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ON SOME USEFUL LINKS BETWEEN EXTERNAL POSITIVITY, KALMAN-YAKUBOVICH-POPOV LEMMA AND POSITIVE REAL LEMMA FOR LINEAR TIME-INVARIANT SYSTEMS WITH CONSTANT POINT DELAYS

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Abstract. *This paper deals with the conditions on external positivity of linear time-invariant systems subject to a finite number of constant point delays. The relations and comparisons between the state and output of the whole delayed system with those of its associated delay-free and delay-free dynamics auxiliary systems are investigated. "Ad hoc" versions of the Kalman-Yakubovich-Popov Lemma and of the Positive Real Lemma for the general delayed system are established and proved. The first one is seen to be related to an "a priori" prescribed boundedness of the transfer matrix norm in the* H_∞ *sense while the second one is related to the passivity property of a modified time-delay system with attenuated output, equivalently, to the positive realness of its associate input-output operator. Most of the stated and proved results are independent of the sizes of the delays. However, the obtained "ad hoc" Positive Real Lemma is, in general, dependent on the delays since the attenuation gain is piecewise constant time-varying and dependent on the delays.*

Keywords: Positivity, External positivity, Stability, Time-delay systems, Kalman-Yakubovich-Popov Lemma (KYPL), Positive Real Lemma (PRL)

1. **Introduction.** Positive systems, also called internally positive systems, are those whose state and output variables are never negative under non-negative initial conditions and non-negative controls. Such systems are relevant to describe biological models, models of population evolution, economic models, etc. See, for instance, [1-3] and some references therein. If only the output components are non-negative through time under non-negative initial conditions and non-negative controls, then the system is said to be externally positive.

A very relevant result is that the transfer function of an externally positive Single-Input Single-Output (SISO) system has necessarily to have at least one real dominant pole [1,2]. It turns out that, in order to keep such a property in the MIMO case, the above property has to hold for each entry of the matrix transfer function, that is, for each individual transfer function which links each input component with each output component. In the case of presence of internal point delays, the stability of the associated systems resulting from the removal either of all delayed dynamics (giving relevance only to the undelayed dynamics) or the removal of all delays (giving relevance only to the whole sets of dynamics but associated to zero-delay) is needed for the achievement of the

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global stability of the whole delayed system independent of the delays sizes. In order to jointly guarantee the external positivity and the asymptotic stability of the mentioned auxiliary systems as necessary conditions for the same properties to hold in the whole delayed system, some results are based on a combination of Eneström-Kakeya theorem of allocation of zeros of polynomials (see, for instance, [4-8]) and the mentioned above property on smaller sizes of real zeros versus real dominant poles in order to maintain the external positivity of the whole delayed system. On the other hand, it can be pointed out that delays are very common in the dynamics of real world processes, for instance, in diffusion problems, economic problems, war/peace models, economic models, chemical engineering problems, hear exchanger dynamics, combustion models, and Volterra-type integral equations, [9,10]. Delays can be internal delays in the state variables or external delays in the outputs and/or in the inputs. They can be also point delays, time-varying bounded or unbounded delays, distributed delays, etc.

This paper deals, in the most general case, with a linear dynamic Multiple-Input Multiple-Output (MIMO) system subject to a finite number of finite delays in the state (internal delays) including the zero-delay associated with the delay-free dynamics. Two related auxiliary systems are also considered for purposes of comparison of the solutions with that of the delayed system. Such systems are a) the delay-free one which has all delays fixed to zero while the sum of all the corresponding matrices of dynamics contributes additively to the delay-free dynamics, and b) the auxiliary system of delayed-free dynamics in which the delays together with their associated matrices of dynamics are deleted so that the unique matrix of dynamics is the corresponding one to the zero delay. See, for instance, [9-11] and some of the references therein.

On the other hand, it is well-known that the Kalman-Yakubovich-Popov Lemma (KYPL), also referred to often as the Yakubovich-Kalman-Popov Lemma (YKPL), links the global asymptotic stability through a storage function to a prescribed RH_∞ norm of the transfer matrix while the Positive Real Lemma (PRL) links the passivity, or positive realness, to a storage function. The main objective of this paper is the study of some basic relations between the external positivity property of, in general, Multi-Input Multi-Output (MIMO) globally asymptotically stable linear time-invariant time-delay systems which possess a finite number of finite point delays with the KYPL and the PRL. Previous related studies on those lemmas for the case of systems with eventual absence of delays in the dynamics and which are not required to be externally positive are abundant in the background literature. For instance, in [12], a presentation is performed with the stabilization problem via designs based on matrix inequalities. Classical versions of the KYPL and the PRL are described in a pedagogical way. In [13], a parameterization is provided of a sequence of cone positive Lyapunov functions through positive semidefinite matrices for linear systems subject to delays. On the other hand, a proof of the multivariable KYPL is provided in [14] based on finite dimensional convexity tools. An extended version of the above lemma is given and proved in [15] for positive systems based on non-strict matrix inequalities. In [16], dissipativity theory is provided for the discrete stochastic case together with an "ad hoc" version of such a lemma. On the other hand, the design of positive real transfer functions is discussed in [17-19] including practical techniques for performing such a design. Positive realness of a feed-forward linear loop, that is a controlled linear plant, is needed for the closed-loop hyperstability of a feedback connection of such a linear plant with any member of a so-called hyperstable controller class. In that way, issues of hyperstability theory and some related applications are discussed, for instance, in [20,21] and some of the references therein.

The rest of the content of this paper is organized as follows. Section 2 deals with the dynamic delayed system under study and its stability and some positivity and external positivity properties based on the parallel properties on both auxiliary systems since those systems can exhibit those properties if the whole system is stable or externally positive independent of the delays, respectively. Furthermore, under external positivity and non-negative initial conditions and inputs, it is found that the state (respectively, the output) trajectory solution of the whole delayed system is not smaller at any time instant than that the state (respectively, the output) trajectory solution of that of the auxiliary delay-free system under the same non-negative input. It is also proved that the state (respectively, the output) trajectory solution of the auxiliary delay-free system is not smaller at any time instant than that of the auxiliary system of delay-free dynamics. It is also seen that the global asymptotic stability of the whole delayed system is guaranteed under the global asymptotic stability of any of the two auxiliary undelayed systems plus another robustness constraint in terms of RH_∞ norms of the error transfer matrices between that of the whole system and that of the corresponding auxiliary system.

On the other hand, it is known that the KYPL gives an equivalence between a prescribed upper-bound of the RH_∞-norm of the transfer matrix and the conditions for globally asymptotically stable state-space realizations of such transfer matrices. In this way, Section 3 of this paper extends the classical KYPL for the studied class of externally positive time-delay globally asymptotically stable independent of the delays systems. In parallel, two inequalities are given for the RH_∞-norms of the delayed externally positive system which are linked to the KYPL, and the corresponding auxiliary system of delayed free-dynamics. On the other hand, the closely related PRL links the global asymptotic stability of a dynamic system to its passivity, or to the positive realness of its associated transfer matrix. A version of the PRL is also given in this section independent of the delays for the whole delayed system. However, such a result is not directly applicable independent of the delays if the system is externally positive. Section 4 is devoted to an "ad hoc" extension of the PRL for externally positive globally asymptotically stable systems which have a finite number of finite point delays and the same number of inputs and outputs. A time-varying piecewise constant time-varying pre-compensator matrix gain is designed for the transfer matrix to modify the system output under some rather weak conditions. Such a gain generates a modified output from the injected non-negative system output. Such a modified output can have negative values at certain intervals and an eventual attenuation of the output level. Finally, conclusions end the paper.

Notation and nomenclature. $\boldsymbol{R}_+ = \{r \in \boldsymbol{R} : r > 0\}$; $\boldsymbol{R}_{0+} = \{r \in \boldsymbol{R} : r \geq 0\}$; $\boldsymbol{R}_- = \{r \in \boldsymbol{R} : r < 0\}$ and $\boldsymbol{R}_{-0} = \{r \in \boldsymbol{R} : r \leq 0\}$ are subsets of the set $\boldsymbol{R}$ of the real numbers.

$\boldsymbol{Z}_+ = \{r \in \boldsymbol{Z} : r > 0\}$; $\boldsymbol{Z}_{0+} = \{r \in \boldsymbol{Z} : r \geq 0\}$; $\boldsymbol{Z}_- = \{r \in \boldsymbol{Z} : r < 0\}$ and $\boldsymbol{Z}_{-0} = \{r \in \boldsymbol{Z} : r \leq 0\}$ are subsets of the set $\boldsymbol{Z}$ of the integer numbers.

$\boldsymbol{C}_+ = \{z \in \boldsymbol{C} : Re\, z > 0\}$; $\boldsymbol{C}_{0+} = \{z \in \boldsymbol{C} : Re\, z \geq 0\}$; $\boldsymbol{C}_- = \{z \in \boldsymbol{C} : Re\, z < 0\}$ and $\boldsymbol{C}_{-0} = \{z \in \boldsymbol{C} : Re\, z \leq 0\}$ are subsets of the set $\boldsymbol{C}$ of the complex numbers.

$\bar{n} = \{1, 2, \ldots, n\} \subset \boldsymbol{Z}_+$.

An internally positive (or, simply, positive) dynamic system is that whose both the n-state vector and the p-output vector satisfy $x(t) \in \boldsymbol{R}_{0+}^n$ and $y(t) \in \boldsymbol{R}_{0+}^p$ for all time $t \geq 0$ provided that the initial conditions and the m-vector control $u(t) \in \boldsymbol{R}_{0+}^m$ have non-negative components for all time $t \geq 0$. An externally positive dynamic system is the one whose output vector has non-negative components for all time under zero initial conditions and non-negative control components for all time.

A real matrix $A \in \boldsymbol{R}^{n\times q}$ whose entries are all non-negative is referred to as a non-negative matrix and it is denoted by $A \in \boldsymbol{R}_{0+}^{n\times q}$ or, in short, by $A \succeq 0$.

A real matrix $A \in \boldsymbol{R}^{n\times q}$ whose entries are all non-negative and at least one is positive is referred to as a positive matrix and it is denoted by $A \in \boldsymbol{R}_{0+}^{n\times q}$, with $A \neq 0$, or by $A \succ 0$.

A real matrix $A \in \boldsymbol{R}^{n\times q}$ whose entries are all positive is referred to as a strictly positive matrix and it is denoted by $A \in \boldsymbol{R}_{+}^{n\times q}$ or by $A \succ\succ 0$.

A real vector $v \in \boldsymbol{R}^n$ is non-negative if all its components are non-negative (denoted by $v \in \boldsymbol{R}_{0+}^n$ or $v \succeq 0$). It is positive if $v(\neq 0) \in \boldsymbol{R}_{0+}^n$, that is, at least one of its components is positive, denoted also in short with $v \succ 0$, and strictly positive if $v \in \boldsymbol{R}_{+}^n$ (that is, all its components are positive), denoted also in short by $v \succ\succ 0$.

A real square matrix $A = (a_{ij}) \in \boldsymbol{R}^{n\times n}$ is a Metzler matrix, denoted by $A \in M_E^{n\times n}$, if $a_{ij} \geq 0$; $\forall i,\ j(\neq i) \in \bar{n}$, that is all its off-diagonal entries are non-negative. If $A \in M_E^{n\times n}$, then $e^{At} \succ 0$ and non-singular for all $t \geq 0$.

A monomial, or generalized permutation, matrix is a square matrix with only a non-zero entry per row and per column. The inverse of a positive permutation matrix is also positive.

The notations $A \succeq B$, $A \succ B$ and $A \succ\succ B$, and $v \succeq z$, $v \succ z$ and $v \succ\succ z$, for, respectively, dimensionally compatible real matrices and vectors, stand for $A - B \succeq 0$, $A - B \succ 0$ and $A - B \succ\succ 0$, and respectively, $v - z \succeq 0$, $v - z \succ 0$ and $v - z \succ\succ 0$.

I_n is the identity matrix of order n.

0_n is a square zero matrix of order n and $0_{n\times q}$ is a rectangular matrix of n rows and q columns.

$\boldsymbol{i} = \sqrt{-1}$ is the imaginary complex unity.

$\boldsymbol{L}$ and $\boldsymbol{L}^{-1}$ stand for the Laplace transform and inverse Laplace transform if they exist. If $F(s)$ is the Laplace of a matrix function $f(t)$, we denote $F(s) = \boldsymbol{L}f(t)$ and $f(t) = \boldsymbol{L}^{-1}F(s)$.

$\delta(t)$ denotes the Dirac impulse distribution.

$\mathbf{1}(t)$ denotes the Heaviside step function.

If $A = (a_{ij})$, then $|A| = (|a_{ij}|) \succeq 0$.

