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A DESIGN METHOD FOR EXTENDED SEMI-STRONGLY STABILIZING CONTROLLERS WITH A POLE AT THE ORIGIN AND TWO PAIRS OF POLES ON THE IMAGINARY AXIS

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Abstract. *In this paper, we examine the design method of the control system for the extended semi-strongly stabilizing controller. Several researchers have studied a strongly stabilization problem. A strongly stabilization is the control method to make control systems stable using stable controllers. Using this method, it is possible to construct a highly reliable control system because it has robustness. On the other hand, in some cases, the controller needs to have poles on the imaginary axis; for example, the control systems follow sinusoidal signals, and the self-repairing control systems with faulty sensors using resonance. To design the controller having poles at the origin and the others in the open left half plane, a semi-strongly stabilizing controller is proposed. On the other hand, this method remains an error between the output and the reference input. Expanding the semi-strongly controller, an extended semi-strongly stabilizing controller, in which a controller has a pole at the origin and two pairs of poles on the imaginary axis, is proposed. According to Kimura et al., the parameterization of all extended semi-strongly stabilizable plants and all extended semi-strongly stabilizing controllers is clarified. However, the proof of the parameterization of all extended semi-strongly stabilizable plants and all extended semi-strongly stabilizing controllers is not shown. In this paper, we show the proof of the parameterization of all extended semi-strongly stabilizable plants and all extended semi-strongly stabilizing controllers. In addition, we propose a design procedure of the extended semi-strongly stabilizing controllers.*

Keywords: Extended semi-strongly stabilization, Parameterization

1. **Introduction.** This paper concerns the parameterization of all extended semi-strongly stabilizing controllers with a pole at the origin and two pairs of poles on the imaginary axis. The parameterization is a method of finding all stabilizing controllers for a given

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plant [1, 2]. Using the parametrization, the controller to guarantee the stability of the control system is obtained [1, 2, 3]. Various researchers have studied about parameterization problems: PID control [4], two-degree-of-freedom stabilizing controller [5], disturbance observer [6], modified Smith predictor [7], and internally stabilizing controller for minimum phase systems [8].

However, the stability of the controller obtained in this parametrization is not taken into account. It is important to consider the stability of the controller. If the controller is unstable, the control system will be highly sensitive when parameters under control change [3, 9]. To be lowly sensitive to a parameter change, we had better use stable controllers. In addition, the unstable controller causes the degradation of target tracking performance [10, 11]. Toward this problem, there exists a control method called the strongly stabilization. This is a method to stabilize the control system by using stable controllers. The condition that there exists strongly stabilizing controllers is known as the p.i.p. (parity interlacing property) condition [9, 12]. Wakaiki et al. examined the sensitivity reduction problem with stable controllers for the linear time-invariant multi-input/multi-output distributed parameter system [13, 14]. However, they do not clarify the class of strongly stabilizable plants. If the class of strongly stabilizable plants is clarified, we have a possibility to obtain the parameterization of all stable stabilizing controllers. In addition, we can clarify the characteristics of strongly stabilizable plants. From this viewpoint, Hoshikawa et al. clarified the class of all strongly stabilizable plants [15]. [16] clarified the parameterization of all two degrees of freedom strongly stabilizing controllers.

Although the control system that the semi-strongly stabilizing controller has a good performance such as robustness when parameters under control change, the strongly stabilizing controller cannot make the output follow the step reference without steady-state error for the plant with uncertainty or disturbances. In many actual control systems, the output must follow the step reference input without steady-state error, even if uncertainty in the plant or a step disturbance exists. Thus, the controller must have a pole at the origin. From this viewpoint, Hoshikawa et al. expanded the concept of strongly stabilization and proposed a concept of semi-strongly stabilization using a controller that has a pole at the origin and the others in the open left half-plane [17]. The class of semi-strongly stabilizable plants and a controller design method are proposed [18]. However, the method by [18] fails to place the poles on the imaginary axis. The controller is often required to have a pair of poles on the imaginary axis for several purposes such as sinusoidal reference tracking. Sinusoidal reference tracking is required in important practical applications [19]. The sinusoidal reference tracking is applied for Robotics [20, 21], and so on. In addition, from the internal model principle, the control system using the semi-strongly stabilizing controller cannot attenuate the sinusoidal disturbances. This is an important problem because, in several nonlinear industrial processes (power distribution, robotics, etc.), disturbances can be sinusoidal signals [22]. Sinusoidal signals also use the part of a controller in the self-repairing control system [23]. Thus, it is important to solve the stabilization problem using a controller that has a pole on the imaginary axis and the others in the open left half-plane. Kimura et al. defined an extended semi-strongly stabilizing controller that has a pair of poles on the imaginary axis and no other unstable poles [24]. In [24], the parameterizations of all extended semi-strongly stabilizing controllers are clarified. Using the result of [24], Niiyama et al. proposed a design method of a control system for failure detection method using the extended semi-strongly stabilizing controller and faulty sensors proposed by Takahashi et al. [25].

