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RESEARCH ON COMMON DUE-WINDOW ASSIGNMENT SCHEDULING WITH POSITIONAL DEPENDENT PROCESSING TIME AND GROUP TECHNOLOGY

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Abstract. *This paper considers the group scheduling problem on a single-machine with the positional dependent processing time and common due-window assignment. For positional dependent processing time under group technology, we mean that in a group the processing time of a job depends on its position. About the due-window assignment, we mainly discuss the common due-window assignment. The group setup time is suffered no matter when single-machine transfers job from one to another group. Each group is allocated to an assignable common due-window. We attempt to determine the optimal job sequence, group sequence and common due-window to minimize the weighted sum of earliness, tardiness, the start time and size of common due-window, where the weights only depend on their positions in a sequence (i.e., position-dependent). We demonstrate that the problem is polynomially solvable by using optimal solution properties.*

Keywords: Scheduling, Common due-window assignment, Positional dependent processing time, Group technology, Single-machine

1. Introduction.

1.1. Scheduling problem.

Scheduling problems and models are very important in industrial engineering, logistics and operational management.

Recently, more and more attention has been paid to due window assignment, i.e., both early and late costs are incurred for the due-window assignment (Rolim and Nagano [1]) and intelligent control system (Octavian et al. [2]). They developed an intelligent control system where a human can dynamically change drone swarm maneuvers by performing some hand gestures movement. Liman et al. [3] first considered the common due-window scheduling problem on single machine. For the maximum lead minimization without tardy jobs, they proved that the problem was non-deterministic polynomial-hard (abbreviated as NP-hard). Mosheiov and Sarig [4] extended the problem raised by Liman et al. [3] to the case of job-processing time dependence. They assumed that the processing time of job is a function of its position in the job sequence.

Mor and Mosheiov [5] studied single-machine scheduling with a maintenance activity and due-window assignment. Yin et al. [6] solved a single-machine scheduling problem for batch delivery, where jobs have an assignable common due-window. They showed that the problem can be solved in polynomial time. Yin et al. [7] studied the single-machine due-window scheduling with controllable job processing time. They proved that five versions

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of the problem can be solved in polynomial time. Wang et al. [8] investigated the single machine scheduling with common due-window assignment. For the weighted sum of earliness and tardiness, due-window starting time, and due-window size minimization, they proposed a polynomial time algorithm. Xu et al. [9] considered the single-machine multitasking scheduling problems. Under the common due-window assignment, they proved that some problems are polynomially solvable. Tian [10] studied the single-machine due-window assignment scheduling with resource allocation. Under common and slack due-window assignments, she proved that a general earliness/tardiness cost minimization is polynomially solvable. Teng et al. [11] considered the single-machine due-window assignment scheduling with deteriorating jobs. They showed that the earliness/tardiness cost minimization is polynomially solvable.

1.2. **Research scope.** Group technology is a manufacturing and engineering management approach that aims to improve efficiency by exploiting similarities in production/execution of different products and activities. In terms of part manufacturing, its main idea is to identify similar parts and classify them into groups, and eventually to configure the unit specifically for the production of a specific group of parts.

The first paper on group scheduling was published by Yoshida et al. [12]. Subsequently, more and more attention has been paid to group scheduling. In recent years, there have been a number of articles on the application of group technique. Guo and Wang [13] studied the single-machine scheduling problem with group technology. Under the deteriorating jobs, they proved that the makespan minimization problem is polynomial solvable. Wu and Lee [14] investigated the group scheduling problem on single-machine with deteriorating setup and processing times. For the maximal completion time and total completion time minimizations, they showed that the problem is polynomial solvable. Zhu et al. [15] studied the single-machine scheduling with learning effects, resource allocation and group technology simultaneously. Lu et al. [16] resolved the single-machine group scheduling with a time-dependent processing time. They proved that the makespan minimization with release date is polynomial solvable. Bajwa et al. [17] explored the single-machine problem with group technology. For the objective of minimizing the number of tardy jobs, they proposed a hybrid heuristic. Wang and Ye [18] discussed the single-machine scheduling problems with learning effects and group technology. Liu and Wang [19] considered due-date assignment single-machine scheduling with resource allocation and group technology. The objective function is to minimize the weighted sum of scheduling cost and resource-allocation cost. For a special case, they proved that the problem is polynomially solvable.

1.3. **Research background.** Liu [20] proposed a new scheduling model, which grouped jobs in advance according to their processing similarity and processed all jobs in the same group continuously. The set time was generated when a single machine transferred job processing from one group to another. The processing time of a job was determined by its processing position in the job sequence of the group it belonged to. Each group is assigned an assignable common delay window, and it is proved to be polynomially solvable. Ren and Yang [21] studied the single-machine group scheduling with resource allocation and common due-window assignment. For the sum of the maximal costs minimization, they proved that the problem is polynomially solvable. Chen et al. [22] explored the single-machine group scheduling with due window assignment and learning effects. The objective is to minimize the sum of due window-related penalty costs and investment cost. They proved that the problem can be solved in polynomial time.

1.4. **Research framework.** In this paper, the group technology and common due-window assignment is combined, i.e., it is natural and interesting to continue the work of Liu [20] but to explore the group scheduling problem with position-dependent weights (Wang et al. [23]) that includes the one given in Liu [20] as a special case. The rest of this article is sorted in the following order. Section 2 gives notations and problem formulation. Section 3 solves the common due-window assignment and gives the algorithm. Section 4 gives an example. Section 5 presents the conclusions.