The set $BAPC\ ([-h, 0]; \boldsymbol{R}^n)$ is the one of real absolutely piecewise continuous functions from $[-h, 0]$ to $\boldsymbol{R}^n$.

A complex matrix $G(s) : \boldsymbol{C} \to \boldsymbol{C}^{n\times n}$ is in the Banach space H_∞, so-called Hardy space, if it is analytic in $\boldsymbol{C}_{0+}$, $\lim_{\sigma\to 0^+} G(\sigma + \boldsymbol{i}\omega) = G(\boldsymbol{i}\omega)$ and $\|G\|_{H_\infty} = \sup_{s\in\boldsymbol{C}_+} \bar{\sigma}(G(s)) = ess\sup_{\omega\in\boldsymbol{R}} \bar{\sigma}(G(\boldsymbol{i}\omega)) < +\infty$, where $\bar{\sigma}(.)$ stands for the singular values. If $G(s)$ has rational entries, then the corresponding subspace of H_∞ which contains such matrix functions is denoted by RH_∞. In particular, the stable transfer matrices belong to RH_∞ and because of the symmetry of their frequency responses with respect to the imaginary axis, $\|G\|_{RH_\infty} = \sup_{s\in\boldsymbol{C}_+} \bar{\sigma}(G(s)) = \sup_{\omega\in\boldsymbol{R}_{0+}} \bar{\sigma}(G(\boldsymbol{i}\omega)) < +\infty$.

The transpose conjugate matrix of the complex matrix $G(s)$ is $G^*(s) = G^T(\bar{s})$, where $s = \sigma + \boldsymbol{i}\omega$ of conjugate $\bar{s} = \sigma - \boldsymbol{i}\omega$. In particular, $G^*(\boldsymbol{i}\omega) = G^T(-\boldsymbol{i}\omega)$. A square complex matrix is Hermitian if $G^*(s) = G(s)$.

If $A \in \boldsymbol{R}^{n\times m}$, then $\|A\|_2 = \lambda_{\max}^{1/2}(A^T A)$, where $\lambda_{\max}(.)$ stands for the maximum (real) eigenvalue of the (.)-symmetric matrix.

A^T is the transpose of a real or complex matrix A and A^{-T} is the transpose of the inverse of a real or complex non-singular matrix A.

$G^*(s)$ is the complex conjugate matrix of $G(s) \in \boldsymbol{C}^{p\times q}$. In particular, $G^*(\boldsymbol{i}\omega) = G^T(-\boldsymbol{i}\omega)$.

$L_2^p[a, b]$ is the set of square-integrable real or complex-valued vector functions of dimension p on $[a, b]$. In particular, $L_2^p \equiv L_2^p[0, +\infty)$. If $p = 1$ (scalar square-integrable functions), the superscript p is omitted. $< f, g >_{L_2^p[a,b]} = \int_a^b f^*(t)g(t)dt$ is the inner product of $f, g \in L_2^p[a, b]$.

2. **Dynamic Linear Time-Invariant System with Point Delays and Its External Positivity and Stability.** Consider the subsequent linear dynamic delayed system of order n with p outputs q inputs subject to m finite point delays:

$$\dot{x}(t) = \sum_{i=0}^{m} A_i x(t - h_i) + Bu(t) = \sum_{i=0}^{m} A_i x(t) + \sum_{i=1}^{m} A_i(x(t - h_i) - x(t)) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) + Du(t) \tag{2}$$

where $x : [-h, 0) \to \boldsymbol{R}^n$ is the state vector function with $h_0 = 0$ and $h = \max_{i \in \bar{m}} h_i$, and $u : [-h, 0) \in \boldsymbol{R}^q$ and $y : [-h, 0) \in \boldsymbol{R}^p$ are, respectively, the control and output functions, and $A_i \in \boldsymbol{R}^{n \times n}$, $B \in \boldsymbol{R}^{n \times q}$, $C \in \boldsymbol{R}^{p \times n}$ and $D \in \boldsymbol{R}^{p \times q}$ are, respectively, the matrices of undelayed (A_0 for $i = 0$), delayed (A_i for $i \in \bar{m}$) dynamics, control and output vectors and direct input-output interconnection matrix.

We will refer to (1) as a differential system and to (1)-(2) as a dynamic system. In other words, the solution of the differential system describes the state evolution through time which together with (2) describes jointly the evolutions of the state and output, that is, the evolution of the dynamic system.

Assume without a loss in generality and the sake of exposition simplicity that the constant point delays satisfy $0 = h_0 < h_1 < \cdots < h_m = h < +\infty$. It is also assumed that the differential system (1) is subject to any given bounded absolutely continuous function of initial conditions $\varphi : [-h, 0] \to \boldsymbol{R}^n$ with $\varphi(0) = x(0) = x_0$ so that $x(t) \equiv \varphi(t)$ for $t \in [-h, 0]$. Under the above conditions, the solution $x : [-h, +\infty) \to \boldsymbol{R}^n$, which satisfies (1) on $[0, +\infty)$ is unique for each given $\varphi : [-h, 0] \to \boldsymbol{R}^n$ provided that $u : [-h, 0) \in \boldsymbol{R}$ is piecewise continuous.

Relevant particular cases of (1), subject to (2), which do not involve delays are

$$\dot{x}_{dfd}(t) = A_0 x_{dfd}(t) + Bu(t) \tag{3}$$

$$\dot{x}_{df}(t) = \sum_{i=0}^{m} A_i x_{df}(t) + Bu(t) \tag{4}$$

which are referred to, respectively, as the auxiliary system of delayed-free dynamics or infinite delay system (that is, $A_i = 0$, or equivalently $h_i = +\infty$, for $i \in \bar{m}$) and the auxiliary delay-free system or delayed-free dynamics system. It turns out directly that the solutions of (1), (3) and (4) are, respectively,

$$\begin{aligned} x(t) &= e^{A_0 t}\left[x_0 + \int_0^t \left(e^{-A_0 \tau}\left[\sum_{i=1}^{m} A_i x(\tau - h_i) + Bu(\tau)\right] d\tau\right)\right] \\ &= e^{\sum_{i=0}^{m} A_i t}\left[x_0 + \int_0^t \left(e^{-\sum_{i=0}^{m} A_i \tau}\left[\sum_{i=1}^{m} A_i(x(\tau - h_i) - x(\tau)) + Bu(\tau)\right] d\tau\right)\right] \\ &= x_{dfd}(t) + \sum_{i=1}^{m} \int_0^t e^{A_0(t-\tau)} A_i x(\tau - h_i) d\tau \\ &= x_{df}(t) + \sum_{i=1}^{m} \int_0^t e^{A_0(t-\tau)} A_i(x(\tau - h_i) - x(\tau)) d\tau, \ t > 0; \ x(t) = \varphi(t), \ t \in [-h, 0] \end{aligned} \tag{5}$$

$$x_{dfd}(t) = e^{A_0 t}\left[x_0 + \int_0^t e^{-A_0 \tau} Bu(\tau) d\tau\right], \quad t > 0 \tag{6}$$

$$
\begin{aligned}
x_{df}(t) &= e^{\sum_{i=0}^{m} A_i t}\left[x_0 + \int_0^t e^{-\sum_{i=0}^{m} A_i t} Bu(\tau)d\tau\right] \\
&= x_{dfd}(t) + \sum_{i=1}^{m} \int_0^t e^{A_0(t-\tau)} A_i x(\tau)d\tau, \quad t > 0 \qquad (7)
\end{aligned}
$$

provided that $x_{dfd}(0) = x_{df}(0) = x(0) = \varphi(0) = x_0$.

The following result is very obvious and already known from the differential systems (1), (3) and (4) and their solutions (5)-(7).

Proposition 2.1. *The following properties hold.*

(i) The delayed system (1)-(2) is positive independent of the delays if $A_0 \in M_E^{n\times n}$, $A_i \succeq 0$ *for* $i \in \bar{m}$, $B \succeq 0$, $C \succeq 0$ *and* $D \succeq 0$.

(ii) The auxiliary system of delayed-free dynamics (3)-(2) is positive if $A_0 \in M_E^{n\times n}$, $B \succeq 0$, $C \succeq 0$ *and* $D \succeq 0$.

(iii) The auxiliary delay-free system (4)-(2) is positive if $\sum_{i=0}^{m} A_i \in M_E^{n\times n}$, $B \succeq 0$, $C \succeq 0$ *and* $D \succeq 0$.

Note that $A_0 \in M_E^{n\times n}$ and $A_i \succeq 0$ for $i \in \bar{m}$ imply that $\sum_{i=0}^{m} A_i \in M_E^{n\times n}$. Then, the following result is direct.

Proposition 2.2. *If the delayed system (1)-(2) is positive independent of the delays, then the auxiliary system of delayed-free dynamics (3)-(2) and the auxiliary delay-free system (4)-(2) are both positive. The converse assertions are not true, in general.*

The outputs and transfer matrices of the delayed system and those of the two auxiliary ones are now given which are unique for each admissible function of initial conditions and each admissible control.

a) Output of the delayed system

$$
\begin{aligned}
y(t) &= Ce^{A_0 t}\left[x_0 + \int_0^t \left(e^{-A_0\tau}\left[\sum_{i=1}^{m} A_i x(\tau - h_i) + Bu(\tau)\right] d\tau\right)\right] + Du(t) \\
&= Ce^{\sum_{i=0}^{m} A_i t}\left[x_0 + \int_0^t \left(e^{-\sum_{i=0}^{m} A_i\tau}\left[\sum_{i=1}^{m} A_i(x(\tau - h_i) - x(\tau)) + Bu(\tau)\right] d\tau\right)\right], \ t > 0 \quad (8)
\end{aligned}
$$

and the system transfer matrix is

$$
G(s) = C\left(sI_n - \sum_{i=0}^{m} A_i e^{-h_i s}\right)^{-1} B + D = \frac{N_G\left(s, e^{-h_1 s}, \ldots, e^{-h_m s}\right)}{D_G\left(s, e^{-h_1 s}, \ldots, e^{-h_m s}\right)} \qquad (9)
$$

where $N_G(.)$ and $D_G(.)$ are its numerator and denominator quasi-polynomials of respective degrees n' and n in s with $n' \le n$ ($n' = n$ if $D \ne 0$).

b) Output of the auxiliary system of delayed-free dynamics

$$
\begin{aligned}
y_{dfd}(t) &= Ce^{A_0 t}\left[x_0 + \int_0^t e^{-A_0\tau} Bu(\tau)d\tau\right] + Du(t) \\
&= y(t) - \int_0^t Ce^{A_0(t-\tau)} A_i x(\tau - h_i)d\tau, \quad t > 0 \qquad (10)
\end{aligned}
$$

and the system transfer matrix is

$$
G_{dfd}(s) = C(sI_n - A_0)^{-1}B + D = \frac{N_{G_{dfd}}(s)}{D_{G_{dfd}}(s)} \qquad (11)
$$

of respective numerator and denominator polynomials $N_{G_{dfd}}(s)$ and $D_{G_{dfd}}(s)$.

c) Output of the auxiliary delay-free system

$$\begin{aligned} y_{df}(t) &= Ce^{\sum_{i=0}^{m} A_i t}\left[x_0 + \int_0^t e^{-\sum_{i=0}^{m} A_i \tau} Bu(\tau)d\tau\right] + Du(t) \\ &= Ce^{A_0 t}\left[x_0 + \int_0^t e^{-A_0 \tau} Bu(\tau)d\tau\right] + Du(t) + \sum_{i=1}^{m} \int_0^t Ce^{A_0(t-\tau)} A_i x(\tau)d\tau \\ &= y_{dfd}(t) + \sum_{i=1}^{m} \int_0^t Ce^{A_0(t-\tau)} A_i x(\tau)d\tau, \quad t > 0 \end{aligned} \tag{12}$$

and the system transfer matrix is

$$G_{df}(s) = C\left(sI_n - \sum_{i=0}^{m} A_i\right)^{-1} B + D = \frac{N_{G_{df}}(s)}{D_{G_{df}}(s)} \tag{13}$$

of respective numerator and denominator polynomials $N_{G_{df}}(s)$ and $D_{G_{df}}(s)$.

Note that $e^{A_0 t}$ is the fundamental matrix function of the auxiliary system of delay-free dynamics and $e^{\sum_{i=0}^{m} A_i t}$ is the fundamental matrix function of the auxiliary delay-free system. The expression (8) gives the state trajectory of the delayed system based upon the fundamental matrix functions of the two auxiliary systems. The solution can also be given as a function of its own fundamental matrix $\Psi : [-h, +\infty) \to \boldsymbol{R}^{n\times n}$ as follows.