However, the extended semi-strongly stabilizing controller proposed by [24] does not have a pole at the origin. This is a problem that the control system proposed by [25] could not eliminate the steady-state error from the output. Kimura et al. considered this problem and redefined the extended semi-strongly stabilizing controller as the controller having no unstable poles except for a pole at the origin and a pair of poles on the imaginary axis [26]. In [26], the parameterization of all extended semi-strongly stabilizing controllers is also clarified. The semi-strongly stabilizing controller redefined by [26] differs from the PID controller because the semi-strongly stabilizing controller has a pole at the origin and a pole on the imaginary axis. Having a pole at the origin and a pole on the imaginary axis, the semi-strongly stabilizing controller can have the tracking performance for the sinusoidal signals and the disturbance attenuation property for the sinusoidal disturbances.

Applying the extended semi-strongly stabilization controller redefined in [26], the control system proposed by [23] can eliminate the steady-state error from the output. On the other hand, sinusoidal signals such as the auxiliary input to the control system in [23] negatively affect the output. To attenuate the effect of sinusoidal signals on the output, Kimura et al. proposed the extended semi-strongly stabilizing controller which has a pole at the origin and two pairs of poles on the imaginary axis, and stable poles [27]. In [27], the parameterization of all extended semi-strongly stabilizable plants and the parameterization of all extended semi-strongly stabilizing controllers are also shown [27]. Although results of [27] considered the one-degree-of-freedom control system, the semi-strongly stabilizing controller has been expected to expand the feedback controller in the two-degree-of-freedom control system. However, Kimura et al. do not show proof of the parameterization of all extended semi-strongly stabilizable plants and all extended semi-strongly stabilizing controllers.

In this paper, we clarify complete proof of the parameterization of all extended semi-strongly stabilizable plants and the parameterization of all extended semi-strongly stabilizing controllers. In addition, we present a design procedure for the extended semi-strongly stabilizing controller. Using the design procedure proposed in this paper, we make the extended semi-strongly controller easy. This paper is organized as follows. In Section 2, the problem considered in this paper is explained. In Section 3, the proof of the parameterization of all extended semi-strongly stabilizable plants is clarified. In Section 4, the proof of the parameterization of all extended semi-strongly stabilizing controllers for the extended semi-strongly stabilizable plants is clarified. In Section 5, the design procedure of the extended semi-strongly stabilizing controllers is shown. In Section 6, a numerical example is illustrated to show the effectiveness of the proposed method. Section 7 gives concluding remarks.

2. **Problem Formulation.**

Consider the control system

$$\begin{cases} y(s) = G(s)u(s) + d(s) \\ u(s) = C(s)(r(s) - y(s)) \end{cases}, \tag{1}$$

where $G(s) \in R(s)$ is the plant, $C(s) \in R(s)$ is the controller, $y(s) \in R$ is the output, $u(s) \in R$ is the control input, $r(s) \in R$ is the reference input, and $d(s) \in R$ is the disturbance.

According to [27], the extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis is defined as shown in Definition 2.1.

Definition 2.1. *(Extended semi-strongly stabilizing controller with a pole at the origin and two pairs of poles on the imaginary axis)* [27]. *The controller $C(s)$ is called an extended semi-strongly stabilizing controller with a pole at the origin and two pairs of poles on the imaginary axis if the following expressions hold true.*

1) $C(s)$ makes the control system in (1) internally stable.

2) $C(s)$ has a pole at the origin and two pairs of poles on the imaginary axis. The other poles of $C(s)$ are in the open left-half plane.

That is, if $C(s)$ in (1) is written as

$$C(s) = \frac{Q_1(s)}{N_c(s)}, \tag{2}$$

then we call $C(s)$ in (1) the extended semi-strongly stabilizing controller with a pole at the origin and two pairs of poles on the imaginary axis. Here, $N_c(s) \in RH_\infty$ is written as

$$N_c(s) = \frac{s\left(s^2+\omega^2\right)^2}{n_{cd}(s)}, \tag{3}$$

where $\omega \in R$ is any constant, $n_{cd}(s)$ is any Hurwitz polynomial of 5 degree and $Q_1(s) \in RH_\infty$ is any function satisfying

$$Q_1(s)|_{s=0,\pm j\omega} \neq 0. \tag{4}$$

Note that any plants are not necessarily stabilized by extended semi-strongly stabilizing controllers with a pole at the origin and two pairs of poles on the imaginary axis. Therefore, we define extended semi-strongly stabilizable plants as follows.

Definition 2.2. *(Extended semi-strongly stabilizable plant)* [27]. *When the plant $G(s)$ in (1) can be stabilized by the extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis $C(s)$ in (2), the plant $G(s)$ is called an extended semi-strongly stabilizable plant.*

The problem considered in this paper is to clarify complete proof of the parameterizations of all extended semi-strongly stabilizable plants and of all extended semi-strongly stabilizing controllers with a pole at the origin and two pairs of poles on the imaginary axis. In addition, we present a design procedure of extended semi-strongly stabilizing controllers with a pole at the origin and two pairs of poles on the imaginary axis.

3. Parameterization of All Extended Semi-Strongly Stabilizable Plants.

In this section, we clarify the proof of parameterization of all extended strongly stabilizable plants.