2. **Problem Assumptions.** There are m independent jobs $\breve{J} = \left\{\breve{J}_1, \breve{J}_2, \ldots, \breve{J}_m\right\}$ without preemption to be handled on a single-machine and classified into q groups, and all jobs are available at time 0. The machine can only process one job at a time. Assume that setup time is not needed between two sequential jobs within the same group. Nevertheless, the setup time $\breve{s}_l$ of group is required to handle the group $\breve{G}_l$. In addition, the practical processing time of job $\breve{J}_{\breve{\xi}_{lh}}$ when it is scheduled in the kth position within processing sequence of group $\breve{G}_l$ is formulated by

$$\breve{p}_{\breve{\xi}_{lh(k)}} = \breve{p}_{\breve{\xi}_{lh}} \breve{g}_l(k), \tag{1}$$

where $\breve{g}_l : [1, +\infty) \to (0, +\infty)$ with $\breve{g}_l(1) = 1$. It is assumed that all jobs in the same group share an allocated common due-window with start time $\tilde{d}_l$, finish time $\tilde{\tilde{d}}_l$ and window size $\breve{D}_l = \tilde{\tilde{d}}_l - \tilde{d}_l$, where $\tilde{d}_l$ and $\tilde{\tilde{d}}_l$ (such $\breve{D}_l$) are group related decision variables. Let $\breve{E}_{\breve{\xi}_{lh}} = \max\left\{\tilde{d}_l - \breve{C}_{\breve{\xi}_{lh}}, 0\right\}$ (resp. $\breve{T}_{\breve{\xi}_{lh}} = \max\left\{\breve{C}_{\breve{\xi}_{lh}} - \tilde{\tilde{d}}_l, 0\right\}$) be the earliness (resp. tardiness) of job $\breve{J}_{\breve{\xi}_{lh}}$. The aim is to determine (I) the optimal group sequence, (II) the optimal job sequence, (III) the common due-window start time, (IV) the common due-window size, so as to minimize the following target (objective) function:

$$\breve{Z} = \sum_{l=1}^{q} \sum_{h=1}^{m_l} \left(\breve{\alpha}_{lh} \breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh} \breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l \tilde{d}_l + \breve{\vartheta}_l \breve{D}_l \right). \tag{2}$$

Using the three-parameter notation scheme, the problem can be denoted as

$$1|GT, PDPT, CODW| \sum_{l=1}^{q} \sum_{h=1}^{m_l} \left(\breve{\alpha}_{lh} \breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh} \breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l \tilde{d}_l + \breve{\vartheta}_l \breve{D}_l \right), \tag{3}$$

where GT denotes group technology, $PDPT$ denotes position-dependent processing time (1), and $CODW$ denotes the common due-window. This problem's complexity will be proved, and the complexity is $O(m^3)$ time in the following Theorem 3.1.

3. **Main Results.** Obviously, for $1|GT, PDPT, CODW| \sum_{l=1}^{q} \sum_{h=1}^{m_l} \left(\breve{\alpha}_{lh} \breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh} \breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l \tilde{d}_l + \breve{\vartheta}_l \breve{D}_l \right)$, there exists an optimal sequence so that all jobs are processed continuously at time 0, and there is no idle time between two consecutive jobs. Therefore, the following Lemmas are presented, which are also mentioned in Wang et al. [23].

Lemma 3.1. *For* $1|GT, PDPT, CODW| \sum_{l=1}^{q} \sum_{h=1}^{m_l} \left(\breve{\alpha}_{lh} \breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh} \breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l \tilde{d}_l + \breve{\vartheta}_l \breve{D}_l \right)$, *there exists an optimal sequence such that the optimal* $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}}$ *and* $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l)}}$, *with* $0 \le e_l \le t_l \le m_l$.

Proof: To receive this, there are three cases to think.

Case 1: $\tilde{d}_l > \breve{C}_{\breve{\xi}_{l(m_l)}}$, all jobs are early. Decreasing both $\tilde{d}_l$ and $\tilde{\tilde{d}}_l$ by the amount $\tilde{d}_l - \breve{C}_{\breve{\xi}_{l(m_l)}}$, and the target function value is going to be reduced by $\sum_{l=1}^{q}\left(\sum_{h=1}^{m_l}\breve{\alpha}_{lh} + m_l\breve{\theta}_l\right)\left(\tilde{d}_l - \breve{C}_{\breve{\xi}_{l(m_l)}}\right)$.

Case 2: $\tilde{\tilde{d}}_l > \breve{C}_{\breve{\xi}_{l(m_l)}}$, no jobs are late. Decreasing $\tilde{\tilde{d}}_l$ by the amount $\tilde{\tilde{d}}_l - \breve{C}_{\breve{\xi}_{l(m_l)}}$, and the target function value is going to be reduced by $\sum_{l=1}^{q}\left(m_l\breve{\vartheta}_l\right)\left(\tilde{\tilde{d}}_l - \breve{C}_{\breve{\xi}_{l(m_l)}}\right)$.