Under identically null control, define $\Psi : [-h, +\infty) \to \boldsymbol{R}^{n\times n}$ with $\Psi(0) = 0$ for $t \in [-h, 0)$ and $\Psi(0) = I_n$ to satisfy (1) on $[0, +\infty)$, that is,

$$\dot{\Psi}(t) = \sum_{i=0}^{m} A_i \Psi(t - h_i) = A_0 \Psi(t) + \sum_{i=1}^{m} A_i \Psi(t - h_i), \quad t \geq 0 \tag{14}$$

whose unique solution is

$$\Psi(t) = e^{A_0 t}\left[I_n + \sum_{i=1}^{m} \int_0^t e^{-A_0 \tau} A_i \Psi(\tau - h_i) \mathbf{1}(\tau - h_i)d\tau\right]; \quad t \geq 0 \tag{15}$$

which is the fundamental matrix function of the delay-free system (1). Thus, the subsequent result holds.

Proposition 2.3. *The state trajectory solution (5) and the output trajectory solution (8) of (1) on* $[0, +\infty)$ *can be rewritten equivalently as follows:*

$$x(t) = \Psi(t)x_0 + \sum_{i=1}^{m} \int_0^{h_i} \Psi(t - \tau) A_i \varphi(\tau - h_i)d\tau + \int_0^t \Psi(t - \tau) Bu(\tau)d\tau; \quad t \geq 0 \tag{16}$$

$$\begin{aligned} y(t) = C&\left(\Psi(t)x_0 + \sum_{i=1}^{m} \int_0^{h_i} \Psi(t - \tau) A_i \varphi(\tau - h_i)d\tau + \int_0^t \Psi(t - \tau) Bu(\tau)d\tau\right) \\ &+ Du(t); \quad t \geq 0 \end{aligned} \tag{17}$$

Note that since the impulse response $g(t)$ of a dynamic system is the output matrix function on $[0, \infty)$ under zero initial conditions of a unity Dirac impulse at $t = 0$ which is also the Laplace inverse transform of the system transfer matrix transfer matrix. One concludes from (17), (10) and (12) that the impulse responses of the system with delays and those of the two auxiliary systems are those given in the result which follows.

Proposition 2.4. *The impulse response matrices of the delayed system, auxiliary system of delayed dynamics and delay-free system are, respectively,*

$$g(t) = \int_0^t C\Psi(t-\tau)Bd\tau + D\delta(t), \quad t \geq 0 \tag{18}$$

$$g_{dfd}(t) = Ce^{A_0 t}B + D\delta(t), \quad t \geq 0 \tag{19}$$

$$g_{df}(t) = Ce^{\sum_{i=0}^{m} A_i t}B + D\delta(t), \quad t \geq 0 \tag{20}$$

A dynamic system is externally positive if its impulse response matrix is non-negative for all time. This guarantees that for any non-negative input the output is non-negative for all time. It also holds that if the system is (internally) positive, then it is externally positive as well.

The following result is concerned with global asymptotic stability and some relative comparisons of output sizes between the above three systems, namely, the delayed system and the two auxiliary ones, provided that they are positive.

Theorem 2.1. *The following properties hold.*

(i) Assume that A_0 is a stability matrix and

$$\left\| (sI - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-h_i s} \right) \right\|_{RH_\infty} = \sup_{\omega \in \boldsymbol{R}_{0+}} \bar{\sigma} \left((\boldsymbol{i}\omega I - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i} h_i \omega} \right) \right) < 1 \tag{21}$$

Then, the unforced delayed system (1)-(2) is globally asymptotically stable independent of the delays sizes and then the auxiliary and delayed-free dynamics and delay-free systems (3)-(2) and (4)-(2) are also globally asymptotically stable and then A_0 and $\sum_{i=0}^{m} A_i$ are stability matrices.

(ii) Assume that $A_0 \in M_E^{n\times n}$, $A_i \succeq 0$ for $i \in \bar{m}$, $B \succeq 0$, $C \succeq 0$, $D \succeq 0$, $\varphi : [-h, 0] \to \boldsymbol{R}_{0+}^n$ and $u : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}$. Then, the delayed system and the two auxiliary systems are positive and, furthermore, $x(t) \succeq x_{dfd}(t)$, $y(t) \succeq y_{dfd}(t)$, $x_{df}(t) \succeq x_{dfd}(t)$ and $y_{df}(t) \succeq y_{dfd}(t)$ for $t \geq 0$.

(iii) Under the assumptions of Property (ii), the impulse response matrices satisfy the constraints $g(t) \succeq g_{dfd}(t) \succeq 0$, $g_{df}(t) \succeq g_{dfd}(t) \succeq 0$ for $t \geq 0$.

(iv) Assume that $A_0 \in M_E^{n\times n}$, $\left(-A_0^{-1}\right) \succ 0$, $A_i \succeq 0$ for $i \in \bar{m}$, $B \succeq 0$, $C \succeq 0$, $D \succeq 0$ and that (21) holds. Then,

$$\begin{aligned} \|G(s)\|_{RH_\infty} = \|G(0)\|_2 = \|G_{df}(s)\|_{RH_\infty} = \|G_{df}(0)\|_2 = \lambda_{\max}^{1/2} \left[\left(\sum_{i=0}^{m} A_i \right)^{-T} \left(\sum_{i=0}^{m} A_i \right)^{-1} \right] \\ \geq \|G_{dfd}(s)\|_{RH_\infty} = \|G_{dfd}(0)\|_2 = \lambda_{\max}^{1/2} \left(A_0^{-T} A_0^{-1} \right) \end{aligned} \tag{22}$$

Proof: If A_0 is a stability matrix and (21) holds, then all the characteristic zeros of the delay system (1) are in $\boldsymbol{C}_-$ independent of the delays sizes since

$$G(s) = \left[(sI_n - A_0) \left(I_n - (sI_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-h_i s} \right) \right) \right]^{-1} \in RH_\infty \tag{23}$$

provided that A_0 is a stability matrix, so that $(sI_n - A_0)^{-1}$ is in RH_∞, and $\left\| (sI_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-h_i s} \right) \right\|_{RH_\infty} < 1$. Therefore, both auxiliary differential systems (3)-(4) are also globally asymptotically stable since they are particular versions for infinity and zero delay sizes. Also, $\sum_{i=0}^{m} A_i$ and A_0, which are the matrices of dynamics under zero delay and in the absence of delayed dynamics, respectively, are stability matrices. Property (i) has been proved.

Property (ii) follows directly from (10) and (12) since, in view of the given assumptions, one has the positivity of the delayed dynamic system and the two auxiliary ones, that is, that $x(t) \succeq 0$, $y(t) \succeq 0$, $x(t) \succeq 0$ for $t \in [-h, \infty)$ with $x(t) = \varphi(t)$ for $t \in [-h, 0]$.

Furthermore, one has from (5)-(6) and (10) and from (7)-(6) and (12) that $x(t) \succeq x_{dfd}(t)$, $y(t) \succeq y_{dfd}(t)$, $x_{df}(t) \succeq x_{dfd}(t)$ and $y_{df}(t) \succeq y_{dfd}(t)$ for $t \in [-h, \infty)$. Property (ii) has been proved.

Property (iii) follows from the inequalities of Property (ii) through the given assumptions applied to (18)-(20), the fact that $\Psi : [0, \infty) \to \boldsymbol{R}_+^{n\times n}$ from (15) and the fact that each $i(\in \bar{q})$-th column of the impulse response matrix is the p-output vector function of time for a Dirac impulse at the i-th input component. Then the non-negativity inequalities of the Property (iii) follow for all entries and column vectors of the respective impulse responses.

To prove Property (iv), first note that if $A_0 \in M_E^{n\times n}$ and $\left(-A_0^{-1}\right) \succ 0$, then A_0 is stability matrix. If, in addition, $A_i \succeq 0$ for $i \in \bar{m}$ and (21) holds, then the delay differential systems (1), (3) and (4) are both positive and globally asymptotically stable from Property (i). If, furthermore, $B \succeq 0$, $C \succeq 0$ and $D \succeq 0$, then the dynamic systems (1)-(2), (3)-(2) and (4)-(2) are jointly positive and globally asymptotically stable. On the other hand, note that

$$\begin{aligned} |G(\boldsymbol{i}\omega)| &= \left|\int_0^t g(t)e^{-\boldsymbol{i}\omega t}dt\right| \le \int_0^t |g(t)|\left|e^{-\boldsymbol{i}\omega t}\right| dt \le \int_0^t |g(t)|dt = |G(0)| \\ &= \int_0^t g(t)dt = G(0), \quad \omega \in \boldsymbol{R} \end{aligned} \tag{24}$$

which implies that $\sup_{\omega\in\boldsymbol{R}_{0+}} |G(\boldsymbol{i}\omega)| = |G(0)| = G(0)$ so that since the delayed system is globally asymptotically stable, all the poles of its transfer matrix are in $\boldsymbol{C}_-$, then

$$\begin{aligned} \|G(s)\|_{RH_\infty} &= \|G(0)\|_2 = \left\|\left(-\sum_{i=0}^m A_i\right)^{-1}\right\|_2 = \lambda_{\max}^{1/2}\left[\left(\sum_{i=0}^m A_i\right)^{-T}\left(\sum_{i=0}^m A_i\right)^{-1}\right] \\ &= \|G_{df}(0)\|_2 \end{aligned} \tag{25}$$

In the same way, $\|G_{dfd}(s)\|_{RH_\infty} = \|G_{dfd}(0)\|_2 = \lambda_{\max}^{1/2}\left(A_0^{-T}A_0^{-1}\right)$. Also, it follows from Property (iii) that $g_{df}(t) \succeq g_{dfd}(t) \succeq 0$ for $t \ge 0$

$$\begin{aligned} &\Rightarrow \|G_{df}(s)\|_{RH_\infty} = \|G_{df}(0)\|_2 = \int_0^t g_{df}(t)dt \succeq \int_0^t g_{dfd}(t)dt = \|G_{dfd}(0)\|_2 \\ &= \|G_{dfd}(s)\|_{RH_\infty} \end{aligned} \tag{26}$$

and (22) has been proved. □

Remark 2.1. *It is known from [9-11] that a) the transfer matrices of time-delay systems like (1)-(2) have infinitely-many poles; b) the maximum value of such poles is finite and the total number of poles in the complex plane* $Re\, s \in [-\sigma, +\infty)$ *is finite for any positive real constant* σ *since there is a finite set of poles in each vertical strip of the complex plane. This implies that infinitely many poles fulfil* $Re\, s \to -\infty$. *The above facts apply to any characteristic equation of the same type as the equation of characteristic zeros (that is, the poles of the transfer matrix) so that they apply also to the quasi-polynomial defining the zeros of a time-delay system.*

Note for the second identity of (1) that another alternative sufficiency-type condition for the global asymptotic stability independent of the delays of the whole delayed system follows from that of the auxiliary delay-free system under the joint conditions that $\sum_{i=0}^m A_i$ is a stability matrix and the RH_∞ norm of the transfer matrix $\bigl(sI_n - \sum_{i=0}^m A_i\bigr)^{-1}\bigl(\sum_{i=0}^m A_i$

$\big(e^{-h_i s}-1\big)\big)$ be less than unity, for which it suffices that $\big\|\big(\sum_{i=0}^{m} A_i\big)\big\|_2 < 1\Big/\Big(2\Big\|\Big(\boldsymbol{i}\omega I_n - \sum_{i=0}^{m} A_i\Big)^{-1}\Big\|_2\Big)$.

For the Single-Input Single-Output (SISO) case of one input and an output, i.e., $p = q = 1$, $C \to c \in \boldsymbol{R}^n$, $B \to b \in \boldsymbol{R}^n$ and $D \to d(>0)$, the following result is of interest for the case of realizable inverse system, that is, $d > 0$.

Theorem 2.2. *Assume that (1)-(2) is an SISO externally positive system for an array* $\hat{h} = (h_0 = 0, h_1, \ldots, h_m = h)$ *of increasing strictly ordered delays which has a biproper transfer function, i.e.,* $c^T\Psi(t)b \geq 0$ *for* $t \geq 0$ *and* $d > 0$ *from (18) and that the transfer function* $G(s) = c^T\left(sI_n - A_0 - \sum_{i=1}^{m} A_i e^{-h_i s}\right)^{-1} b + d$ *has no zero-pole cancellation. Assume also that the sets of distinct real zeros and real poles of (1)-(2) fulfilling* $Re\, s \geq -\sigma$ *for a given positive real constant* σ*, are the respective sets* $Z_{G\sigma} = \{z_1, z_2, \ldots, z_\alpha\} \supset [\sigma, z_\alpha]$ *and* $P_{G\sigma} = \{p_1, p_2, \ldots, p_\beta\} \supset [\sigma, z_\alpha]$ *of finite cardinals, with eventual multiplicities being unity or higher, satisfying also the strict ordering conditions* $z_i < z_{i+1}$, $p_j < p_{j+1}$ *for* $i \in \overline{\alpha - 1}$ *and* $j \in \overline{\beta - 1}$, $z_i \neq p_j$ *for* $i \in \bar{\alpha}$ *and* $j \in \bar{\beta}$, z_α *and* p_β *being finite.*

Then, the following constraints hold.