According to [27], this parameterization is summarized in the following theorem.

Theorem 3.1. *The plant $G(s)$ is extended semi-strongly stabilizable if and only if the plant $G(s)$ is written as*

$$G(s) = \frac{N_b(s) + N_c(s)Q_2(s)}{\frac{1-N_b(s)Q_1(s)}{N_c(s)} - Q_1(s)Q_2(s)}, \tag{5}$$

where $N_b(s) \in RH_\infty$ is any function satisfying

$$\left.\frac{1}{s^2+\omega^2}\{1 - N_b(s)Q_1(s)\}\right|_{s=0,\pm j\omega} = 0, \tag{6}$$

$$N_b(s)|_{s=0,\pm j\omega} \neq 0. \tag{7}$$

$Q_2(s) \in RH_\infty$ is any function, and $Q_1(s) \in RH_\infty$ is any function satisfying (4).

The proof of Theorem 3.1 requires the following lemma.

Lemma 3.1. [9] *Assume that $A(s) \in RH_\infty^{m\times n}$, $B(s) \in H_\infty^{q\times p}$, $C(s) \in RH_\infty^{m\times p}$ and*

$$\text{rank} \left[\begin{array}{cc} A^T(s) & B^T(s) \end{array} \right] = \gamma \tag{8}$$

are satisfied. There exist $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ satisfying

$$X(s)A(s) + Y(s)B(s) = C(s) \tag{9}$$

if and only if there exists $U(s) \in \mathcal{U}$ *satisfying*

$$\begin{bmatrix} A(s) \\ B(s) \\ C(s) \end{bmatrix} = U(s) \begin{bmatrix} A(s) \\ B(s) \\ O \end{bmatrix}. \tag{10}$$

When $X_0(s) \in RH_\infty$ *and* $Y_0(s) \in RH_\infty$ *are solutions to (9), then all solutions to (9) are given by*

$$\begin{bmatrix} X(s) & Y(s) \end{bmatrix} = \begin{bmatrix} X_0(s) & Y_0(s) \end{bmatrix} + Q(s) \begin{bmatrix} W_1(s) & W_2(s) \end{bmatrix}, \tag{11}$$

where $W_1(s)$ *and* $W_2(s)$ *satisfy*

$$W_1(s)A(s) + W_2(s)B(s) = 0 \tag{12}$$

and

$$\text{rank} \begin{bmatrix} W_1(s) & W_2(s) \end{bmatrix} = n + q - \gamma \tag{13}$$

and $Q(s) \in RH_\infty^{p\times(N+q-\gamma)}$ *is any function.*

Using Lemma 3.1, we will prove Theorem 3.1.

Proof: First, the necessity is shown. That is, we show that if $C(s)$ in (2) makes the control system in (1) stable, then $G(s)$ takes the form of (5). From the assumption that $C(s)$ in (2) is stable,

$$N(s)Q_1(s) + D(s)N_c(s) = 1 \tag{14}$$

holds true, where $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G(s)$ satisfying

$$G(s) = \frac{N(s)}{D(s)}. \tag{15}$$

A pair of solutions $N_0(s)$ and $D_0(s)$ to (14) is

$$N_0(s) = N_b(s) \tag{16}$$

and

$$D_0(s) = \frac{1 - N_b(s)Q_1(s)}{N_c(s)}, \tag{17}$$

respectively. Here $N_b(s) \in RH_\infty$ is any function and satisfies (6), (7) because $D(s) \in RH_\infty$. From Lemma 3.1, all solutions $N(s)$ and $D(s)$ satisfying (14) are written as

$$N(s) = N_b(s) + N_c(s)Q_2(s) \tag{18}$$

and

$$D(s) = \frac{1 - N_b(s)Q_1(s)}{N_c(s)} - Q_1(s)Q_2(s), \tag{19}$$

respectively, since

$$N_b(s)Q_1(s) + \frac{1 - N_b(s)Q_1(s)}{N_c(s)} N_c(s) = 1 \tag{20}$$

and

$$N_c(s)Q_1(s) - Q_1(s)N_c(s) = 0, \tag{21}$$

where $N_b(s) \in RH_\infty$ is any function and satisfies (6), (7), N_c is written by (3) and $Q_2(s) \in RH_\infty$ is any function. Substituting (18) and (19) for (15), we have (5). Thus, the necessity has been shown.