Case 3: $\breve{C}_{\breve{\xi}_{l(e_l-1)}} < \tilde{d}_l < \breve{C}_{\breve{\xi}_{l(e_l)}}$ or $\breve{C}_{\breve{\xi}_{l(t_l-1)}} < \tilde{\tilde{d}}_l < \breve{C}_{\breve{\xi}_{l(t_l)}}$ for some $0 \le e_l \le t_l \le m_l$. First $\tilde{d}_l$ is shifted either to the left or to the right such as $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l-1)}}$ or $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}}$, respectively, if we shift to the left, $\tilde{d}_l$ decreases by $\tilde{d}_l - \breve{C}_{\breve{\xi}_{l(e_l-1)}}$, and then the objective function will be reduced by $\left(\sum_{h=1}^{e_l-1}\breve{\alpha}_{lh} + m_l\breve{\theta}_l - m_l\breve{\vartheta}_l\right)\left(\tilde{d}_l - \breve{C}_{\breve{\xi}_{l(e_l-1)}}\right)$, if we shift to the right, $\tilde{d}_l$ increases by $\breve{C}_{\breve{\xi}_{l(e_l)}} - \tilde{d}_l$, and then the objective function will be reduced by $\left(m_l\breve{\theta}_l - \sum_{h=1}^{e_l-1}\breve{\alpha}_{lh} - m_l\breve{\vartheta}_l\right)\left(\breve{C}_{\breve{\xi}_{l(e_l)}} - \tilde{d}_l\right)$. Then $\tilde{\tilde{d}}_l$ is shifted either to the left or to the right such as $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l-1)}}$ or $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l)}}$, respectively, if we shift to the left, $\tilde{\tilde{d}}_l$ decreases by $\tilde{\tilde{d}}_l - \breve{C}_{\breve{\xi}_{l(t_l-1)}}$, and then the objective function will be reduced by $\sum_{l=1}^{q}\left(m_l\breve{\vartheta}_l - \sum_{h=t_l}^{m_l}\breve{\beta}_{lh}\right)\left(\tilde{\tilde{d}}_l - \breve{C}_{\breve{\xi}_{l(t_l-1)}}\right)$, if we shift to the right, $\tilde{\tilde{d}}_l$ increases by $\breve{C}_{\breve{\xi}_{l(t_l)}} - \tilde{\tilde{d}}_l$, and then the objective function will be reduced by $\sum_{l=1}^{q}\left(\sum_{h=t_l}^{m_l}\breve{\beta}_{lh} - m_l\breve{\vartheta}_l\right)\left(\breve{C}_{\breve{\xi}_{l(t_l)}} - \tilde{\tilde{d}}_l\right)$.

To sum up, in any case, when the start time of due-window $\tilde{d}_l$ either equals 0 or $\breve{C}_{\breve{\xi}_{l(e_l)}}$ and the finish time of due-window $\tilde{\tilde{d}}_l$ either equals 0 or $\breve{C}_{\breve{\xi}_{l(t_l)}}$, with $0 \le e_l \le t_l \le m_l$, the schedule is optimal. Thus, Lemma 3.1 holds. □

Lemma 3.2. *For* $1|GT, PDPT, CODW|\sum_{l=1}^{q}\sum_{h=1}^{m_l}\left(\breve{\alpha}_{lh}\breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh}\breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l\tilde{d}_l + \breve{\vartheta}_l\breve{D}_l\right)$, *there exists an optimal sequence such that* $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}}$ *and* $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l)}}$, *where* $\sum_{h=1}^{e_l-1}\breve{\alpha}_{lh} \le m_l\left(\breve{\vartheta}_l - \breve{\theta}_l\right)$ *and* $\sum_{h=t_l+1}^{m_l}\breve{\beta}_{lh} \le m_l\breve{\vartheta}_l$.

Proof: When the start time of due-window $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}}$, $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l)}}$, we have $\breve{Z} = \sum_{l=1}^{q}\left(\sum_{h=1}^{e_l-1}\breve{\alpha}_{lh}\left(\breve{C}_{l[e_l]} - \breve{C}_{\breve{\xi}_{l(h)}}\right) + \sum_{h=t_l+1}^{m_l}\breve{\beta}_{lh}\left(\breve{C}_{\breve{\xi}_{l(h)}} - \breve{C}_{\breve{\xi}_{l(t_l)}}\right) + m_l\breve{\theta}_l\breve{C}_{l[e_l]} + m_l\breve{\vartheta}_l\left(\breve{C}_{\breve{\xi}_{l(t_l)}} - \breve{C}_{\breve{\xi}_{l(e_l)}}\right)\right)$. Applying the typical small perturbation technique, we have the following two cases.

Case 1: To e_l: When $\tilde{d}_l$ reduces by ε, and $\tilde{\tilde{d}}_l$ is unchangeable, i.e., $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}} - \varepsilon$, $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l)}}$, we have

$$\breve{Z}_1 = \sum_{l=1}^{q}\left(\sum_{h=1}^{e_l-1}\breve{\alpha}_{lh}\left(\breve{C}_{\breve{\xi}_{l(e_l)}} - \varepsilon - \breve{C}_{\breve{\xi}_{l(h)}}\right) + \sum_{h=t_l+1}^{m_l}\breve{\beta}_{lh}\left(\breve{C}_{\breve{\xi}_{l(h)}} - \breve{C}_{\breve{\xi}_{l(t_l)}}\right)\right.$$
$$\left. + m_l\breve{\theta}_l\left(\breve{C}_{\breve{\xi}_{l(e_l)}} - \varepsilon\right) + m_l\breve{\vartheta}_l\left(\breve{C}_{\breve{\xi}_{l(t_l)}} - \left(\breve{C}_{\breve{\xi}_{l(e_l)}} - \varepsilon\right)\right)\right).$$

When $\tilde{d}_l$ increases by ε, and $\tilde{\tilde{d}}_l$ is unchangeable, i.e., $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}} + \varepsilon, \tilde{\tilde{d}}_l$, we have

$$\breve{Z}_2 = \sum_{l=1}^{q} \left(\sum_{h=1}^{e_l} \breve{\alpha}_{lh} \left(\breve{C}_{\breve{\xi}_{l(e_l)}} + \varepsilon - \breve{C}_{\breve{\xi}_{l(h)}} \right) + \sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} \left(\breve{C}_{\breve{\xi}_{l(h)}} - \breve{C}_{\breve{\xi}_{l(t_l)}} \right) + m_l \breve{\theta}_l \left(\breve{C}_{\breve{\xi}_{l(e_l)}} + \varepsilon \right) + m_l \breve{\vartheta}_l \left(\breve{C}_{\breve{\xi}_{l(t_l)}} - \left(\breve{C}_{\breve{\xi}_{l(e_l)}} + \varepsilon \right) \right) \right).$$