(i) $z_\alpha < p_\beta$.

(ii) The inverse system of transfer function of real zeros and real poles $Z_G^I = \bigcup_{\sigma\in\boldsymbol{R}_{0+}} Z_{G\sigma}^I \supset Z_{G\sigma}^I \equiv P_{G\sigma} = \{p_1, p_2, \ldots, p_\beta\}$ *and* $P_G^I = \bigcup_{\sigma\in\boldsymbol{R}_{0+}} P_{G\sigma}^I \supset P_{G\sigma}^I \equiv Z_{G\sigma} = \{z_1, z_2, \ldots, z_\alpha\}$ *cannot be externally positive.*

Now assume, furthermore, that $0 \notin Z_G \cup P_G$. *Define the* σ*-reciprocal system of (1)-(2) as the one which has a transfer function whose zeros and poles are, respectively, the inverses of the poles and zeros of the transfer function of (1)-(2), so that the sets of real ordered zeros and ordered real poles are given by* $Z_{G\sigma}^R = \left\{p_\beta^{-1}, p_{\beta-1}^{-1}, \ldots, p_1^{-1}\right\} \subset \left[p_\beta^{-1}, \sigma^{-1}\right]$ *and* $P_{G\sigma}^R = \left\{z_\alpha^{-1}, z_{\alpha-1}^{-1}, \ldots, z_1^{-1}\right\} \subset \left[p_\beta^{-1}, \sigma^{-1}\right]$ *for any given real constant* $\sigma > 0$. *Then, the following further properties hold.*

(iii) If the reciprocal system is externally positive for some $\sigma > 0$*, then* $z_1 > p_1$.

(iv) If Property (iii) holds, then

$\big(z_i > p_j,\ \textit{some } i \in \overline{\alpha-1}, j \in \overline{\beta-1}\big)$

$\Rightarrow (p_\beta > z_\alpha > z_{\alpha-1} > \cdots > z_{i+1} > z_i > p_j > p_{j-1} > \cdots > p_1 > z_1 > p_1 \geq \sigma)$

(v) If Property (iii) holds, then

$\big(z_i < p_j,\ \textit{some } i \in \overline{\alpha-1}, j \in \overline{\beta-1}\big)$

$\Rightarrow (\sigma \leq p_1 < z_1 < \cdots < z_{i-1} < z_i < \min(z_{i+1}, p_j) < \min(z_{i+2}, p_{j+1}) < \cdots < \min(z_\alpha, p_\beta) = z_\alpha < p_\beta)$

for some existing positive integer numbers ℓ, δ *with* $\alpha > \ell > i$ *and* $1 \leq \delta \leq \alpha - \ell$.

Proof: Since the transfer function of the SISO dynamic system (1)-(2) is biproper then $d \neq 0$ and since it is externally positive, then $c^T\Psi(t)b \geq 0$ for $t \geq 0$ and $d > 0$ from (18). Note that $Z_{G\sigma}$ and $P_{G\sigma}$ have finite cardinals and are bounded for any given finite $\sigma > 0$ (see Remark 2.1), so that p_β and z_α are finite. Also, for any finite $\sigma' > \sigma$, $Z_{G\sigma'} \subset Z_{G\sigma}$ and $P_{G\sigma'} \subset P_{G\sigma}$ and their upper-members are the same p_β and z_α as above. Now, Property (i) follows since a necessary condition for a system to be externally positive is that any real zero of its transfer function be smaller than the dominant pole [1,2]. Otherwise, assume that $z \in Z_{G\sigma}$ is a real zero of $G(s)$ in $Z_{G\sigma}$ and p_β is its real dominant pole in $P_{G\sigma}$. Thus, if $z \geq p_\beta$, then z lies in the region defined by the radius of convergence of $G(s)$ so that, for $s = z$, $G(z)$ satisfies

$$0 = G(z) = \int_0^\infty g(t)e^{-zt}dt = \int_0^\infty |g(t)|e^{-zt}dt > 0 \tag{27}$$

which is a contradiction. Then, $z < p_\beta$ for $z \in Z_{G\sigma}$ and any real $\sigma > 0$ and, thus, $z_\alpha < p_\beta$. Property (i) has been proved.

Since $G(s) = N_G\left(s, e^{-h_1 s}, \ldots, e^{-h_m s}\right) / D_G\left(s, e^{-h_1 s}, \ldots, e^{-h_m s}\right)$ is biproper by assumption, its numerator and denominator polynomials have the same degree and $G^{-1}(s) = D_G\left(s, e^{-h_1 s}, \ldots, e^{-h_m s}\right) / N_G\left(s, e^{-h_1 s}, \ldots, e^{-h_m s}\right)$ is state-space realizable with sets of real ordered zeros and real ordered poles $Z^I_{G\sigma} \equiv P_{G\sigma} = \{p_1, p_2, \ldots, p_\beta\}$ and $P^I_{G\sigma} \equiv Z_{G\sigma} = \{z_1, z_2, \ldots, z_\alpha\}$ for any given real constant $\sigma > 0$. If the inverse system is externally positive, then $p_\beta < z_\alpha$ from Property (i) applied to the inverse system. This contradicts the external positivity of (1)-(2) under the necessary condition $z_\alpha < p_\beta$ of Property (i). Property (ii) has been proved.

Property (iii) is proved from the ordering condition of the elements of $Z^R_{G\sigma}$ and $P^R_{G\sigma}$ and Property (i).

Properties (iv) and (v) follow from the joint external positivity necessary conditions for the system (1)-(2) and its reciprocal system $z_\alpha < p_\beta$ and $z_1 > p_1$ by taking account of the ordering of elements in the sets $Z_{G\sigma}$, $P_{G\sigma}$, $Z^R_{G\sigma}$ and $P^R_{G\sigma}$. ☐

Remark 2.2. *Although the delayed system has infinitely many poles and it can have infinitely many zeros, their numbers become finite constrained to any half-closed bounded* $\boldsymbol{C}_s$ *subset of* $\boldsymbol{C}$. *Such numbers depend trivially on each such possible defined bounded subset* $\boldsymbol{C}_s$ *of* $\boldsymbol{C}$ *(see Remark 2.1). Note that the necessary condition* $z_1 > p_1$ *of external positivity of the* σ*-reciprocal system depends on the given* $m > 0$ *in the sense that if, for some* σ' *with* $0 < \sigma' < \sigma$, $Z^R_{G\sigma'} = \left\{p_\beta^{-1}, p_{\beta-1}^{-1}, \ldots, p_1^{-1}, p_0^{-1}\right\}$ *and* $P^R_{G\sigma'} = \left\{z_\alpha^{-1}, z_{\alpha-1}^{-1}, \ldots, z_1^{-1}, z_0^{-1}\right\}$, *where* $p_0 < p_1$ *and* $z_0 < z_1$, *then the necessary condition of external positivity of the* m' *reciprocal system would be* $p_0 < z_0$. *If* $z_0 \in Z^R_{G\sigma'}$ *and* $P^R_{G\sigma'} = P^R_{G\sigma}$, *then the condition becomes to be* $p_1 < z_0$. *If* $Z^R_{G\sigma'} = Z^R_{G\sigma}$ *and* $P^R_{G\sigma'} = \left\{z_\alpha^{-1}, z_{\alpha-1}^{-1}, \ldots, z_1^{-1}\right\}$, *then the external positivity condition becomes* $p_0 < z_1$. *If* $\sigma' \neq \sigma$ *but* $Z^R_{G\sigma'} = Z^R_{G\sigma}$ *and* $P^R_{G\sigma'} = P^R_{G\sigma}$, *then the necessary condition becomes unchanged, that is,* $p_0 < z_0$.

Note then that the necessary condition of external positivity of the σ*-reciprocal system depends on the value of* σ *although the transfer function any considered* σ*-reciprocal system has a finite number of zeros and poles. However, note also that the external positivity necessary condition for the delayed system (1)-(2) does not depend on* σ *since it involves the maximum real zero* z_α *and pole* p_β *of* $G(s)$ *only, which do not depend on* σ, *although (1)-(2) has infinitely many poles and can have infinitely many zeros. Note also that Theorem 2.2 also stands for both SISO auxiliary systems, for which* $\sigma = 0$ *trivially since their transfer functions have finite numbers of poles since they have neither delays nor delayed dynamics. Therefore, their transfer functions have a finite number of zeros and poles. Therefore, the 0-reciprocal systems of the auxiliary systems can simply be referred to as "the reciprocal delay-free system" and as "the reciprocal delayed-free dynamics system".*

Remark 2.3. *Note that for a complex zero of* $G(s)$, $z = \rho \pm \boldsymbol{i}\omega$, *with* $\omega > 0$, *then the second equality of (27) does not hold since* $\mathrm{e}^{-zt} = e^{-t\rho}(\cos(t\omega) \pm \boldsymbol{i}\sin(t\omega))$ *implies that (27) generates two real equations with periodic integrands so that (1) has not a defined sign for all time. Thus, the involvement of (27) to get the contradiction implying, by its refusal, that the real zeros of the transfer function have to be smaller than the dominant pole as a necessary condition for external positivity does not apply for complex zeros. On the other hand, the eventual application of the same reasoning to* $G(|z|)$ *is not valid since the modulus of a complex number being less than the dominant pole is not necessarily a zero of the transfer function. Therefore, Theorem 2.2 does not apply to complex zeros of the transfer functions with non-null imaginary part.*

Remark 2.4. *For the linear discrete case, consider that* $G(z) = \sum_{n=0}^{\infty} g_n z^{-n}$ *is the z-transform of the impulse response sequence* $\{g_n\}_{n=0}^{\infty}$*. The above series is a Laurent series of convergence of the type* $|z| > R$*. The elements of the sum have not a common sign for negative real values of z so that the series is oscillatory even if the impulse response is non-negative. Therefore, Theorem 2.2 is not applicable either for real or for complex zeros of* $G(z)$*, and then it does not apply to the discrete case.*

The following results are concerned with the instability of the externally positive auxiliary systems under certain conditions which imply that the delayed system is not globally stable independent of the delays sizes. The results are based on a combination of Eneström-Kakeya theorem on the allocation of the zeros of polynomials (see, for instance, [4-8]) and Theorem 2.2(i) on the sizes of real zeros versus the dominant poles. Firstly, we recall that the celebrated Eneström-Kakeya theorem [5,6] establishes that if $P(z) = \sum_{j=0}^{n} p_j z^j$ is a polynomial of degree $n \geq 1$ and real coefficients and the (non-strict) ordering condition of all the coefficients $p_j \geq p_{j-1} \geq 0$; $j \in \bar{n} \cup \{0\}$ is fulfilled, then all the zeros of $P(z)$ lie in the closed complex unit circle centred at zero $|z| \leq 1$; i.e., the polynomial is (non-strictly) stable. If the above ordering condition is strict rather than non-strict, then all the zeros of $P(z)$ lie in the open complex unit circle centred at zero $|z| < 1$; i.e., the polynomial is Schur, that is, strictly stable in the discrete framework, which is the well-known counterpart of a Hurwitz polynomial in the continuous framework. Many extensions of the above theorem are available in the background literature by considering polynomials with complex coefficients or their stability properties in complex regions non-necessarily being coincident with the unit complex circle.

