in Example 3.4, $b \uplus c = c \notin \Upsilon \oslash \eth$. Thus, $\Upsilon \oslash \eth$ is not a W2RI of $\mathscr{B}$.*

Theorem 4.3. *If Υ is an LI and $\eth$ is a W2RI of $\mathscr{B}$, then*
(i) $\Upsilon \uplus \eth$ is a W2ID of $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ is a W1LI of $\mathscr{B}$.

Proof: Let Υ be an LI and $\eth$ be a W2RI of $\mathscr{B}$.
(i) For $x, y \in \Upsilon \uplus \eth$ and $s \in \mathscr{B}$,

$$
\begin{aligned}
x \oslash y &= (\zeta \uplus \xi) \oslash (\zeta_1 \uplus \xi_1) \\
&= [\zeta \oslash (\zeta_1 \uplus \xi_1)] \uplus [\xi \oslash (\zeta_1 \uplus \xi_1)] \\
&= s \uplus [(\xi \oslash \zeta_1) \uplus (\xi \oslash \xi_1)] \\
&= s \uplus (\zeta_2 \uplus \xi_2) \\
&= (s \uplus \zeta_2) \uplus \xi_2 \\
&= \zeta_3 \uplus \xi_2 \in \Upsilon \uplus \eth, \\
s \uplus x &= s \uplus (\zeta \uplus \xi) \\
&= (s \uplus \zeta) \uplus \xi \\
&= \zeta' \uplus \xi \in \Upsilon \uplus \eth, \\
x \uplus s &= (\zeta \uplus \xi) \uplus s \\
&= \zeta \uplus (\xi \uplus s) \\
&= \zeta \uplus \xi' \in \Upsilon \uplus \eth.
\end{aligned}
$$

Therefore, $\Upsilon \uplus \eth$ is a W2ID of $\mathscr{B}$.
(ii) For $x, y \in \Upsilon \oslash \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \uplus y &= (\zeta \oslash \xi) \uplus (\zeta_1 \oslash \xi_1) \\ &= [\zeta \uplus (\zeta_1 \oslash \xi_1)] \oslash [\xi \uplus (\zeta_1 \oslash \xi_1)] \\ &= s \oslash [(\xi \uplus \zeta_1) \oslash (\xi \uplus \xi_1)] \\ &= s \oslash (\zeta'' \oslash \xi'') \\ &= (s \oslash \zeta'') \oslash \xi'' \\ &= \zeta''' \oslash \xi'' \in \Upsilon \oslash \eth, \\ s \oslash x &= s \oslash (\zeta \oslash \xi) \\ &= (s \oslash \zeta) \oslash \xi \\ &= \zeta_4 \oslash \xi \in \Upsilon \oslash \eth. \end{aligned}$$

Therefore, $\Upsilon \oslash \eth$ is a W1LI of $\mathscr{B}$. □

Remark 4.4.
(i) The $\uplus$ product of LIs with a W2RI need not be an ID of $\mathscr{B}$,
(ii) The $\oslash$ product of LIs with a W2RI need not be a W1ID of $\mathscr{B}$.

Example 4.4. $\Upsilon = \{\zeta_1, \zeta_2, \zeta_4\}$ *is an LI and* $\eth = \{\zeta_1, \zeta_3\}$ *is a W2RI of a b-semiring* $\mathscr{B}$ *as in Example 3.4,* $\zeta_1 \oslash \zeta_5 = \zeta_5 \notin \Upsilon \uplus \eth$. *Thus,* $\Upsilon \uplus \eth$ *is not a W1RI of* $\mathscr{B}$. *By taking* $C = \{\zeta_1, \zeta_2, \zeta_3, \zeta_5\}$ *as an LI and* $D = \{\zeta_1, \zeta_2\}$ *as a W2RI of* $\mathscr{B}$, $C \oslash D$ *fails to be a W1RI of* $\mathscr{B}$ *by* $\zeta_2 \oslash \zeta_4 = \zeta_4 \notin C \oslash D$.

Theorem 4.4. *If* Υ *is an LI and* $\eth$ *is a W2LI of* $\mathscr{B}$, *then*
(i) $\Upsilon \uplus \eth$ *is a W2LI of* $\mathscr{B}$,
(ii) $\Upsilon \oslash \eth$ *is an LI of* $\mathscr{B}$.