By using the typical small perturbation technique, we have

$$\breve{Z} - \breve{Z}_1 = \sum_{l=1}^{q} \left(\sum_{h=1}^{e_l - 1} \breve{\alpha}_{lh} \varepsilon + m_l \breve{\theta}_l \varepsilon - m_l \breve{\vartheta}_l \varepsilon \right) \leq 0$$

and

$$\breve{Z} - \breve{Z}_2 = \sum_{l=1}^{q} -\breve{\alpha}_{l e_l} \varepsilon - \left(\sum_{h=1}^{e_l - 1} \breve{\alpha}_{lh} \varepsilon + m_l \breve{\theta}_l \varepsilon - m_l \breve{\vartheta}_l \varepsilon \right) \leq 0.$$

Hence, $\sum_{h=1}^{e_l-1} \breve{\alpha}_{lh} \leq m_l \left(\breve{\vartheta}_l - \breve{\theta}_l \right) \leq \sum_{h=1}^{e_l-1} \breve{\alpha}_{lh} + \breve{\alpha}_{l e_l}$. Since $\breve{\alpha}_{l e_l}$ is a non-negative number, it can obviously see that $\sum_{h=1}^{e_l-1} \breve{\alpha}_{lh} \leq m_l \left(\breve{\vartheta}_l - \breve{\theta}_l \right)$.

Case 2: To t_l: When $\tilde{\tilde{d}}_l$ reduces by ε, and $\tilde{d}_l$ is unchangeable, i.e., $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}}$, $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l)}} - \varepsilon$, we have

$$\breve{Z}_3 = \sum_{l=1}^{q} \left(\sum_{h=1}^{e_l - 1} \breve{\alpha}_{lh} \left(\breve{C}_{\breve{\xi}_{l(e_l)}} - \breve{C}_{\breve{\xi}_{l(h)}} \right) + \sum_{h=t_l}^{m_l} \breve{\beta}_{lh} \left(\breve{C}_{\breve{\xi}_{l(h)}} - \left(\breve{C}_{\breve{\xi}_{l(t_l)}} - \varepsilon \right) \right) + m_l \breve{\theta}_l \breve{C}_{\breve{\xi}_{l(e_l)}} + m_l \breve{\vartheta}_l \left(\breve{C}_{\breve{\xi}_{l(t_l)}} - \varepsilon - \breve{C}_{\breve{\xi}_{l(e_l)}} \right) \right).$$

When $\tilde{\tilde{d}}_l$ increases by ε, and $\tilde{d}_l$ is unchangeable, i.e., $\tilde{d}_l = \breve{C}_{\breve{\xi}_{l(e_l)}}$, $\tilde{\tilde{d}}_l = \breve{C}_{\breve{\xi}_{l(t_l)}} + \varepsilon$, we have

$$\breve{Z}_4 = \sum_{l=1}^{q} \left(\sum_{h=1}^{e_l - 1} \breve{\alpha}_{lh} \left(\breve{C}_{\breve{\xi}_{l(e_l)}} - \breve{C}_{\breve{\xi}_{l(h)}} \right) + \sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} \left(\breve{C}_{\breve{\xi}_{l(h)}} - \left(\breve{C}_{\breve{\xi}_{l(t_l)}} + \varepsilon \right) \right) + m_l \breve{\theta}_l \breve{C}_{\breve{\xi}_{l(e_l)}} + m_l \breve{\vartheta}_l \left(\breve{C}_{\breve{\xi}_{l(t_l)}} + \varepsilon - \breve{C}_{\breve{\xi}_{l(e_l)}} \right) \right).$$

By using the typical small perturbation technique, we have

$$\breve{Z} - \breve{Z}_3 = \sum_{l=1}^{q} \left(-\breve{\beta}_{l t_l} \varepsilon - \left(\sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} - m_l \breve{\vartheta}_l \right) \varepsilon \right) \leq 0$$

and

$$\breve{Z} - \breve{Z}_4 = \sum_{l=1}^{q} \left(\sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} \varepsilon - m_l \breve{\vartheta}_l \varepsilon \right) \leq 0.$$

Hence, $\sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} \leq m_l \breve{\vartheta}_l \leq \sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} + \breve{\beta}_{l t_l}$. Since $\breve{\beta}_{l t_l}$ is a non-negative number, it can obviously see that $\sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} \leq m_l \breve{\vartheta}_l$. Thus, Lemma 3.2 holds. □

Next, the target function will be discussed in two cases.

Case 1: $\breve{\vartheta}_l \le \breve{\theta}_l$. According to Lemmas 3.1 and 3.2,

$$\tilde{d}_l = 0, \tag{4}$$

$$\breve{D} = \tilde{\tilde{d}}_l = \breve{S}_l + \breve{s}_l + \sum_{k=1}^{t_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k), \tag{5}$$

$$\breve{E}_{\breve{\xi}_{l(h)}} = 0, \tag{6}$$

$$\breve{T}_{\breve{\xi}_{l(h)}} = \begin{cases} \sum_{k=t_l+1}^{h} \breve{p}_{\breve{\xi}_{l(k)}} \breve{g}_l(k), & h > t_l, \\ 0, & h \le t_l. \end{cases} \tag{7}$$