Proposition 2.5. *Assume that* $G_{dfd}(s) = B_{dfd}(s)/A_{dfd}(s)$ *is the proper transfer function of an SISO externally positive linear continuous-time auxiliary system of delayed-free dynamics, with* $B_{dfd}(s) = \sum_{j=0}^{n_B} b_j s^j$ *and* $A_{dfd}(s) = \sum_{j=0}^{n_A} a_j s^j$ *being coprime polynomials of respective degrees* $1 \leq n_B \leq n_A$ *and* n_A *such that* $0 \leq b_0 \leq b_1 \leq \cdots \leq b_{n_B}$ *and* $B_{dfd}(s)$ *is not Hurwitz and all its zeros are real. Then, the dominant real zero* ρ_{Adfd} *of* $A_{dfd}(s)$ *satisfies* $\rho_{Adfd} > 0$ *so that the auxiliary system of delayed-free dynamics is unstable. As a result, the delayed system is not globally stable independent of the delays sizes.*

If $G_{dfd}(s) \to G_{df}(s)$ *under similar conditions as the above ones then the auxiliary delay-free system is unstable and the delayed system is not globally stable independent of the delays sizes.*

Proof: From Eneström-Kakeya theorem, all the zeros of $B_{dfd}(s)$ lie in $|s| \leq 1$ for the given constraint for the coefficients of $B_{dfd}(s)$ [4-6]. Since $B_{dfd}(s)$ is not Hurwitz, at least one of its real zeros z satisfies $0 \leq z \leq 1$. Thus, the (real) dominant pole of $G_{dfd}(s)$, which is the dominant zero ρ_{Adfd} of $A_{dfd}(s)$, satisfies $\rho_{Adfd} > 0$ from Theorem 2.2(i). Thus, the auxiliary system of delayed-free dynamics is unstable so that the delayed system is not globally stable independent of the delays sizes since it is not stable as delays tend to infinity. The same instability conclusion arises for the delay-free system under similar conditions the whole delayed system is not globally stable independent of the delays sizes since it is not for zero delay. □

Proposition 2.6. *Assume that* $G_{dfd}(s) = B_{dfd}(s)/A_{dfd}(s)$ *is the proper transfer function of an SISO externally positive linear continuous-time auxiliary system of delayed-free dynamics, with* $B_{dfd}(s) = \sum_{j=0}^{n_B} b_j s^j$ *and* $A_{dfd}(s) = \sum_{j=0}^{n_A} a_j s^j$ *being coprime polynomials of respective degrees* $1 \leq n_B \leq n_A$ *and* n_A *whose coefficients are ordered as non-decreasing sequences according to* $b_0 \leq b_1 \leq \cdots \leq b_m$ *and* $a_0 \leq a_1 \leq \cdots \leq a_n$*. If* $\frac{b_m - b_0 + |b_0|}{|b_m|} < \frac{a_0 - a_m - |a_0|}{|a_m|}$*, then all the real zeros of* $B_{dfd}(s)$ *are smaller than the dominant zero of* $A_{dfd}(s)$ *and then the auxiliary system of delayed-free dynamics is externally positive.*

Proof: From the ordering rules for the coefficients of $B_{dfd}(s)$ and $A_{dfd}(s)$, the zeros of $B_{dfd}(s)$ and those of $A_{dfd}(s)$ are, respectively, in the circles [6]:

$$|s| \leq \frac{b_m - b_0 + |b_0|}{|b_m|}, \quad |s| \leq \frac{a_m - a_0 + |a_0|}{|a_m|} \tag{28}$$

Since $G(s)$ is externally positive, it has a dominant real pole ρ_{Adfd}, that is, a dominant zero of $A_{dfd}(s)$, satisfying

$$\rho_{Adfd} \in \left[\frac{a_0 - a_m - |a_0|}{|a_m|}, \frac{a_m - a_0 + |a_0|}{|a_m|}\right] \tag{29}$$

From Theorem 2.1, all the real zeros of $B_{dfd}(s)$, if any, have to be smaller than ρ_{Adfd}, since $G_{dfd}(s)$ is externally positive, what is guaranteed if

$$\frac{b_m - b_0 + |b_0|}{|b_m|} < \frac{a_0 - a_m - |a_0|}{|a_m|} \tag{30}$$

□

Proposition 2.7. *Assume that* $G_{dfd}(s) = B_{dfd}(s)/A_{dfd}(s)$ *is the proper transfer function of an SISO externally positive linear auxiliary system of delayed-free dynamics, with* $B_{dfd}(s) = \sum_{j=0}^{n_B} b_j s^j$ *and* $A_{dfd}(s) = \sum_{j=0}^{n_A} a_j s^j$ *being coprime polynomials of respective degrees* n_B *and* n_A *with* $1 \leq n_B \leq n_A$*. If the real dominant zero* ρ_{Adfd} *of* $A_{dfd}(s)$ *satisfies* $\rho_{Adfd} > \eta$*, where* η *is the unique positive real zero of* $Q(s) = s^{n_B} - \sum_{j=0}^{n_B-1}(b_j/b_m)s^j$*, or if* $\rho_{Adfd} \geq 1 + \max_{i \in \overline{n_B - 1} \cup \{0\}} |b_i|$*, then the necessary condition for external positivity consisting of the real zeros of* $B_{dfd}(s)$ *being smaller than* ρ_{Adfd} *is guaranteed.*

Proof: Define the monic polynomial $B'_{dfd}(s) = \sum_{j=0}^{n_B} b'_j s^j = \sum_{j=0}^{n_B}(b_j/b_n)s^j$, that is, its leading coefficient b'_{n_B} is unity, which is got from $B_{dfd}(s)$ and let $Q(s) = s^{n_B} - \sum_{j=0}^{n_B-1} b'_j s^j$ be a monic polynomial such that $b = \max_{i \in \overline{n_B - 1} \cup \{0\}} |b_i|$. Note that $B'_{dfd}(s)$ and $B_{dfd}(s)$ of Theorem 2.2 have the same zeros since $b'_j = b_j/b_{n_B}$; $j \in \bar{n}_B \cup \{0\}$. It is known from [7,8] that $Q(s)$ has a unique positive real root η. Define the open and closed balls $B(r) = \{s : |s| < r\}$ and $\bar{B}(r) = \{s : |s| \leq r\}$. Then, the zeros of $B(s)$ and $B'(s)$ are in $\bar{B}(\eta) \subset B(1 + b)$. Since $G_{dfd}(s)$ is externally positive, all its real zeros are smaller than its real dominant pole ρ_{Adfd} which is guaranteed if $\rho_{Adfd} > \eta$ or if $\rho_{Afdf} \geq 1 + b$. □

3. **The KYPL and the PRL for the Time-Delay System.**

The standard KYPL and the PRL for linear continuous time-invariant delay-free systems are described, for instance, in [12-16]. The following result gives firstly an extended KYPL for the stable delayed system (1)-(2) which gives an equivalence between a prescribed upper-bound of the transfer matrix RH_∞-norm and a Lyapunov-type stability of an associate state-space realization of such a transfer matrix. The result is applicable too if the system is externally positive. Also, two inequalities are given for the RH_∞ norms of the delayed system and its auxiliary one of delayed free-dynamics for the case of external positivity.

Theorem 3.1. *The following properties hold.*

(i) The following conditions are equivalent:

1) The RH_∞*-norm of the delayed system satisfies the constraint* $\|G\|_{RH_\infty} \leq \gamma$ *independent of the delays.*

2) There exist $(m+1)$ *real symmetric matrices* $P\,(\in \boldsymbol{R}^{n\times n}) > 0$, $S_i\,(\in \boldsymbol{R}^{n\times n}) > 0$ *for* $i \in \bar{m}$ *such that*

$$\hat{Q} = \hat{Q}_1 + \gamma^{-1}\hat{Q}_2$$

$$\begin{aligned}
\hat{Q}_1 &= \begin{bmatrix} \bar{Q} & \bar{P}\bar{B} \\ \bar{B}^T\bar{P} & -\gamma I_q \end{bmatrix} \in \boldsymbol{R}^{((m+1)n+q)\times((m+1)n+q)} \\
\hat{Q}_2 &= \begin{bmatrix} \bar{C}^T \\ D^T \end{bmatrix} \begin{bmatrix} \bar{C} & D \end{bmatrix} \in \boldsymbol{R}^{((m+1)n+q)\times((m+1)n+q)} \\
\bar{C} &= (C, 0_{q\times nm}) \in \boldsymbol{R}^{p\times(m+1)n} \\
\bar{B}^T &= \left(B^T, 0_{q\times mn}\right) \in \boldsymbol{R}^{q\times(m+1)n} \\
\bar{Q} &= \begin{bmatrix} A_0^T P + PA_0 + \sum_{i=1}^m S_i & -PA_1 & \cdots & -PA_m \\ -A_1^T P & -S_1 & & 0 \\ \vdots & & \ddots & \\ -A_m^T P & 0 & & -S_m \end{bmatrix} \in \boldsymbol{R}^{(m+1)n\times(m+1)n} \\
\bar{P} &= Block\ diag(P, 0_{mn}) \in \boldsymbol{R}^{(m+1)n\times(m+1)n}
\end{aligned} \tag{31}$$

(ii) Assume that the delayed system is externally positive independent of the delays with $\|G\|_{RH_\infty} \le \gamma$ for some $\gamma \in \boldsymbol{R}_+$. Then, the following properties hold:

$$\left\| C\left(-\sum_{i=0}^m A_i\right)^{-1} B + D \right\|_2 \le \gamma; \quad \left\|C(-A_0)^{-1}B + D\right\|_2 \le \gamma_{dfd} \tag{32}$$

for some $\gamma_{dfd} \in \boldsymbol{R}_+$, and

$$\begin{aligned}
\|G(s)\|_{RH_\infty} &\le \frac{1}{1-\hat{g}} \|C\|_2 \|B\|_2 \left\|(sI_n - A_0)^{-1}\right\|_{RH_\infty} + \|D\|_2 \\
\|G_{dfd}(s)\|_{RH_\infty} &\le \|C\|_2 \|B\|_2 \left\|(sI_n - A_0)^{-1}\right\|_{RH_\infty} + \|D\|_2 \\
\|G\|_{RH_\infty} - \|G_{dfd}\|_{RH_\infty} &\le \frac{\hat{g}\|C\|_2}{1-\hat{g}} \left\|(sI_n - A_0)^{-1}B\right\|_{RH_\infty} \\
&= \frac{\hat{g}\|C\|_2}{1-\hat{g}} \left\|(\boldsymbol{i}\omega I_n - A_0)^{-1}B\right\|_2
\end{aligned} \tag{33}$$

where

$$\hat{g} = \sup_{\omega \in \boldsymbol{R}_{0+}} \left\| (\boldsymbol{i}\omega I_n - A_0)^{-1} \left(\sum_{i=1}^m A_i e^{-\boldsymbol{i}h_i\omega} \right) \right\|_2 < 1 \tag{34}$$

Proof: First, note that if $\|G\|_{RH_\infty} \le \gamma$ independent of the delays, in the sense that the first inequality of (33) jointly with (35) is valid for any sizes of the delays, then the unforced time-delay system is globally asymptotically stable. Note also that the matrix inequality $\hat{Q} = \hat{Q}_1 + \gamma^{-1}\hat{Q}_2 < 0$ implies, since $\hat{Q}_2$ is singular because of its structure, that $\hat{Q}_1 < 0$, which implies that $\bar{Q} < 0$, which implies also that $A_0^T P + PA_0 + \sum_{i=1}^m S_i < 0$ which is a linear matrix inequality which trivially guarantees the global asymptotic stability of the unforced auxiliary system of delayed-free dynamics. Now, consider the Krasovskii-Lyapunov functional:

$$V(x_t) = x^T(t)Px(t) + \sum_{i=1}^m \int_{-h_i}^0 x^T(t+\tau)S_i x(t+\tau)d\tau \tag{35}$$

(see, for instance, [9]) considered as s a storage function which depends on the control input vector and on the function of initial conditions in Φ such that $\Phi = \{\varphi \in$

$BAPC([-h,0];\boldsymbol{R}^n)\}$. Thus,

$$\begin{aligned}\dot{V}(x_t) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + \sum_{i=1}^{m}\frac{d}{dt}\int_{t-h_i}^{t}\left(x^T(\tau)S_i x(\tau)d\tau\right)d\tau \\ &= \bar{x}^T(t)\bar{Q}\bar{x}(t) + 2x^T(t)PBu(t) \qquad (36)\end{aligned}$$

where $\bar{x}^T(t) = \left(x^T(t), x^T(t-h_1), \ldots, x^T(t-h_m)\right)$. Since the matrix inequality $\hat{Q} < 0$ is strict, $\hat{x}^T(t)\hat{Q}\hat{x}(t) < 0$ for any $\hat{x}(t) \neq 0$, where $\hat{x}^T(t) = \left(\bar{x}^T(t), u^T(t)\right)$, and, furthermore, there exists $\varepsilon \in \boldsymbol{R}_+$ such that

$$\begin{aligned}\vartheta(t) &= \hat{x}^T(t)\left(\hat{Q} + \varepsilon Block\ diag(O_{mn}, I_q)\right)\hat{x}(t) \\ &= \hat{x}^T(t)\left[\left(\hat{Q}_1 + \varepsilon Block\ diag(O_{mn}, I_q)\right) + \gamma^{-1}\hat{Q}_2\right]\hat{x}(t) < 0 \text{ if } \hat{x}(t) \neq 0.\end{aligned}$$

Also, note that

$$y(t) = Cx(t) + Du(t) = \bar{C}\bar{x}(t) + Du(t) \qquad (37)$$

so that

$$\begin{aligned}\gamma^{-1}\hat{x}^T(t)\hat{Q}_2\hat{x}(t) &= \gamma^{-1}\left(\bar{x}^T(t), u^T(t)\right)\hat{Q}_2\left(\bar{x}^T(t), u^T(t)\right)^T \\ &= \gamma^{-1}\left(y^T(t), u^T(t)\right)\hat{Q}_2\left(y^T(t), u^T(t)\right)^T \qquad (38)\end{aligned}$$

and

$$\begin{aligned}\vartheta(t) &= \hat{x}^T(t)\hat{Q}_{1\varepsilon}\hat{x}(t) + \gamma^{-1}y^T(t)y(t) \\ &= \bar{x}^T(t)\bar{Q}\bar{x}(t) - (\gamma - \varepsilon)\|u(t)\|^2 + \gamma^{-1}\|y(t)\|^2 \\ &= \dot{V}(x_t) - (\gamma - \varepsilon)\|u(t)\|^2 + \gamma^{-1}\|y(t)\|^2 < 0, \qquad (39)\end{aligned}$$

since $\hat{Q}_1 < 0$, and $\bar{Q} < 0$, where $\hat{Q}_{1\varepsilon} = \hat{Q}_1 + \varepsilon Block\ diag(O_{mn}, I_q) < 0$.