Proof: Let Υ be an LI and $\eth$ be a W2LI of $\mathscr{B}$.
(i) For $x, y \in \Upsilon \uplus \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \oslash y &= (\zeta \uplus \xi) \oslash (\zeta_1 \uplus \xi_1) \\ &= \xi' \oslash (\zeta_1 \uplus \xi_1) \\ &= (\xi' \oslash \zeta_1) \uplus (\xi' \oslash \xi_1) \\ &= \zeta' \uplus \xi'' \in \Upsilon \uplus \eth, \\ s \uplus x &= s \uplus (\zeta \uplus \xi) \\ &= (s \uplus \zeta) \uplus \xi \\ &= \zeta_2 \uplus \xi \in \Upsilon \uplus \eth. \end{aligned}$$

Therefore, $\Upsilon \uplus \eth$ is a W2LI of $\mathscr{B}$.
(ii) For $x, y \in \Upsilon \oslash \eth$ and $s \in \mathscr{B}$,

$$\begin{aligned} s \oslash x &= s \oslash (\zeta \oslash \xi) \\ &= (s \oslash \zeta) \oslash \xi \\ &= \zeta'' \oslash \xi \in \Upsilon \oslash \eth, \\ s \uplus x &= s \uplus (\zeta \oslash \xi) \\ &= (s \uplus \zeta) \oslash (s \uplus \xi) \\ &= \zeta_3 \oslash \xi_2 \in \Upsilon \oslash \eth. \end{aligned}$$

Therefore, $\Upsilon \oslash \eth$ is an LI of $\mathscr{B}$. □

Remark 4.5. *The $\uplus$ product of LIs with a W2LI need not be an ID of $\mathscr{B}$.*

Example 4.5. $\Upsilon = \{\zeta_1, \zeta_3\}$ *is an LI and* $\partial = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ *is a W2LI of* $\mathscr{B}$ *as in Example 3.4,* $\zeta_3 \oslash \zeta_4 = \zeta_4 \notin \Upsilon \uplus \partial$. *Thus,* $\Upsilon \uplus \partial$ *is not a W1RI of* $\mathscr{B}$.

Corollary 4.2. *If* Υ *is an LI and* ∂ *is a W2ID of* $\mathscr{B}$*, then*
(i) $\Upsilon \uplus \partial$ *is a W2ID of* $\mathscr{B}$*,*
(ii) $\Upsilon \oslash \partial$ *is an LI of* $\mathscr{B}$*.*

Proof: Let Υ be an LI and ∂ be a W2ID of $\mathscr{B}$.
(ii) By Theorem 4.4, $\Upsilon \oslash \partial$ is an LI of $\mathscr{B}$.
(i) By Theorem 4.4, $\Upsilon \uplus \partial$ is a W2LI of $\mathscr{B}$. For $x \in \Upsilon \uplus \partial$ and $s \in \mathscr{B}$,

$$\begin{aligned} x \uplus s &= (\zeta \uplus \xi) \uplus s \\ &= \zeta \uplus (\xi \uplus s) \\ &= \zeta \uplus \xi' \in \Upsilon \uplus \partial. \end{aligned}$$

Therefore, $\Upsilon \uplus \partial$ is a W2ID of $\mathscr{B}$. □

Remark 4.6. *If* Υ *is an LI and* ∂ *is a W2ID of* $\mathscr{B}$*, then* $\Upsilon \uplus \partial$ *need not be an ID of* $\mathscr{B}$*.*

Example 4.6. $\Upsilon = \{\zeta_1, \zeta_2, \zeta_4\}$ *is an LI and* $\partial = \{\zeta_1, \zeta_3\}$ *is a W2ID of b-semiring* $\mathscr{B}$ *as in Example 3.4,* $\zeta_1 \oslash \zeta_5 = \zeta_5 \notin \Upsilon \uplus \partial$. *Thus,* $\Upsilon \uplus \partial$ *is not a W1RI of* $\mathscr{B}$.

5. **Conclusion.** Several weak ideals of b-semirings are introduced in this paper and several ideals are identified. A product of several ideals and right ideals was also discussed. Our discussion also included the product of left ideals and various ideals. The reverse implications are not true in the examples indicated. We proved the $\uplus$ product of left ideal and right ideal is a weak-2 ideal, also the $\oslash$ product of left ideal and right ideal is a weak-1 ideal. A weak-2 right ideal that need not be an ideal is exemplified by the $\uplus$ product of left ideals instance. Also, the $\oslash$ product of left ideals with a weak-2 right ideal need not be a weak-1 ideal of b-semirings. Future research will focus on b-semirings, ternary semirings and hyper semirings using quasi-ideals, tri-ideals and bi-quasi-ideals. We will try to develop b-semirings to hyper b-semirings using various weak ideals and tri-ideals.

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