Combining Equations (4)-(7), the total cost of all jobs in the $\breve{G}_l$ under ξ can be given by

$$\begin{aligned} \breve{Z}_l\left(\breve{\xi}_l\right) &= \sum_{h=1}^{m_l} \breve{\beta}_{lh} \breve{T}_{\breve{\xi}_{l(h)}} + m_l \breve{\vartheta}_l \tilde{\tilde{d}}_l \\ &= \sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} \left(\sum_{k=t_l+1}^{h} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k) \right) + m_l \breve{\vartheta}_l \left(\breve{S}_l + \breve{s}_l + \sum_{k=1}^{t_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k) \right) \\ &= \sum_{h=t_l+1}^{m_l} \left(\sum_{k=h}^{m_l} \breve{\beta}_{lk} \right) \breve{P}_{\breve{\xi}_{l(h)}} \breve{g}_l(h) + m_l \breve{\vartheta}_l \left(\breve{S}_l + \breve{s}_l + \sum_{k=1}^{t_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k) \right) \\ &= \sum_{h=1}^{t_l} m_l \breve{\vartheta}_l \breve{g}_l(h) \breve{P}_{\breve{\xi}_{l(h)}} + \sum_{h=t_l+1}^{m_l} \left(\sum_{k=h}^{m_l} \breve{\beta}_{lk} \right) \breve{P}_{\breve{\xi}_{l(h)}} \breve{g}_l(h) + m_l \breve{\vartheta}_l \left(\breve{S}_l + \breve{s}_l \right) \\ &= \sum_{h=1}^{m_l} \breve{\Phi}_{\breve{\xi}_{l(h)}} \breve{P}_{\breve{\xi}_{l(h)}} + m_l \breve{\vartheta}_l \left(\breve{S}_l + \breve{s}_l \right), \end{aligned} \tag{8}$$

where

$$\breve{\Phi}_{\breve{\xi}_{l(h)}} = \begin{cases} m_l \breve{\vartheta}_l \breve{g}_l(k), & h \le t_l, \\ \left(\sum_{k=h}^{m_l} \breve{\beta}_{lk} \right) \breve{g}_l(k), & h > t_l. \end{cases} \tag{9}$$

Case 2: $\breve{\vartheta}_l > \breve{\theta}_l$. According to Lemmas 3.1 and 3.2,

$$\tilde{d}_l = \breve{S}_l + \breve{s}_l + \sum_{k=1}^{e_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k), \tag{10}$$

$$\tilde{\tilde{d}}_l = \breve{S}_l + \breve{s}_l + \sum_{k=1}^{t_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k), \tag{11}$$

$$\breve{D}_l = \tilde{\tilde{d}}_l - \tilde{d}_l = \sum_{k=e_l+1}^{t_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k), \tag{12}$$

$$\breve{E}_{\breve{\xi}_{l(h)}} = \begin{cases} \sum_{k=h+1}^{e_l} \breve{p}_{\breve{\xi}_{l(k)}} \breve{g}_l(k), & h < e_l, \\ 0, & h \ge e_l, \end{cases} \tag{13}$$

$$\breve{T}_{\breve{\xi}_{l(h)}} = \begin{cases} \sum_{k=t_l+1}^{h} \breve{p}_{\breve{\xi}_{l(k)}} \breve{g}_l(k), & h > t_l, \\ 0, & h \le t_l. \end{cases} \tag{14}$$

Combining Equations (10)-(14), the total cost of all jobs in the $\breve{G}_l$ under $\breve{\xi}_l$ can be given by

$$\begin{aligned}
\breve{Z}_l\left(\breve{\xi}_l\right) &= \sum_{h=1}^{m_l} \breve{\alpha}_{lh} \breve{E}_{\breve{\xi}_{l(h)}} + \sum_{h=1}^{m_l} \breve{\beta}_{lh} \breve{T}_{\breve{\xi}_{l(h)}} + m_l \breve{\theta}_l \tilde{d}_l + m_l \breve{\vartheta}_l \left(\tilde{\tilde{d}}_l - \tilde{d}_l\right) \\
&= \sum_{h=1}^{e_l-1} \breve{\alpha}_{lh} \left(\sum_{k=h+1}^{e_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k)\right) + \sum_{h=t_l+1}^{m_l} \breve{\beta}_{lh} \left(\sum_{k=t_l+1}^{h} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k)\right) \\
&\quad + m_l \breve{\theta}_l \left(\breve{S}_l + \breve{s}_l + \sum_{k=1}^{e_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k)\right) + m_l \breve{\vartheta}_l \sum_{k=e_l+1}^{t_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k) \\
&= \sum_{h=1}^{e_l} \left(\sum_{k=1}^{h-1} \breve{\alpha}_{lk}\right) \breve{P}_{\breve{\xi}_{l(h)}} \breve{g}_l(h) + \sum_{h=t_l+1}^{m_l} \left(\sum_{k=h}^{m_l} \breve{\beta}_{lk}\right) \breve{P}_{\breve{\xi}_{l(h)}} \breve{g}_l(h) \\
&\quad + m_l \breve{\theta}_l \left(\breve{S}_l + \breve{s}_l + \sum_{k=1}^{e_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k)\right) + m_l \breve{\vartheta}_l \sum_{k=e_l+1}^{t_l} \breve{P}_{\breve{\xi}_{l(k)}} \breve{g}_l(k) \\
&= \sum_{h=1}^{e_l} \left(m_l \breve{\theta}_l + \left(\sum_{k=1}^{h-1} \breve{\alpha}_{lk}\right)\right) \breve{P}_{\breve{\xi}_{l(h)}} \breve{g}_l(h) + \sum_{h=e_l+1}^{t_l} m_l \breve{\vartheta}_l \breve{P}_{\breve{\xi}_{l(h)}} \breve{g}_l(h) \\
&\quad + m_l \breve{\theta}_l \left(\breve{S}_l + \breve{s}_l\right) + \sum_{h=t_l+1}^{m_l} \left(\sum_{k=h}^{m_l} \breve{\beta}_{lk}\right) \breve{P}_{\breve{\xi}_{l(h)}} \breve{g}_l(h) \\
&= \sum_{h=1}^{m_l} \breve{\varPhi}_{\breve{\xi}_{l(h)}} \breve{P}_{\breve{\xi}_{l(h)}} + m_l \breve{\theta}_l \left(\breve{S}_l + \breve{s}_l\right),
\end{aligned} \tag{15}$$