Note from (39) that

$$\int_0^T \vartheta(t)dt = V(T) - V(0) + \int_0^T \left(\gamma^{-1}\|y(t)\|^2 - (\gamma - \varepsilon)\|u(t)\|^2\right)dt < 0 \qquad (40)$$

Then, for any $u \in L_2[0,+\infty)$, one has

$$\limsup_{T\to+\infty}\frac{\int_0^T \vartheta(t)dt}{\int_0^T \|u(t)\|^2dt} \leq \limsup_{T\to+\infty}\frac{V(x_T) - V(\varphi)}{\int_0^T \|u(t)\|^2dt} + \gamma^{-1}\|G\|_{RH_\infty}^2 + \varepsilon - \gamma \qquad (41)$$

$$\begin{aligned}&\inf_{\phi\in\Phi, u\in L_2[0,+\infty)}\limsup_{T\to+\infty}\frac{\int_0^T \vartheta(t)dt}{\int_0^T \|u(t)\|^2dt} \\ \leq &\inf_{\phi\in\Phi, u\in L_2[0,+\infty)}\limsup_{T\to+\infty}\frac{V(x_T) - V(\varphi)}{\int_0^T \|u(t)\|^2dt} + \gamma^{-1}\|G\|_{RH_\infty}^2 - \gamma \leq 0 \qquad (42)\end{aligned}$$

Note that $(V(T) - V(0))\Big/\int_0^T \|u(t)\|^2dt$ is bounded for any $T \geq 0$, any nonzero and piecewise continuous $u \in L_2[0,+\infty)$ and any admissible function of initial conditions $\varphi \in \Phi$ since the unforced systems are globally asymptotically stable. Irrespectively of the particular $u \in L_2[0,+\infty)$ and $\varphi \in \Phi$, the above condition leads to $\gamma^{-1}\|G\|_{RH_\infty}^2 \leq \gamma + v_u$, where

$$v_u = \inf_{\phi\in\Phi, u\in L_2[0,+\infty)}\limsup_{T\to+\infty}\left[(V(x_T) - V(\varphi))\Big/\int_0^T \|u(t)\|^2dt\right] = 0 \qquad (43)$$

Thus, $\|G\|_{RH_\infty} \leq \gamma$ if the condition 2) holds. Thus, the condition 2) implies the condition 1). Conversely, if $\|G\|_{RH_\infty} \leq \gamma$, $u \equiv 0$ and $\hat{Q} < 0$ guaranteed by the condition

2), then $V(x_T)$ is bounded for any function of admissible initial conditions, $\dot{V}(x_T) < 0$ if the strip x_T is non-null, and $V(x_T) \to 0$ as $T \to \infty$. Thus, the candidate $V(x_T)$ is a Krasovskii-Lyapunov functional. Then, the condition 1) implies the condition 2). Property (i) has been proved. On the other hand, Property (ii) follows directly from the subsequent relations:

$$\|G\|_{RH_\infty} = \|G_{df}\|_{RH_\infty} = \|G(0)\|_2 = \|G_{df}(0)\|_2 = \left\| C\left(-\sum_{i=0}^{m} A_i\right)^{-1} B + D \right\|_2 \tag{44}$$

$$\|G_{dfd}\|_{RH_\infty} = \|G_{dfd}(0)\|_2 = \|C(-A_0)^{-1}B + D\|_2 \tag{45}$$

Since the delayed system is externally positive independent of the delays what implies that a) the auxiliary free-delay system and that of delayed-free dynamics are both also externally positive; and b) both auxiliary systems are also RH_∞-stable what implies that the Metzler matrices of dynamics of both auxiliary systems are stability matrices, non-singular with positive negative inverses. Now, note that

$$\begin{aligned}
&\|G(s)\|_{RH_\infty}\\
&= \left\| C\left(-\sum_{i=0}^{m} A_i\right)^{-1} B + D \right\|_2\\
&= \sup_{s\in\boldsymbol{C}_{0+}} \left\| C\left[(sI_n - A_0)\left(I_n - (sI_n - A_0)^{-1}\left(\sum_{i=1}^{m} A_i e^{-h_i s}\right)\right)\right]^{-1} B + D \right\|_2\\
&= \sup_{\omega\in\boldsymbol{R}_{0+}} \left\| C\left[(\boldsymbol{i}\omega I_n - A_0)\left(I_n - (\boldsymbol{i}\omega I_n - A_0)^{-1}\left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega}\right)\right)\right]^{-1} B + D \right\|_2\\
&= \sup_{\omega\in\boldsymbol{R}_{0+}} \left\| C\left(I_n - (\boldsymbol{i}\omega I_n - A_0)^{-1}\left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega}\right)\right)^{-1} (\boldsymbol{i}\omega I_n - A_0)^{-1} B + D \right\|_2 \le \gamma
\end{aligned} \tag{46}$$

$$\begin{aligned}
&\|G_{dfd}(s)\|_{RH_\infty}\\
&= \sup_{s\in\boldsymbol{C}_{0+}} \left\| C(sI_n - A_0)^{-1} B + D \right\|_2\\
&= \sup_{\omega\in\boldsymbol{R}_{0+}} \left\| C(\boldsymbol{i}\omega I_n - A_0)^{-1} B + D \right\|_2 \le \gamma_{dfd}
\end{aligned} \tag{47}$$

Note that for γ to be finite, it is needed that γ_{dfd} be finite and that $\hat{g} = \sup_{\omega\in\boldsymbol{R}_+} \left\|(\boldsymbol{i}\omega I_n - A_0)^{-1}\left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega}\right)\right\|_2 < 1$. This last constraint implies that

$$\begin{aligned}
&\left(I_n - (\boldsymbol{i}\omega I_n - A_0)^{-1}\left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega}\right)\right)^{-1}\\
&= I_n + \sum_{k=1}^{\infty}\left((\boldsymbol{i}\omega I_n - A_0)^{-1}\left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega}\right)\right)^k
\end{aligned} \tag{48}$$

such that the series of the right-hand-side is convergent for any ω so that

$$\sum_{k=1}^{\infty}\left((\boldsymbol{i}\omega I_n - A_0)^{-1}\left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega}\right)\right)^k$$

$$= \left((\boldsymbol{i}\omega I_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega} \right) \right) \left(\sum_{k=0}^{\infty} \left((\boldsymbol{i}\omega I_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega} \right) \right)^{k} \right) \tag{49}$$

and

$$\sup_{\omega \in \boldsymbol{R}_{0+}} \left\| \sum_{k=1}^{\infty} \left((\boldsymbol{i}\omega I_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega} \right) \right)^{k} \right\|_2 = \frac{\hat{g}}{1-\hat{g}} \tag{50}$$

Then,

$$\begin{aligned}
&\|G(s)\|_{RH_\infty} \\
&= \left\| C \left(-\sum_{i=0}^{m} A_i \right)^{-1} B + D \right\|_2 \\
&= \sup_{\omega \in \boldsymbol{R}_{0+}} \left\| C \left(I_n + (\boldsymbol{i}\omega I_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega} \right) \right) (\boldsymbol{i}\omega I_n - A_0)^{-1} B + D \right\|_2 \\
&\leq \sup_{\omega \in \boldsymbol{R}_{0+}} \left\| C(\boldsymbol{i}\omega I_n - A_0)^{-1} B + D \right\|_2 \\
&\quad + \sup_{\omega \in \boldsymbol{R}_{0+}} \left\| C \sum_{k=1}^{\infty} \left[(\boldsymbol{i}\omega I_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega} \right) \right]^{k} (\boldsymbol{i}\omega I_n - A_0)^{-1} B \right\|_2 \\
&\leq \|G_{dfd}(s)\|_{RH_\infty} + \sup_{\omega \in \boldsymbol{R}_{0+}} \left\| \sum_{k=1}^{\infty} C \left[(\boldsymbol{i}\omega I_n - A_0)^{-1} \left(\sum_{i=1}^{m} A_i e^{-\boldsymbol{i}h_i\omega} \right) \right]^{k} (\boldsymbol{i}\omega I_n - A_0)^{-1} B \right\|_2 \\
&= \left\| C(-A_0)^{-1} B + D \right\|_2 + \frac{\hat{g}\|C\|_2}{1-\hat{g}} \sup_{\omega \in \boldsymbol{R}_{0+}} \left\| (\boldsymbol{i}\omega I_n - A_0)^{-1} B \right\|_2 \\
&\leq \frac{\|C\|_2 \|B\|_2}{1-\hat{g}} \sup_{\omega \in \boldsymbol{R}_{0+}} \left\| (\boldsymbol{i}\omega I_n - A_0)^{-1} \right\|_2 + \|D\|_2 \\
&= \frac{\|C\|_2 \|B\|_2}{1-\hat{g}} \left\| (sI_n - A_0)^{-1} \right\|_{RH_\infty} + \|D\|_2
\end{aligned} \tag{51}$$

and Inequalities (32)-(34) of Property (ii) follow directly. □

The following result is the PRL, a variant to KYPL, applied to the (non-necessarily) positive time-delay system (1)-(2).

Theorem 3.2. *Assume that $p = q$, that is, the numbers of inputs and outputs coincide. Then, the following properties hold.*

(i) The following conditions are equivalent:

1) The transfer matrix of the delayed system $G(s)$ is passive, i.e., $< u, Gu >_{L_2^q} \geq 0$, independent of the delays, equivalently, $G(s)$ is positive real, i.e., $G(s) + G^*(s) \geq 0$ for all $Re\, s \geq 0$.

2) There exist $(m+1)$ real symmetric matrices $P\,(\in \boldsymbol{R}^{n\times n}) > 0$, $S_i\,(\in \boldsymbol{R}^{n\times n}) > 0$ for $i \in \bar{m}$ such that

$$\hat{Q}_p = \begin{bmatrix} \bar{Q} & \bar{P}\bar{B} - \bar{C}^T \\ \bar{B}^T\bar{P} - \bar{C} & -\left(D + D^T\right) \end{bmatrix} \left(\in \boldsymbol{R}^{((m+1)n+q)\times((m+1)n+q)}\right) \leq 0 \tag{52}$$

under the definitions in (31) of the various involved matrices involved in (52).

(ii) $G(s)$ *is strictly passive, i.e.,* $<u, Gu>_{L_2^q}> 0$ *independent of the delays, equivalently,* $G(s)$ *is strictly positive real, i.e.,* $G(s) + G^*(s) > 0$ *for all* $Re\, s \geq 0$ *if and only if* $\hat{Q}_p < 0$.

Proof: Under the Krasovskii-Lyapunov functional candidate $V(x_t)$ of (35), one has following similar derivation steps and in the proof of Theorem 3.1 and taking account of the sparse characteristics of $\bar{B}$ and $\bar{C}$ related to B and C:

$$\begin{aligned}\vartheta_p(t) &= \bar{x}^T(t)\bar{Q}\bar{x}(t) - 2x^T(t)Cu(t) - u^T(t)\left(D + D^T\right)u(t)\\ &= \dot{V}(x_t) - 2x^T(t)Cu(t) - u^T(t)\left(D + D^T\right)u(t)\\ &= \dot{V}(x_t) - 2y^T(t)u(t) + 2u^T(t)Du(t) - u^T(t)\left(D + D^T\right)u(t)\\ &= \dot{V}(x_t) - 2y^T(t)u(t)\\ &= \left(\bar{x}^T(t), u^T(t)\right)\hat{Q}_p\left(\bar{x}^T(t), u^T(t)\right)^T \leq 0; \quad \forall t \in \boldsymbol{R}_{0+} \end{aligned} \tag{53}$$

instead of (39) in the proof of Theorem 3.1, according to (52). Also, $y^T(t)u(t) \leq 0\ t \geq 0$ in order to fulfil (53) for all $t \geq 0$. Thus, $<u, Gu>_{L_2}\geq 0$, independent of the delays, equivalently $G(s)$ is positive real, that is, $G(s) + G^*(s) \geq 0$. Property (i) has been proved. Property (ii) is direct from Property (i) for the equivalence between strict passivity and strict positive realness of the transfer matrix. □

It can be pointed out that the requisition of external positivity is an "a priori" handicap for guaranteeing furthermore the positive realness of the transfer matrix. Although both the external positivity and the positive realness are properties which are directly related to transfer functions in the linear and time-invariant case, they are different properties by nature. In this way, note for instance, that in the case of a single-input single-output system, the positive realness of a transfer function implies that of the inverse one while the inverse transfer functions of an externally positive system are not related to an externally positive system.