Next, the sufficiency is shown. That is, if $G(s)$ in (1) takes the form of (5), then there exists an extended semi-strongly stabilizing controller to make the control system in (1) stable. A controller is set as (2). From simple manipulation, we have

$$\frac{G(s)C(s)}{1+G(s)C(s)} = Q_1(s)\left(N_b(s) + N_c(s)Q_2(s)\right), \tag{22}$$

$$\frac{G(s)}{1+G(s)C(s)} = N_c(s)\left(N_b(s) + N_c(s)Q_2(s)\right), \tag{23}$$

$$\frac{C(s)}{1+G(s)C(s)} = Q_1(s)\left(\frac{1-N_b(s)Q_1(s)}{N_c(s)} - Q_1(s)Q_2(s)\right) \tag{24}$$

and

$$\frac{1}{1+G(s)C(s)} = 1 - N_b(s)Q_1(s) - Q_1(s)Q_2(s)N_c(s). \tag{25}$$

From $N_b(s)$ satisfying (6), $(1-N_b(s)Q_1(s))/N_c(s) \in RH_\infty$. Since $N_c(s) \in RH_\infty$, $N_b(s) \in RH_\infty$, $Q_1(s) \in RH_\infty$, $Q_2(s) \in RH_\infty$ and $(1-N_b(s)Q_1(s))/N_c(s) \in RH_\infty$, the transfer functions in (22), (23), (24) and (25) are stable. This implies that the control system in (1) is stable. Thus, the sufficiency has been proved.

We have thus proved Theorem 3.1. □

4. Parameterization of All Extended Semi-Strongly Stabilizing Controllers.

In this section, the complete proof of parameterization of all extended semi-strongly stabilizing controllers $C(s)$ for the extended semi-strongly stabilizable plant $G(s)$ in (5) is proposed.

According to [27], this parameterization is summarized in the following theorem.

Theorem 4.1. *The controller $C(s)$ is an extended semi-strongly stabilizing controller with a pole at the origin and two pairs of poles on the imaginary axis for the plant $G(s)$ in (5) if and only if the controller $C(s)$ is written as*

$$C(s) = \frac{Q_1(s) + \left\{\frac{1-N_b(s)Q_1(s)}{N_c(s)} - Q_1(s)Q_2(s)\right\}P(s)}{N_c(s) - \left\{N_b(s) + N_c(s)Q_2(s)\right\}P(s)}, \tag{26}$$

where $P(s) \in RH_\infty$ and $Q(s) \in RH_\infty$ are functions written as

$$P(s) = N_c(s)Q(s), \tag{27}$$

$$Q(s) = \frac{1-\hat{Q}(s)}{N_b(s) + N_c(s)Q_2(s)}, \tag{28}$$

respectively, where $\hat{Q}(s) \in \mathcal{U}$ is a unimodular function that makes $Q(s)$ proper and satisfies

$$\left.\frac{1}{(s-s_i)^{m_i-1}}\left\{1-\hat{Q}(s)\right\}\right|_{s=s_i} = 0 \ \forall i, \tag{29}$$

$s_i \in R$ is unstable zeros of $N_b(s) + N_c(s)Q_2(s)$ and m_i is its multiplicity.

Next, we will prove Theorem 4.1.

Proof: From [9], the parameterization of all stabilizing controllers for $G(s)$ is written as

$$C(s) = \frac{X(s) + D(s)P(s)}{Y(s) - N(s)P(s)}, \tag{30}$$

where $N(s) \in RH_\infty$ and $D(s) \in RH_\infty$ are coprime factors of $G(s)$ on RH_∞ satisfying (15) and $X(s) \in RH_\infty$ and $Y(s) \in RH_\infty$ are a pair of solutions of

$$N(s)X(s) + D(s)Y(s) = 1, \tag{31}$$

and $P(s) \in RH_\infty$ is any function. Since the extended semi-strongly stabilizable plant $G(s)$ takes the form of (5), coprime factors $N(s)$ and $D(s)$ in (15) are written as

$$N(s) = N_b(s) + N_c(s)Q_2(s), \tag{32}$$

$$D(s) = \frac{1 - N_b(s)Q_1(s)}{N_c(s)} - Q_1(s)Q_2(s), \tag{33}$$

respectively. From (32) and (33), a pair of solution $X(s)$ and $Y(s)$ to (31) is given by

$$X(s) = Q_1(s), \tag{34}$$

$$Y(s) = N_c(s), \tag{35}$$

respectively. Substituting (32), (33), (34), and (35) for (30), we have (26).

Next, we show that $C(s)$ in (26) is an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis if and only if $P(s)$ in (26) is given by (27), $Q(s)$ in (27) is given by (28), and $\hat{Q}(s)$ in (28) satisfies $\hat{Q}(s) \in \mathcal{U}$ and (29). First, we show necessity. That is, we show that if $C(s)$ in (26) is an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis, then $P(s)$ in (26) is given by (27), $Q(s)$ in (27) is given by (28), and $\hat{Q}(s)$ in (28) satisfies $\hat{Q}(s) \in \mathcal{U}$ and (29). From the assumption that $C(s)$ in (26) is an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis,

$$\left. \frac{1}{s^2+\omega^2} \left\{ N_c(s) - \left\{ N_b(s) + N_c(s)Q_2(s) \right\} P(s) \right\} \right|_{s=0,\pm j\omega} = 0 \tag{36}$$

is satisfied. From (7), this equation yields

$$\left. \frac{1}{s^2+\omega^2} P(s) \right|_{s=0,\pm j\omega} = 0. \tag{37}$$

The equation implies that $P(s)$ is given by (27), where $Q(s) \in RH_\infty$. Substituting (27) for (26), (26) is then written as

$$C(s) = \frac{1}{N_c(s)} \left\{ Q_1(s) + \frac{Q(s)}{1 - (N_b(s) + N_c(s)Q_2(s))\, Q(s)} \right\}. \tag{38}$$