where

$$\breve{\varPhi}_{\breve{\xi}_{l(h)}} = \begin{cases} \left(m_l \breve{\theta}_l + \left(\sum_{k=1}^{h-1} \breve{\alpha}_{lk}\right)\right) \breve{g}_l(h), & h \le e_l, \\ m_l \breve{\vartheta}_l \breve{g}_l(h), & e_l + 1 \le h \le t_l, \\ \left(\sum_{k=h}^{m_l} \breve{\beta}_{lk}\right) \breve{g}_l(h), & h > t_l. \end{cases} \tag{16}$$

From Equations (8) and (15), $\breve{Z}_l\left(\breve{\xi}_l\right)$ can be represented by a united form as follows:

$$\breve{Z}_l\left(\breve{\xi}_l\right) = \sum_{h=1}^{m_l} \breve{\varPhi}_{\breve{\xi}_{l(h)}} \breve{P}_{\breve{\xi}_{l(h)}} + m_l \min\left\{\breve{\vartheta}_l, \breve{\theta}_l\right\} \left(\breve{S}_l + \breve{s}_l\right). \tag{17}$$

If $\breve{\vartheta}_l \le \breve{\theta}_l$, $\breve{\varPhi}_{\breve{\xi}_{l(h)}}$ is determined by Equation (9); otherwise, $\breve{\varPhi}_{\breve{\xi}_{l(h)}}$ is determined by Equation (16). As can be seen from Equation (17), under the optimal due-window allocation strategy, the first item $\sum_{h=1}^{m_l} \breve{\varPhi}_{\breve{\xi}_{l(h)}} \breve{P}_{\breve{\xi}_{l(h)}}$ only depends on the internal job sequence of group $\breve{G}_l$ and has nothing to do with the processing sequence of other groups, while the second item is independent of the internal work order of each group. Based on the above analysis, combined with the following lemma, the Hungarian method can be applied to determine the optimal job order within each group.

Let $\breve{X}_{lhk} = 1$ if job $\breve{J}_{\breve{\xi}_{lh}}$ is processed in the kth position under group $\breve{G}_l$ and $\breve{X}_{lhk} = 0$ otherwise. For the CODW, from Equations (9) and (16), for the group $\breve{G}_l$, the optimal job sequence $\breve{\xi}_l$ of the problem $1|GT, PDPT, CODW|\sum_{l=1}^{q}\sum_{h=1}^{m_l}\Big(\breve{\alpha}_{lh}\breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh}\breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l\tilde{d}_l$ $+\breve{\vartheta}_l\breve{D}_l\Big)$ can be provided by following assignment problem:

$$\min \breve{Z}_l = \sum_{h=1}^{m_l}\sum_{k=1}^{m_l}\breve{\Omega}_{lhk}\breve{X}_{lhk} \tag{18}$$

$$\text{s.t.}\begin{cases}\displaystyle\sum_{h=1}^{m_l}\breve{X}_{lhk} = 1, \quad h = 1, 2, \ldots, m_l,\\ \displaystyle\sum_{k=1}^{m_l}\breve{X}_{lhk} = 1, \quad k = 1, 2, \ldots, m_l,\\ \breve{X}_{lhk} = 0 \text{ or } 1,\end{cases} \tag{19}$$

in which, if $\breve{\vartheta}_l \le \breve{\theta}_l$

$$\breve{\Omega}_{lhk} = \begin{cases} m_l\breve{v}_l\breve{P}_{\breve{\xi}_{lh}}\breve{g}_l(k), & k \le t_l,\\ \left(\displaystyle\sum_{h=k}^{m_l}\breve{\beta}_{lh}\right)\breve{P}_{\breve{\xi}_{lh}}\breve{g}_l(k), & k > t_l,\end{cases} \tag{20}$$

if $\breve{\vartheta}_l > \breve{\theta}_l$

$$\breve{\Omega}_{lhk} = \begin{cases}\left(m_l\breve{\theta}_l + \left(\displaystyle\sum_{h=1}^{k-1}\breve{\alpha}_{lh}\right)\right)\breve{P}_{\breve{\xi}_{lh}}\breve{g}_l(k), & k \le e_l,\\ m_l\breve{v}_l\breve{P}_{\breve{\xi}_{lh}}\breve{g}_l(k), & e_l + 1 \le k \le t_l,\\ \left(\displaystyle\sum_{h=k}^{m_l}\breve{\beta}_{lh}\right)\breve{P}_{\breve{\xi}_{lh}}\breve{g}_l(k), & k > t_l.\end{cases} \tag{21}$$

Let $\breve{\pi}_l = \sum_{h=1}^{m_l}\breve{P}_{\breve{\xi}_{l(h)}}\breve{g}_l(h)$ for $l = 1, 2, \ldots, q$, $h = 1, 2, \ldots, m_l$. The optimal group sequence can be formulated by the following lemma.

Lemma 3.3. *For the problem* $1|GT, PDPT, CODW|\sum_{l=1}^{q}\sum_{h=1}^{m_l}\Big(\breve{\alpha}_{lh}\breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh}\breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l\tilde{d}_l$ $+\breve{\vartheta}_l\breve{D}_l\Big)$*, the optimal group sequence is assigned the groups by the non-decreasing order of* $\frac{\breve{s}_l + \breve{\pi}_l}{m_l\min\{\breve{v}_l, \breve{\theta}_l\}}$.