4. "Ad hoc" Extension of the PRL for Time Delay Externally Positive Systems.

It can be pointed out that Theorem 3.2 is not directly applicable to positive systems without first introducing an "ad hoc" extension which are now addressed. In particular, it is known that in the delay-free SISO case, the inverse of a positive real transfer function is also positive real [17-19]. This is unfeasible in typical cases of positive systems with at least a real zero and a real pole since the real zero has to be smaller than the dominant pole for the system to be externally positive (Theorem 2.2) and, at the same time, the real pole, which is a zero of the inverse transfer function, has to be smaller than the dominant pole of the inverse transfer function which is one of the zeros of the system transfer function. For instance, if a delay-free transfer function of first order with one real zero and one real pole is positive real, its inverse is positive real too but only one of the two could be associated with an externally positive system by the above reason. If one of the dominant poles and one of the dominant zeros are real, then the above reasoning still holds for transfer functions of any order. It is well-known that the positive realness of the hodograph of a transfer function, or a transfer matrix, on the complex imaginary axis is a necessary condition for positive realness of a transfer function. It can be pointed out that positive realness of a transfer function of a linear system implies the hyperstability, also often referred to as passivity, of its associated linear dynamic system. See, for instance, [20-23]. Such a theory has been useful as well in certain applications, particularly, in electrical machinery control [21-23] and some of the references therein. Some recent research involving passivity and some KYPL applications is as follows. A stability-preserving model reduction approach for a vibro-acoustic element model including poroelastic materials finite is presented in [24]. In [25], a generalized KYPL used to select a frequency for the design of dynamic precompensators for decoupling politopic linear systems. On the other hand, a finite frequency controller is proposed in [26] by using the generalized KYPL to investigate the stability and controller synthesis for discrete linear repetitive processes subject to a polyhedral uncertainty. In [27], a generalized KYPL is used to overcome an active chatter suppression in turning processes. The subsequent result addresses the positive realness on the complex imaginary axis of the Fourier transform of a modified positive impulse response matrix function. The modified impulse response is obtained through a cascade connection the positive transfer function matrix with a weighting designed piecewise-constant time-varying matrix gain.

Lemma 4.1. *Assume that* $p = q$*,* $\hat{h} = (h_0 = 0, h_1, \ldots, h_m = h)$ *is an increasing strictly ordered array of delays and that* $g\left(\hat{h}, t\right) \in \boldsymbol{R}_{0+}^{q\times q}$ *for* $t \geq 0$ *and define the modified Fourier transform* $G_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right) = \int_0^t K_{\hat{h}}(t) g\left(\hat{h}, t\right) e^{-\boldsymbol{i}\omega t}dt$ *of* $g\left(\hat{h}, t\right)$ *through a matrix function* $K_{\hat{h}} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{q\times q}$*. Then, the following properties hold.*

(i) If $g\left(\hat{h}, t\right)$ *is everywhere positive monomial for all* $t \in \boldsymbol{R}_{0+}$ *then there exist, in general, non-unique piecewise constant matrix functions* $K_{\hat{h}} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{q\times q}$ *satisfying the constraint:*

$$K_{\hat{h}}(t) = K_{\hat{h},n(t)} \text{ for } t \in \left[\frac{n(t)\pi}{2\omega}, \frac{(n(t)+1)\pi}{2\omega}\right) \tag{54}$$

with

$$K_{\hat{h},n(t)} \preceq \sum_{k=1}^{n(t)} \sum_{j=k-1}^{k} \left(K_{j-1} \int_{(j-1)\pi/2\omega}^{j\pi/2\omega} g\left(\hat{h}, t\right) \cos\omega t dt\right)\left(\int_{n(t)\pi/2\omega}^{n(t)\pi/2\omega+\xi(t)} g\left(\hat{h}, t\right) dt\right)^{-1} \tag{55}$$

$$n(t) = \{\max z \in \boldsymbol{Z}_{0+} : t \in [z\pi/2\omega, (z+1)\pi/2\omega)\} \tag{56}$$

Then, $G_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right)$ *is not identically zero for all* $(t, \omega) \in \boldsymbol{R}_{0+} \times \boldsymbol{R}_{0+}$ *and it satisfies* $ReG_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right) \succeq 0$ *for all* $t \in \boldsymbol{R}_{0+}$ *and all* $\omega \in \boldsymbol{R}_{0+}$*.*

(ii) Property (i) also holds for non-unique matrix functions $K_{\hat{h}} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{q\times q}$ *which satisfy the general constraint:*

$$\begin{aligned} K_{\hat{h},n(t)} \int_{n(t)\pi/2\omega}^{n(t)\pi/2\omega+\xi(t)} g\left(\hat{h}, t\right) dt &\preceq \sum_{k=1}^{n(t)} \sum_{j=k-1}^{k} \left(K_{j-1} \int_{(j-1)\pi/2\omega}^{j\pi/2\omega} g\left(\hat{h}, t\right) \cos\omega t dt\right) \\ &= \Gamma_{\hat{h},n(t)} \succeq 0 \end{aligned} \tag{57}$$

Proof: Note that $G_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right) = \int_0^t g\left(\hat{h}, t\right) e^{-\boldsymbol{i}\omega t}dt$ which is identical to $G_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right) = \int_0^t K_{\hat{h}}(t) g\left(\hat{h}, t\right) e^{-\boldsymbol{i}\omega t}dt$ if $K_{\hat{h}}(t) = I_p$ so that $ReG_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right) = \int_0^t g\left(\hat{h}, t\right) \cos\omega t dt$ can trivially change of sign through time which implies in that case that $G(s)$ cannot be positive real. Now, note that $ReG_{K_{\hat{h}}(t)}(\boldsymbol{i}\omega) = \int_0^t K_{\hat{h}}(t) g\left(\hat{h}, t\right) \cos\omega t dt$ and put $t = n(t)\pi/2\omega + \xi(t)$, where $n(t) = \{\max z \in \boldsymbol{Z}_{0+} : t \in [z\pi/2\omega, (z+1)\pi/2\omega)\}$ and $\xi(t) = t - n(t)\pi/2\omega \in [0, \pi/2\omega)$. Assume that $K_{\hat{h}} : \boldsymbol{R}_{0+} \to \boldsymbol{R}_{0+}^{p\times p}$ is piecewise constant defined by (54) and (55). Then, since $g\left(\hat{h}, t\right) \succeq 0$ for $t \geq 0$, one has

$$ReG_{K_{\hat{h}}(t)}\left(\hat{h}, 0\right) = G_{K(t)}\left(\hat{h}, 0\right) = \int_0^t g\left(\hat{h}, t\right) dt \succeq 0 \tag{58}$$

for $t \geq 0$, and

$$
\begin{aligned}
ReG_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right) &= ReG_{K_{\hat{h}}(t)}\left(\hat{h}, -\boldsymbol{i}\omega\right) \\
&= \int_0^{n(t)\pi/2\omega+\xi(t)} K_{\hat{h}}(t) g\left(\hat{h}, t\right) \cos\omega t dt \\
&= \sum_{k=1}^{n(t)} \sum_{j=k-1}^{k} \left(K_{j-1} \int_{(j-1)\pi/2\omega}^{j\pi/2\omega} g\left(\hat{h}, t\right) \cos\omega t dt \right) \\
&\quad + K_{\hat{h},n(t)} \int_{n(t)\pi/2\omega}^{n(t)\pi/2\omega+\xi(t)} g\left(\hat{h}, t\right) \cos\omega t dt
\end{aligned} \tag{59}
$$

for $t \geq 0$ and $\omega \neq 0$. Again since $g\left(\hat{h}, t\right) \succeq 0$ for $t \geq 0$ and $\cos\omega t \geq 0$ for $\omega t \in [0, \pi/2]$, that is $n(t) = 0$, and $t \in [0, \pi/2\omega]$, it follows that $ReG_{K(t)}(\boldsymbol{i}\omega) \succeq 0$ for $t \in [0, \pi/2\omega]$ and $\omega \in \boldsymbol{R}_{0+}$.

Now, proceed by complete induction by assuming that $ReG_{K(t)}(\boldsymbol{i}\omega) \succeq 0$ for $t \in [(n(t)-1)\pi/2\omega, n(t)\pi/2\omega)$ and $\omega \in \boldsymbol{R}_{0+}$ for some $n(t) \in \boldsymbol{Z}_{+t}$. Thus, it follows that

$$
\begin{aligned}
&ReG_{K_{\hat{h}}(t)}\left(\hat{h}, \boldsymbol{i}\omega\right) \succeq 0 \text{ for } t \in [(n-1)\pi/2\omega, n\pi/2\omega) \\
&\text{if for } t \in [n(t)\pi/2\omega, (n(t)+1)\pi/2\omega)
\end{aligned} \tag{60}
$$

if and only if

$$
\begin{aligned}
K_{\hat{h},n(t)} \int_{n(t)\pi/2\omega}^{n(t)\pi/2\omega+\xi(t)} g\left(\hat{h}, t\right) \cos\omega t dt &\succeq -\sum_{k=1}^{n(t)} \sum_{j=k-1}^{k} \left(K_{j-1} \int_{(j-1)\pi/2\omega}^{j\pi/2\omega} g\left(\hat{h}, t\right) \cos\omega t dt \right) \\
&= -\Gamma_{\hat{h},n(t)} \preceq 0
\end{aligned} \tag{61}
$$

Since $g\left(\hat{h}, t\right) \succeq 0$ for $t \in [n(t)\pi/2\omega, (n(t)+1)\pi/2\omega)$, then

a) If $n(t)$ is an even positive integer, then $\cos\omega t \geq 0$ for $t \in [n(t)\pi/2\omega, (n(t)+1)\pi/2\omega)$ and $K_{\hat{h},n(t)}$ can be fixed to the identity matrix, or to any other positive real matrix, such that the above inequality holds.

b) If $\cos\omega t \geq 0$ for $t \in [n(t)\pi/2\omega, (n(t)+1)\pi/2\omega)$, that is $n(t)$ is an odd non-negative integer, then the above inequality holds if and only if

$$
\begin{aligned}
-K_{\hat{h},n(t)} \int_{n(t)\pi/2\omega}^{n(t)\pi/2\omega+\xi(t)} g\left(\hat{h}, t\right) |\cos\omega t| dt &\succeq -\sum_{k=1}^{n(t)} \sum_{j=k-1}^{k} \left(K_{j-1} \int_{(j-1)\pi/2\omega}^{j\pi/2\omega} g\left(\hat{h}, t\right) \cos\omega t dt \right) \\
&= -\Gamma_{\hat{h},n(t)} \preceq 0
\end{aligned} \tag{62}
$$

or, equivalently,

$$
\begin{aligned}
K_{\hat{h},n(t)} \int_{n(t)\pi/2\omega}^{n(t)\pi/2\omega+\xi(t)} g\left(\hat{h}, t\right) |\cos\omega t| dt &\preceq \sum_{k=1}^{n(t)} \sum_{j=k-1}^{k} \left(K_{j-1} \int_{(j-1)\pi/2\omega}^{j\pi/2\omega} g\left(\hat{h}, t\right) \cos\omega t dt \right) \\
&= \Gamma_{\hat{h},n(t)} \succeq 0
\end{aligned} \tag{63}
$$

which is guaranteed under the constraint (57) since $g\left(\hat{h}, t\right) \in \boldsymbol{R}_+^{p\times p}$ for $t \geq 0$ is positive monomial then there exists its inverse $g^{-1}\left(\hat{h}, t\right) \in \boldsymbol{R}_+^{p\times p}$ for $t \geq 0$ exists and it is positive monomial too so that (55) holds. Property (i) has been proved. Property (ii) follows since the above condition of a) and b) in this proof, and in particular (63), also hold without the monomial constraint for $g\left(\hat{h}, t\right) \in \boldsymbol{R}_+^{p\times p}$ by "ad hoc" non-unique designs of $K_{\hat{h}}(t)$. □

In the SISO case, i.e., $p = q = 1$ with $K_{n(t)} \to k_{n(t)}$, the constraint (63) becomes

$$k_{\hat{h},n(t)} \preceq \frac{\sum_{k=1}^{n(t)} \sum_{j=k-1}^{k} \left(K_{j-1} \int_{(j-1)\pi/2\omega}^{j\pi/2\omega} g\left(\hat{h}, t\right) \cos \omega t dt \right)}{\int_{n(t)\pi/2\omega}^{n(t)\pi/2\omega+\xi(t)} g\left(\hat{h}, t\right) dt} \succeq 0 \tag{64}$$

Note that Lemma 4.1 does not guarantee that $G_{K_{\hat{h}}(t)}(s)$ is positive real in the sense that $G_{K_{\hat{h}}(t)}(s) + G^*_{K_{\hat{h}}(t)}(s) \geq 0$ for all $t \in (0, +\infty]$ and all $Re\, s \geq 0$. It only guarantees the needed condition for positive realness on the imaginary complex axis. Note also that the design constraints in Lemma 4.1 for the time-varying weighting matrix are trivially compatible with its uniform boundedness for all time. Note also that Lemma 4.1 is dependent of the delays result which is difficult to be reformulated as independent of the delays in the general case. The main reason for that is that the weighting matrix is designed according to dependent of delays conditions.