From the assumption that $C(s)$ in (2) is an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis,

$$N_c(s)C(s) = Q_1(s) + \frac{Q(s)}{1 - (N_b(s) + N_c(s)Q_2(s))\, Q(s)} \in RH_\infty \tag{39}$$

must be satisfied. Since $Q_1(s) \in RH_\infty$ and $Q(s) \in RH_\infty$,

$$1 - (N_b(s) + N_c(s)Q_2(s))\, Q(s) \in \mathcal{U}. \tag{40}$$

Using any function $\hat{Q}(s) \in \mathcal{U}$, (40) is then written as

$$\hat{Q}(s) = 1 - (N_b(s) + N_c(s)Q_2(s))\, Q(s). \tag{41}$$

Equation (41) corresponds to (28). Since $Q(s) \in RH_\infty$, (29) is satisfied where s_i are unstable zeros of $N_b(s) + N_c(s)Q_2(s)$ and multiplicities of s_i are m_i. Thus, the necessity has been proved.

Next, we show sufficiency. That is, we show that if $P(s)$ in (26) is given by (27), $Q(s)$ in (27) is given by (28), and $\hat{Q}(s)$ in (28) satisfies $\hat{Q}(s) \in \mathcal{U}$ and (29), then $C(s)$ in (26) is an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis. Substituting (27) and (28) for (26), we have

$$C(s)=\frac{1}{N_c(s)}\left\{Q_1(s)+\frac{1-\hat{Q}(s)}{\left(N_b(s)+N_c(s)Q_2(s)\right)\hat{Q}(s)}\right\}=\frac{1}{N_c(s)}\left\{Q_1(s)+\frac{Q(s)}{\hat{Q}(s)}\right\}. \quad (42)$$

From (42), $Q_1(s)\in RH_\infty$, $\hat{Q}(s)\in\mathcal{U}$, and $Q(s)\in RH_\infty$, the controller $C(s)$ in (26) has a pole at the origin and two pairs of poles on the imaginary axis and the other poles of $C(s)$ are in the closed left-half plane. Next, we show that $C(s)$ in (42) makes the control system in (1) stable. By simple manipulation, we have

$$\frac{G(s)C(s)}{1+G(s)C(s)}=1-\left(1-N_b(s)Q_1(s)-N_c(s)Q_1(s)Q_2(s)\right)\hat{Q}(s), \quad (43)$$

$$\frac{G(s)}{1+G(s)C(s)}=N_c(s)\left(N_b(s)+N_c(s)Q_2(s)\right)\hat{Q}(s), \quad (44)$$

$$\frac{C(s)}{1+G(s)C(s)}=\left(\frac{1-N_b(s)Q_1(s)}{N_c(s)}-Q_1(s)Q_2(s)\right)\left(Q_1(s)\hat{Q}(s)+Q(s)\right) \quad (45)$$

and

$$\frac{1}{1+G(s)C(s)}=\left(1-N_b(s)Q_1(s)-N_c(s)Q_1(s)Q_2(s)\right)\hat{Q}(s). \quad (46)$$

From $N_b(s)$ satisfying (6), $(1-N_b(s)Q_1(s))/N_c(s)\in RH_\infty$. Since $N_c(s)\in RH_\infty$, $N_b(s)\in RH_\infty$, $Q_1(s)\in RH_\infty$, $Q_2(s)\in RH_\infty$, $\hat{Q}(s)\in\mathcal{U}$, $Q(s)\in RH_\infty$ and $(1-N_b(s)Q_1(s))/N_c(s)$ $\in RH_\infty$, the transfer functions in (43), (44), (45) and (46) are stable. This implies that the control system in (1) is stable. Thus, the sufficiency has been shown.

We have thus proved Theorem 4.1. □

5. **Design Procedure.** In this section, a design procedure for extended semi-strongly stabilizing controllers with a pole on the origin and two pairs of poles on the imaginary axis is presented. We extend the design procedure proposed by Hoshikawa et al. in [18] which cannot design an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis. For the plant $G(s)$ satisfying Theorem 3.1, we have an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis. From Theorem 4.1, to design an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis $C(s)$, $\hat{Q}(s)$ in (28) needs to be $\hat{Q}(s)\in\mathcal{U}$ satisfying (29) and to make $Q(s)$ in (28) proper. Therefore, we first show a design procedure for $\hat{Q}(s)$ that satisfies these conditions. A design procedure for the semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis $C(s)$ using $Q(s)$, $\hat{Q}(s)$ that satisfies these conditions is shown as follows.

Step 1) We factorize

$$\tilde{Q}(s)=N_b(s)+N_c(s)Q_2(s)\in RH_\infty \quad (47)$$

as

$$N_b(s)+N_c(s)Q_2(s)=\tilde{Q}_i(s)\tilde{Q}_o(s), \quad (48)$$

where $\tilde{Q}_i(s)\in RH_\infty$ is an inner function satisfying $\tilde{Q}_i(0)=1$ and $\tilde{Q}_o(s)\in RH_\infty$ is an outer function.