Proof: Let $\breve{\varsigma}$ be the optimal group schedule for the problem

$$1|GT, PDPT, CODW|\sum_{l=1}^{q}\sum_{h=1}^{m_l}\left(\breve{\alpha}_{lh}\breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh}\breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l\tilde{d}_l + \breve{\vartheta}_l\breve{D}_l\right)$$

and it dissatisfies this lemma. Let group $\breve{G}_i(\breve{\varsigma})$ and group $\breve{G}_j(\breve{\varsigma})$ be two adjacent groups, supposed that $\breve{G}_i(\breve{\varsigma})$ is prior to $\breve{G}_j(\breve{\varsigma})$, and the start time of group under ς be $\breve{S}$. Exchanging the groups $\breve{G}_i$ and $\breve{G}_j$ to construct another schedule $\breve{\varsigma}^1$ while leaving the other groups unchanged. Then by Equation (17), we have

$$\breve{Z}_i(\breve{\varsigma}) + \breve{Z}_j(\breve{\varsigma}) - \left(\breve{Z}_i\left(\breve{\varsigma}^1\right) + \breve{Z}_j\left(\breve{\varsigma}^1\right)\right)$$

$$= m_i \min\left\{\breve{\vartheta}_i, \breve{\theta}_i\right\}\left(\breve{S} + \breve{s}_i\right) + m_j \min\left\{\breve{\vartheta}_j, \breve{\theta}_j\right\}\left(\breve{S} + \breve{s}_i + \breve{\pi}_i + \breve{s}_j\right)$$
$$- \left(m_j \min\left\{\breve{\vartheta}_j, \breve{\theta}_j\right\}\left(\breve{S} + \breve{s}_j\right) + m_i \min\left\{\breve{\vartheta}_i, \breve{\theta}_i\right\}\left(\breve{S} + \breve{s}_j + \breve{\pi}_j + \breve{s}_i\right)\right)$$
$$= m_j \min\left\{\breve{\vartheta}_j, \breve{\theta}_j\right\}(\breve{s}_i + \breve{\pi}_i) - m_i \min\left\{\breve{\vartheta}_i, \breve{\theta}_i\right\}(\breve{s}_j + \breve{\pi}_j).$$

To show that $\breve{\varsigma}$ is better than $\breve{\varsigma}^1$, we have

$$\breve{Z}_i(\breve{\varsigma}) + \breve{Z}_j(\breve{\varsigma}) - \left(\breve{Z}_i\left(\breve{\varsigma}^1\right) + \breve{Z}_j\left(\breve{\varsigma}^1\right)\right) \leq 0,$$

then

$$m_j \min\left\{\breve{\vartheta}_j, \breve{\theta}_j\right\}(\breve{s}_i + \breve{\pi}_i) - m_i \min\left\{\breve{\vartheta}_i, \breve{\theta}_i\right\}(\breve{s}_j + \breve{\pi}_j) \leq 0.$$

This completes the proof. □

Based on the above analysis, the optimal algorithm can be described as follows.

Algorithm 1

Input: $\breve{p}_{lh}$, $\breve{g}_l(k)$, $\breve{\alpha}_{lh}$, $\breve{\beta}_{lh}$, $\breve{\theta}_l$, $\breve{\vartheta}_l$, $\breve{s}_l$ $(l = 1, 2, \ldots, q,\ h = 1, 2, \ldots, m_l,\ k = 1, 2, \ldots, m_l)$
Output: The optimal sequence e_l, t_l, $\breve{\xi}_l^*$, $\breve{\varsigma}^*$, $\breve{Z}_l^*$, $\breve{Z}^*$, $\tilde{d}_l^*$, $\breve{D}_l^*$

Step 1. According to Lemmas 3.1 and 3.2, calculate the e_l, t_l.

Step 2. According to Equations (20) and (21), calculate the $\breve{\Omega}_{hk}$ for $(h = 1, 2, \ldots, m_l,$ $k = 1, 2, \ldots, m_l)$.

Step 3. The optimal job sequence $\breve{\xi}^*(l)$ can be provided by Hungarian method (18)-(21) within each group.

Step 4. The optimal group sequence $\breve{\varsigma}^*$ can be provided by Lemma 3.3 between groups.

Step 5. Calculate $\breve{Z}_l^*$ by using Equation (17).

Step 6. Calculate $\tilde{d}_l^* = \breve{C}_{\breve{\xi}_{l(e_l)}}$ and $\breve{D}_l^*$.

Theorem 3.1. *The problem* $1|GT, PDPT, CODW|\sum_{l=1}^{q}\sum_{h=1}^{m_l}\left(\breve{\alpha}_{lh}\breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh}\breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l\tilde{d}_l + \breve{\vartheta}_l\breve{D}_l\right)$ *can be solved by Algorithm 1 in* $O(m^3)$ *time.*

Proof: In Algorithm 1, Step 1 needs $O(qm)$ time; Steps 2 and 3 for each group, solving the assignment problem requires $O(m^3)$ time; in Step 4 for each group, solving the problem requires $O(m\log(m))$ time; qm and $m\log(m)$ are less than m^3. Hence, the time complexity of Algorithm 1 is $O(m^3)$. □

4. Example.

In this section, Algorithm 1 is applied to solve

$$1|GT, PDPT, CODW|\sum_{l=1}^{q}\sum_{h=1}^{m_l}\left(\breve{\alpha}_{lh}\breve{E}_{\breve{\xi}_{l(h)}} + \breve{\beta}_{lh}\breve{T}_{\breve{\xi}_{l(h)}} + \breve{\theta}_l\tilde{d}_l + \breve{\vartheta}_l\breve{D}_l\right).$$

Example 4.1. *Assume a common due-window assignment problem with* $m = 10$ *jobs which are classified into* $q = 2$ *groups and the relevant parameters of jobs are given in Tables 1-4, where* $\breve{g}_1(k) = k$ *and* $\breve{g}_2(k) = k^{-2}$ *for* $k = 1, 2, \ldots, m$.