The following result combines Theorem 3.2 and Lemma 4.1 to give an "ad hoc" extended version of the PRL dependent on the delays.

Theorem 4.1. *Assume that $p = q$ and that the following conditions hold.*

1) $g\left(\hat{h}, t\right) \in \boldsymbol{R}_{0+}^{q\times q}$ for all $t \geq 0$ is the impulse response of the delayed system with array $\hat{h} = (h_0 = 0, h_1, \ldots, h_m = h)$ of increasing strictly ordered delays whose transfer matrix is $G\left(\hat{h}, s\right) = \boldsymbol{L} g\left(\hat{h}, t\right)$.

2) $G\left(\hat{h}, s\right)$ is stable, i.e., analytic in $Re\, s > 0$, and, furthermore, any imaginary pole $\boldsymbol{i}\omega$, if any, is simple and its associate residue matrix $\lim_{s\to\boldsymbol{i}\omega}(s - \boldsymbol{i}\omega) G\left(\hat{h}, s\right)$ is positive semidefinite Hermitian.

3) $G_{K_{\hat{h}}(t)}(s)\left(\hat{h}, \boldsymbol{i}\omega\right) = \int_0^t K_{\hat{h}}(t) g\left(\hat{h}, t\right) e^{-\boldsymbol{i}\omega t} dt$ such that the weighting matrix $K_{\hat{h}} : \boldsymbol{R}_{0+} \to \boldsymbol{R}^{q\times q}$ satisfies the conditions of Lemma 4.1 and it is bounded for all $t \geq 0$.

Then, the following properties hold.

(i) The operator of the delayed system $G_{K_{\hat{h}}(t)}(s)\left(\hat{h}, s\right)$, of delays array $\hat{h}$, is passive in the sense that $< u, G_{K_{\hat{h}}(t)}(s) u >_{L_2^q} \geq 0$, equivalently, $G_{K_{\hat{h}}(t)}(s)\left(\hat{h}, s\right)$ is positive real in the sense that $G_{K_{\hat{h}}(t)}(s)\left(\hat{h}, s\right) + G^*_{K_{\hat{h}}(t)}(s)\left(\hat{h}, s\right) \geq 0$ for all $Re\, s \geq 0$ and all $t \geq 0$. Equivalently, there exist $(m + 1)$ real symmetric matrices $P\, (\in \boldsymbol{R}^{n\times n}) > 0$, $S_i\, (\in \boldsymbol{R}^{n\times n}) > 0$ for $i \in \bar{m}$ such that

$$\hat{Q}_{K_{\hat{h}}(t)} = \begin{bmatrix} \bar{Q} & \bar{P}\bar{B} - \bar{C}^T_{K_{\hat{h}}(t)} \\ \bar{B}^T \bar{P} - \bar{C}_{K_{\hat{h}}(t)} & -\left(D + D^T\right) \end{bmatrix} \left(\in \boldsymbol{R}^{((m+1)n+q)\times((m+1)n+q)}\right) \leq 0 \tag{65}$$

for all $t \geq 0$, under the definitions in (31), of the various involved matrices involved in (65), with C being pre-multiplied by $K_{\hat{h}}(t)$.

(ii) If, furthermore, the positivity relation (55) for the weighting gain matrix $K_{\hat{h}}(t)$ is strict and $G\left(\hat{h}, s\right)$ is strictly stable, i.e., analytic in $Re\, s \geq 0$, then $G_{K_{\hat{h}}(t)}\left(\hat{h}, s\right)$ is strictly passive, i.e., $< u, G_{K_{\hat{h}}(t)} u >_{L_2^q} \geq 0$. Equivalently, $G_{K_{\hat{h}}(t)}\left(\hat{h}, s\right)$ is strictly positive real, i.e., $G_{K_{\hat{h}}(t)}(s)\left(\hat{h}, s\right) + G^*_{K_{\hat{h}}(t)}(s)\left(\hat{h}, s\right) \geq 0$ for all $Re\, s \geq 0$ if and all $t \geq 0$ if and only if $\hat{Q}_{K_{\hat{h}}(t)} < 0$ for all $t \geq 0$.

Note that Theorem 4.1 is dependent of delay result since $\hat{Q}_{K_{\hat{h}}(t)}$ is typically dependent on the delays, since the extended output matrix $\bar{C}_{K_{\hat{h}}(t)}$ is dependent of delays, as it is the modified output matrix $C_{K_{\hat{h}}}(t)$, via the weighting matrix $K_{\hat{h}(t)}$.

5. **Conclusions.** This paper has dealt with the statements and their proofs of an "ad hoc" extended Kalman-Yakubovich-Popov Lemma and an "ad hoc" extended Positive Real Lemma for MIMO externally positive linear time-invariant time-delay systems subject to a finite number of finite point delays. The first lemma is stated independent of the sizes of the delays. The second lemma requires the design of a feed-forward piecewise constant compensator matrix under some weak assumptions for its feasible application to such externally positive time-delay systems and it is, in general, dependent on the sizes of the delays. While the output is kept positive for all time for everywhere non-negative inputs as a consequence of the external positivity property, the modified attenuated output through the feed-forward compensator can have negative values along some time-intervals. Some preceding useful results on asymptotic stability, positivity and external positivity of time-delay systems are proved which compare through time the levels of the states and the outputs of the whole delay systems with their counterparts of two auxiliary undelayed systems. Such auxiliary systems are a) the delay-free one which is obtained as a particular case of the whole delayed system when all delays are identically zero; b) the system of undelayed dynamics, that is, the one where all the delayed matrices of dynamics of the whole delayed system are removed. It is also seen as expected that both auxiliary systems are positive/externally positive if the whole delayed system possesses those properties independent of the delays while they are globally asymptotically stable if the whole delayed system is globally asymptotically stable independent of the delays.

It is foreseen the potential extension of the results given in this paper to dynamic systems including distributed internal delays and to those with mixed point and distributed delays including special cases of Volterra-type integro-differential systems which could be focused on as special cases of dynamic systems with a class of distributed delays. It is also foreseen to extend the results to some discretized systems including those arising from the use of fractional sampling and hold devices [28].

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REFERENCES

[1] C. Grussler, *Model Reduction of Positive Systems*, Master Thesis, Technical University Kaiserslautern, 2012.

[2] L. Farina, Necessary conditions for positive realizability of continuous-time linear systems, *Systems & Control Letters*, vol.25, no.2, pp.121-124, 1995.

[3] L. Farina and S. Rinaldi, *Positive Linear Systems: Theory and Applications*, John Wiley and Sons, New York, 2000.

[4] W. T. Frazier and R. Gardner, An Eneström-Kakeya theorem for new classes of polynomials, *Acta et Commentationes Universitatis Tartuensis de Mathematica*, vol.23, no.1, pp.103-115, 2019.

[5] A. Aziz and Q. G. Mohammad, On the zeros of a certain class of polynomials and related analytic functions, *Journal of Mathematical Analysis and Applications*, vol.75, pp.495-502, 1980.

[6] K. K. Dewan and N. K. Govil, On the Eneström-Kakeya theorem, *Journal of Approximation Theory*, vol.42, pp.239-244, 1984.

[7] A. L. Cauchy, Exercices de mathèmatique, *Oeuvres*, vol.9, p.122, 1829.

[8] V. K. Jain, On Cauchy's bound for zeros of a polynomial, *Turkish Journal of Mathematics*, vol.30, no.6, pp.95-100, 2006.

[9] S.-I. Niculescu, Delay effects on stability. A robust control approach, in *Lecture Notes in Control and Information Sciences*, London, Springer-Verlag, 2001.

[10] H. Gorecki, S. Fuksa, P. Grabowski and A. Korytowski, *Analysis and Synthesis of Time Delay Systems*, John Wiley & Sons, PWN – Polish Scientific Publishers, Warszawa, 1989.

[11] M. De la Sen, Sufficiency-type stability and stabilization criteria for linear time-invariant systems with constant point delays, *Acta Applicandae Mathematicae*, vol.83, no.3, pp.235-256, 2004.

[12] M. M. Peet, *Stabilization via LMIs*, Lecture 22, Illinois Institute of Technology, 2012.

[13] M. M. Peet, A. Papachristodoulou and S. Lall, Positive forms and stability of linear time-delay systems, *SIAM Journal of Control and Optimization*, vol.47, no.6, pp.3237-3258, 2009.

[14] A. Rantzer, On the Kalman-Yakubovich Popov Lemma, *Systems & Control Letters*, vol.28, no.1, pp.7-10, 1996.

[15] A. Rantzer, An extended Kalman-Yakubovich-Popov Lemma for positive systems, *IFAC-Papers Online*, vol.48, no.11, pp.242-245, 2015.

[16] W. M. Haddad and M. Lanchares, Dissipativity theory for discrete-time nonlinear stochastic dynamical systems, Part II: Kalman-Yakubovich-Popov conditions and stability of feedback interconnections, *American Control Conference (ACC)*, pp.838-843, 2022.

[17] M. De la Sen, A method for general design of positive real functions, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol.45, no.7, pp.764-769, 1998.

[18] J. Gregor, On the design of positive real functions, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol.43, no.11, pp.945-947, 1996.

[19] J. J. Marquez and C. J. Damaren, On the design of strictly positive real transfer functions, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol.42, no.4, pp.214-218, 1995.

[20] M. Delasen, Stability of composite systems with an asymptotically hyperstable subsystem, *International Journal of Control*, vol.44, no.6, pp.1769-1775, 1986.

[21] A. Glushchenko, K. Lastochkin and V. Petrov, DC drive adaptive speed controller based on hyperstability theory, *Computation*, vol.10, no.3, Article number 40, 2022.

[22] X. Tu, X. Hou, J. Zhao, S. Yan and Y. Xiong, Speed identification of speed sensorless linear induction motor based on MRAS, *CES Transactions on Electrical Machines and Systems*, vol.7, no.3, pp.294-300, 2023.

[23] S. K. Kakodia and G. Dyanamina, Sliding mode MRAS observer for PMSM-fed electric vehicle control using recurrent neural network-based parallel resistance estimator, *2023 IEEE Transportation Electrification Conference and Expo, Asia-Pacific (ITEC Asia-Pacific)*, Chiang Mai, Thailand, pp.1-6, 2023.

[24] Y. Cai, S. Van Ophem, W. Desmet and E. Deckers, Model order reduction of time-domain vibroacoustic finite element simulations with poroelastic materials, *Computer Methods in Applied Mechanics and Engineering*, vol.426, Article number 116980, 2024.

[25] C. G. M. Angelo and E. S. Tognetti, Design of robust pre-compensators for decoupling polytopic linear systems, *International Journal of Robust and Nonlinear Control*, vol.34, no.4, pp.2496-2514, 2024.

[26] L. Dong, L. Wang, F. Gao and Y. Chen, Static iterative learning control over finite frequency domain for actuator failure uncertain systems, *IEEE Access*, DOI: 0.1109/ACCESS.2024.3364396, 2024.

[27] G. Han, H. Ma, Y. Liu and Q. Song, Model-free finite frequency H_∞ control for active chatter suppression in turning, *Journal of Sound and Vibration*, vol.577, Article number 118342, 2024.

[28] R. Bárcena, M. De la Sen and I. Sagastabeitia, Improving the stability properties of the zeros of the sampled systems with fractional order hold, *IEE Proceedings – Control Theory and Applications*, vol.147, no.4, pp.456-464, 2000.

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