Step 2) Using $\tilde{Q}_o(s)$, $\bar{Q}(s)\in RH_\infty$ is settled by

$$\bar{Q}(s)=\frac{q(s)}{\tilde{Q}_o(s)}, \quad (49)$$

where

$$q(s) = \frac{k}{(\rho s + 1)^m}, \tag{50}$$

where $\rho \in R$ is an arbitrary positive small number, m is an arbitrary positive integer to make $\bar{Q}(s)$ in (49) proper, and $k \in R$ is a real number satisfying $0 < k < 1$ and

$$k \simeq 1. \tag{51}$$

Step 3) Using $\bar{Q}(s)$ in (49), $\hat{Q}(s) \in \mathcal{U}$ is set as

$$\hat{Q}(s) = 1 - (N_b(s) + N_c(s)Q_2(s))\,\bar{Q}(s). \tag{52}$$

Thus, a design procedure for $\hat{Q}(s)$ in (52) satisfying $\hat{Q}(s) \in \mathcal{U}$ and (29) and making $Q(s)$ in (28) proper is shown.

Step 4) Substituting (52) into (28), $Q(s)$ is obtained. And substituting the obtained $Q(s)$ into (27), we obtain $P(s)$ in (27).

Step 5) Finally, substituting the obtained $P(s)$ into (26), an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis $C(s)$ is given by (26).

Next, using the presented procedure, we show that $\hat{Q}(s)$ in (52) satisfies $\hat{Q}(s) \in \mathcal{U}$ and (29) and makes $Q(s)$ in (28) proper. First, we show that $\hat{Q}(s)$ in (52) satisfies $\hat{Q}(s) \in \mathcal{U}$, (29) and makes $\bar{Q}(s)$ in (49) proper. Substituting (47), (49), and (50) for (52), $\hat{Q}(s)$ in (52) is written as

$$\hat{Q}(s) = 1 - \tilde{Q}_i(s)q(s). \tag{53}$$

Because $\tilde{Q}_i(s)$ is an inner function, $\tilde{Q}_i(s)$ is biproper. That is, $\tilde{Q}_i(s)q(s)$ is strictly proper. In addition, from (50) and $0 < k < 1$,

$$\left\| \tilde{Q}_i(s)q(s) \right\|_\infty < 1 \tag{54}$$

is satisfied. This implies that $\hat{Q}(s) \in \mathcal{U}$.

Next, we show that (29) holds true. $s_i \in R$ is unstable zeros of $N_b(s) + N_c(s)Q_2(s)$ and m_i is its multiplicity. Because $\tilde{Q}_i$ is an inner function of $N_b(s) + N_c(s)Q_2(s)$, (29) holds true, and

$$\left. \frac{1}{(s - s_i)^{m_i - 1}} \tilde{Q}_i(s) \right|_{s = s_i} = 0 \ \forall i \tag{55}$$

holds true. From this equation and (50),

$$\left. \frac{1}{(s - s_i)^{m_i - 1}} \tilde{Q}_i(s)q(s) \right|_{s = s_i} = 0 \ \forall i \tag{56}$$

is satisfied.

6. **Numerical Example.**

In this section, we illustrate a numerical example to show the effectiveness of the proposed design method for an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis.

Consider the problem of designing an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis $C(s)$ for the plant $G(s)$ [18], written as

$$G(s) = \frac{90.9 \times 10^3}{(s + 0.117)(s^2 + 3.97s + 2.02 \times 10^3)}. \tag{57}$$

First, we show that the plant $G(s)$ in (57) can be rewritten in the form (5). $N_c(s)$, $N_b(s)$, $Q_1(s)$ and $Q_2(s)$ are settled as

$$N_c(s) = \frac{s(s^2+1)^2}{(s+0.1)^3(1.01s^2+2s+1.01)}, \tag{58}$$

$$N_b(s) = \frac{s^2+400s+1}{1.01s^2+4s+1.01}, \tag{59}$$

$$Q_1(s) = 0.01, \tag{60}$$

and

$$\begin{aligned} &Q_2(s) \\ &= \frac{-(s+z_1)(s+z_2)(s+z_3)(s+z_4)^3(s+z_5)(s+z_6)(s+z_7)(s+z_8)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)(s+p_5)(s+p_6)(s+p_7)(s+p_8)(s+p_9)(s+p_{10})} \end{aligned} \tag{61}$$

where $z_i \in \mathbb{C}$ $(i = 1, 2, \dots, 8)$ and $p_k \in \mathbb{C}$ $(k = 1, 2, \dots, 10)$ are

$$\begin{cases} z_1 = 399.4, & z_2 = -0.5881, & z_3 = 0.2169, \\ z_4 = 0.1, & z_5 = 2.515 + 42.33j, & z_6 = 2.515 - 42.33j, \\ z_7 = 0.99 - 0.1411j, & z_8 = 0.99 - 0.1411j, & \\ p_1 = 3.689, & p_2 = 0.2711, & p_3 = 0.117, \\ p_4 = 0.003568, & p_5 = 1.758 + 44.88j, & p_6 = 1.758 - 44.88j, \\ p_7 = -0.1239 + 1.576j, & p_8 = 0.1239 - 1.576j, & p_9 = 0.1012 + 0.6528j, \\ p_{10} = 0.1012 + 0.6528j. & & \end{cases}$$

Using (59), (58), (60) and (61), the plant $G(s)$ in (57) is rewritten in the form in (5). That is, $G(s)$ in (57) is extended semi-strongly stabilizable.