TABLE 1. Values of group-dependent parameters

l	θ_l	ϑ_l	$\breve{s}_l$
1	5	4	20
2	1	3	10

TABLE 2. Values of $\breve{P}_{lh}$

$\breve{P}_{lh} = l\backslash h$	1	2	3	4	5
1	4	8	10	5	2
2	11	6	2	9	3

TABLE 3. Values of $\breve{\alpha}_{lh}$

$\breve{\alpha}_{lh} = l\backslash h$	1	2	3	4	5
1	6	2	3	4	1
2	2	4	1	7	5

TABLE 4. Values of $\breve{\beta}_{lh}$

$\breve{\beta}_{lh} = l\backslash h$	1	2	3	4	5
1	3	7	5	8	9
2	6	2	5	11	4

Depending on Step 1, for $\breve{G}_1$, $\breve{\vartheta}_1 < \breve{\theta}_1$, thus $e_1 = 0$, on the basis of Lemmas 3.1 and 3.2, $t_1 = 3, 4, 5$. For $\breve{G}_2$, in accordance with $\breve{\vartheta}_2 > \breve{\theta}_2$ and Lemmas 3.1 and 3.2, $e_2 = 0, 1, 2, 3, 4$ and $t_2 = 4, 5$.

Depending on Steps 2 and 3, the following Tables 5 and 6 are given.

TABLE 5. Results of group $\breve{G}_1$

e_1	t_1	$\breve{\xi}(e_1, t_1)$	$\breve{Z}(e_1, t_1)$
0	**3**	$(\boldsymbol{J_{31}}, \boldsymbol{J_{22}}, \boldsymbol{J_{13}}, \boldsymbol{J_{54}}, \boldsymbol{J_{45}})$	**1121**
0	4	$(J_{31}, J_{22}, J_{13}, J_{54}, J_{45})$	1145
0	5	$(J_{31}, J_{22}, J_{43}, J_{14}, J_{55})$	1340

The bold numbers are the optimal solution $\breve{\xi}^*$

TABLE 6. Results of group $\breve{G}_2$

e_2	t_2	$\breve{\xi}(e_2, t_2)$	$\breve{Z}(e_2, t_2)$
0	4	$(J_{31}, J_{52}, J_{23}, J_{44}, J_{15})$	61.4475
1	4	$(J_{31}, J_{52}, J_{23}, J_{44}, J_{15})$	41.4475
2	4	$(J_{31}, J_{22}, J_{53}, J_{44}, J_{15})$	35.6975
3	4	$(J_{31}, J_{52}, J_{23}, J_{44}, J_{15})$	34.2808
4	**4**	$(\boldsymbol{J_{31}}, \boldsymbol{J_{52}}, \boldsymbol{J_{23}}, \boldsymbol{J_{44}}, \boldsymbol{J_{15}})$	**32.5933**
0	5	$(J_{31}, J_{52}, J_{23}, J_{44}, J_{15})$	66.2875
1	5	$(J_{31}, J_{52}, J_{23}, J_{44}, J_{15})$	46.2875
2	5	$(J_{31}, J_{22}, J_{53}, J_{44}, J_{15})$	40.5375
3	5	$(J_{31}, J_{52}, J_{23}, J_{44}, J_{15})$	39.1208
4	5	$(J_{31}, J_{52}, J_{23}, J_{44}, J_{15})$	37.4333

The bold numbers are the optimal solution $\breve{\xi}^*$

Depending on Step 4, the optimal group sequence $\breve{\varsigma}^* = \left\{\breve{G}_1 \to \breve{G}_2\right\}$.

Depending on Steps 5 and 6, $\breve{Z}^* = 1183.5933$, $\tilde{d}_1^* = 0$, $\breve{D}_1^* = 58$, $\tilde{d}_2^* = 76$, $\breve{D}_2^* = 0$.

Remark 4.1. *In this example, the application of Algorithm 1 to solve the problem is demonstrated. Consider a scenario with 10 jobs grouped into 2 groups, each with specific job parameters. By following the steps outlined in the algorithm and using Lemmas 3.1 and 3.2, we determine the optimal due-window start and finish times for each group. See Tables 5 and 6 for details.*

5. **Summary and Future Research.** In this paper, we studied common due-window assignment with position-dependent weights under group scheduling on the single-machine. The objective is to determine the optimal group sequence, the optimal job sequence, and the optimal due-window assignment to minimize the total cost including early and late penalties, common due-window start time, and common due-window size costs. We demonstrated that the common due-window assignment problem can be solved in $O(m^3)$ time. Future research may consider scheduling with learning effects in flow shop setting (Li et al. [24] and Jung and Kim [25]), study parallel machine scheduling with group technology (Yang [26]) or deal with scheduling with deteriorating jobs (Zhang et al. [27] and Lv et al. [28]).

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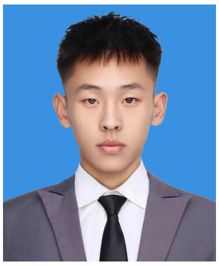

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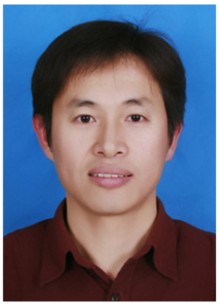

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