For the plant $G(s)$ in (57), we design an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis. Using the method in Section 5, we design $\hat{Q}(s)$. $\tilde{Q}_i(s)$, $\tilde{Q}_o(s)$ and $q(s)$ in (47) are factorized by (48), where

$$\tilde{Q}_i(s) = 1 \tag{62}$$

and

$$\tilde{Q}_o(s) = \tilde{Q}(s). \tag{63}$$

Using (63), $\bar{Q}(s)$ is settled by (49), where

$$q(s) = \frac{0.99}{(s+1)^3}. \tag{64}$$

$\hat{Q}(s)$ is set by (52) and written as

$$\hat{Q}(s) = \frac{(s+0.003345)(s^2+2.997s+2.99)}{(s+1)^3}. \tag{65}$$

Using the above mentioned parameters, an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis $C(s)$ is obtained as

$$C(s) = \frac{0.01011(s+z_{c1})(s+z_{c2})(s+z_{c3})(s+z_{c4})(s+z_{c5})(s+z_{c6})(s+z_{c7})(s+z_{c8})}{(s+p_{c1})(s+p_{c2})^2(s+p_{c3})^2(s+p_{c4})(s+p_{c5})(s+p_{c6})}, \tag{66}$$

where $z_{cp} \in \mathbb{C}$ $(p = 1, 2, \ldots, 8)$ and $p_{cq} \in \mathbb{C}$ $(q = 1, 2, 3, 4, 5, 6)$ are

$$\begin{cases} z_{c1} = 0.117, & z_{c2} = 0.003565, & z_{c3} = -0.06335 + 0.5050j, \\ z_{c4} = -0.06335 - 0.5050j, & z_{c5} = 0.4262 + 0.9478j, & z_{c6} = 0.4262 - 0.9478j, \\ z_{c7} = 2.217 + 1.910j, & z_{c8} = 2.217 - 1.910j, & \\ p_{c1} = 0, & p_{c2} = j, & p_{c3} = -j, \\ p_{c4} = 0.003345j, & p_{c5} = 1.499 + 0.8628j, & p_{c6} = 1.499 - 0.8628j. \end{cases}$$

$C(s)$ in (66) has unstable poles at the origin and $\pm j$. Other poles are located in the open left half plane. If $C(s)$ in (66) stabilizes the control system in (1), then $C(s)$ in (66) is an extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis.

Using $C(s)$ in (66), the response of the output $y(t)$ of the control system in (1) for the reference input $r(t) = 1$ is shown in Figure 1. Here, the dotted line shows $y(t) = 1$, and the solid line shows the response of the output $y(t)$ of the control system in (1) for the reference input $r(t) = 1$. From Figure 1, the control system in (1) is stable. In addition, the output $y(t)$ follows the step reference input $r(t) = 1$ without steady-state error. This is because the extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis has a pole at the origin.

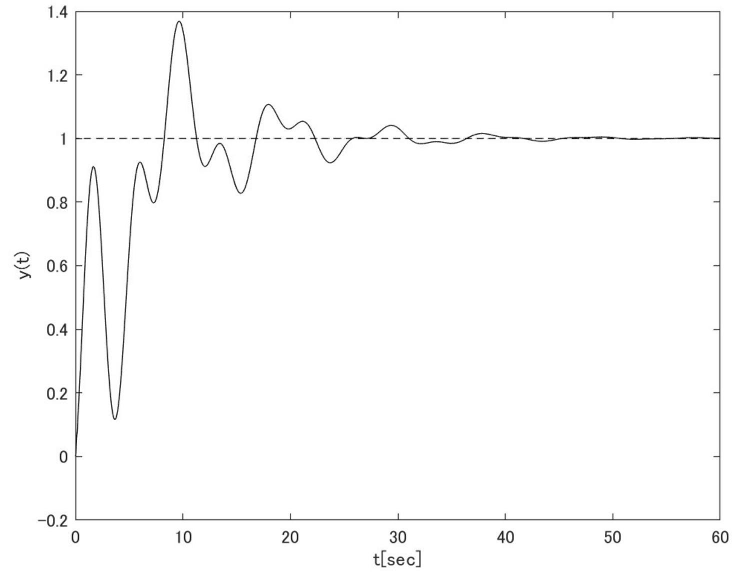

FIGURE 1. Response of the output $y(t)$ for $r(t) = 1$

The response of the output $y(t)$ of the control system in (1) for the sinusoidal disturbance $d(t) = \sin t$ is shown in Figure 2. Here, the dotted line shows $y(t) = 0$, and the solid line shows the response of the output $y(t)$ of the control system in (1) for the sinusoidal disturbance $d(t) = \sin t$. Figure 2 shows that the response of the output $y(t)$ of the control system in (1) for the sinusoidal disturbance $d(t) = \sin t$ is closed to 0. From Figure 2, the sinusoidal disturbance $d(t) = \sin t$ is effectively attenuated. This is the reason why the extended semi-strongly stabilizing controller with a pole on the origin and two pairs of poles on the imaginary axis has poles at $\pm j$ on the imaginary axis that the same poles of disturbance $d(s) = 1/(s^2 + 1)$.

The response of the output $y(t)$ of the control system in (1) for the disturbance $d(t) = t \sin t$ is shown in Figure 3. Figure 3 shows that the sinusoidal disturbance $d(t) = t \sin t$ is effectively attenuated. Because the extended semi-strongly stabilizing controller has two

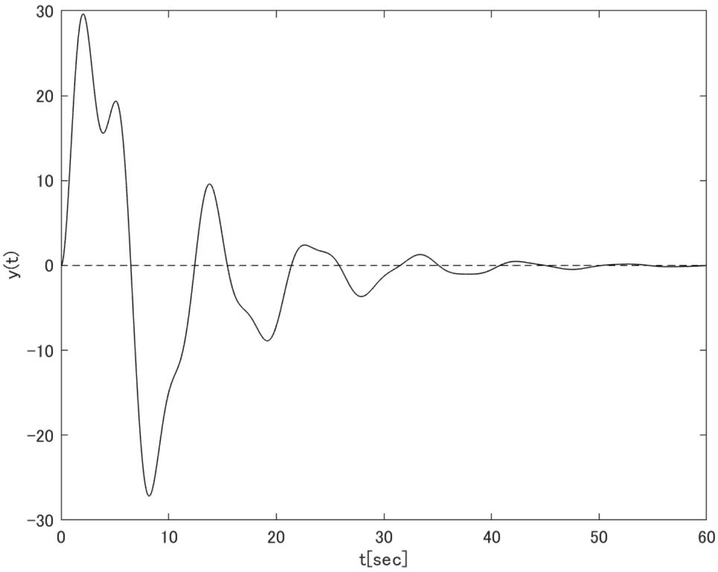

FIGURE 2. Response of the output for $d(t) = \sin t$

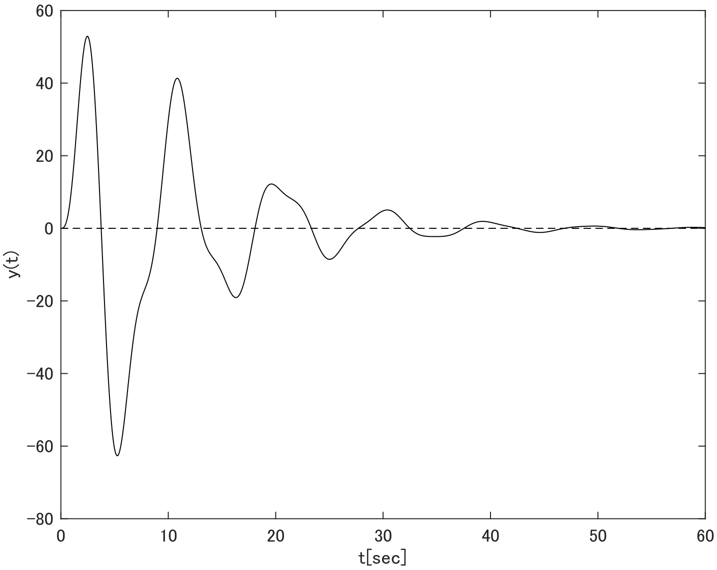

FIGURE 3. Response of the output for $d(t) = t \sin t$

pairs of poles at $\pm j$ on the imaginary axis and disturbance $d(s) = 2s/(s^2 + 1)^2$ has the same two pairs of poles at $\pm j$, the disturbance $d(t) = t \sin t$ is effectively attenuated.

7. **Conclusion.** In this paper, we clarified the proof of the parameterization of all semi-strongly stabilizable plants. In addition, we also clarified the proof of the parameterization of all semi-strongly stabilizing controllers for extended semi-strongly stabilizable plants. In addition, we present a design procedure of the extended semi-strongly stabilizing controller with a pole at the origin and two pairs of poles on the imaginary axis. The results in this paper can expand the results of [27] and apply to the control system with failure detection to eliminate the steady-state error and the effect of the auxiliary input on the output. However, we do not have the art to know whether the performance of the controller satisfies the design specification or not. That is, we do not have considered the evaluation of the

performance of the controller in this paper. To evaluate the performance of the controller, obtaining the performance indicator is important. In addition, the parameterization of all semi-strongly stabilizable plants and the parameterization of all semi-strongly stabilizing controllers for multiple-input/multiple-output systems is not considered. These works will be considered in another paper.

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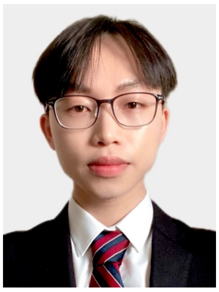

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