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ON THE RADICAL OF IDEALS IN SEMIRINGS

Jatuporn Sanborisoot[1] and Nareupanat Lekkoksung[2,*]

[1]Algebra and Applications Research Unit
Department of Mathematics
Faculty of Science
Mahasarakham University
Khamriang Sub-District, Kantarawichai District, Mahasarakham 44150, Thailand
jatuporn.san@msu.ac.th

[2]Division of Mathematics
Faculty of Engineering
Rajamangala University of Technology Isan, Khon Kaen Campus
150 Srichan Road, Muang District, Khon Kaen 40000, Thailand
[*]Corresponding author: nareupanat.le@rmuti.ac.th

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Abstract. *The radical of a subset A of a semiring S is the set of elements whose exponents are preserved in the same set A. In commutative semirings with unity, it was known that the radical of a two-sided ideal is described by the intersection of all prime ideals and the radical of such a two-sided ideal is still a two-sided ideal. This known result holds if the commutativity and the existence of unity are provided. In the present paper, we consider the behavior of the radicals of various ideals in semirings, not only the radicals of two-sided ideals. Moreover, we skip such two conditions: commutativity and the existence of unity. The interesting point we provide by an example is that the radical of an ideal may not be an ideal. Therefore, by using the properties of such ideals in semirings, we consider the conditions that preserve the ideal properties upon radical formation. Then, we conclude all conditions that the radical of ideals in semirings preserves ideals properties.*
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1. **Introduction.** The concept of semirings was first introduced when von Neumann defined regular semirings (see [31]). However, the formal definition of semirings is credited to Vandiver [30] in 1939. Semirings are a fundamental algebraic structure that consists of a nonempty set and two binary operations with specific properties. The concept of semirings can be thought of as a generalization of rings in the sense that the cancellation law does not hold for multiplication. Therefore, several questions are asked if the properties held in rings also satisfy semirings, such as the prime and weakly prime properties and the construction of quotient semirings (see [3, 7]). Semirings find applications in various scientific fields, including computer science, coding theory, and programming languages (see [8, 11, 16]).

One of the notions used to study semirings is ideals. Ideals are used to investigate algebraic structures of semirings and decompose semirings into small pieces, so-called quotient semirings (see [1, 12, 14]). This method of decomposing semirings has the advantage of studying only a small part of semirings, and then we can understand entire semirings. This illustrates the importance of ideals in semirings. Later, some scientists considered ideals with a particular property, which they called prime ideals (see [7, 8, 27]). The

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concept of prime ideals also shares closed connections with subtractive ideals and maximal ideals that can be used to investigate the structural properties of semirings (see [9]). This shows the importance of prime ideals in semirings. The radicals of numbers are a fundamental concept in number theory. This concept can be extended to the radical of a set in semirings. The radical of a subset A of a semiring S is the set of elements whose exponents are preserved in the same set A. The concept of radicals in semirings is a component when building a new semiring from an existing one using the quotient method (see [3]). Moreover, this concept is used to decompose semirings into classes, especially, k-Archimedean and t-k-Archimedean semirings (see [4, 18, 19]). Exploring the radicals study of ideals holds significance in semirings and across various algebraic systems (see [15, 28, 29]). Delving into this concept offers insights into the broader understanding of algebraic properties, enriching the study of different algebraic structures. In 2018, Nasehpour [22] demonstrated a connection between the radicals of two-sided ideals and the intersection of prime ideals in commutative semirings with unity. It turns out that the radicals of two-sided ideals are again two-sided ideals that can be represented by the intersection of prime ideals in commutative semirings with unity. The radicals concept was also used to study semirings in various directions (see [2, 10, 13, 20, 23]).

The concept of radicals is not only studied in semirings; however, it was first extended to semigroups. Ćirić and Bogdanović firstly took this concept by extending the study to the radicals of sets in semigroups. They explored the conditions under which the radicals of subsemigroups and left (resp., right, bi-) ideals form subsemigroups (see [6]). In 2020, Sanborisoot and Changphas [26] partially expanded upon the previous work by considering the case of radicals of quasi-ideals in semigroups. Notably, the results of Ćirić and Bogdanović were fully extended to (m, n)-ideals by Luangchaisri et al. in 2022 (see [17]).

The previous discussion shows that the radicals concept is important not only for semirings but also for other structures such as semigroups and ordered semigroups. Therefore, it is worth investigating their properties more deeply. This paper explores the radicals of subsemirings and left (resp., right, and bi-) ideals in semirings. Since other investigations considered the radicals in commutative semirings with unity and they considered the radical of ideals in terms of prime property, in the present paper, we reduce the commutativity and the existence of unity with semirings we considered. Furthermore, we concentrate on characterizing the radical of an ideal as an ideal in semirings, utilizing the intrinsic properties of radicals. Through illustrative examples, we demonstrate that the radical of an ideal in a semiring may not necessarily be an ideal itself. This observation differs from the case of commutative semirings with unity since we do not use the prime property to study. Our inquiry then identifies the conditions under which these ideals' radicals preserve their ideals conditions in semirings.

2. **Preliminaries.** This section provides essential foundational knowledge about semirings, their ideals, and the concept of radicals of sets in semirings. Additionally, we introduce some primary results of ideals in semirings.

Definition 2.1 ([30]). *A* semiring *is a structure $\langle S; +, \cdot, 0\rangle$ consisting of a nonempty set S together with two binary operations $+$ (addition) and $\cdot$ (multiplication), and a nullary operation 0 such that $\langle S; +, 0\rangle$ is a commutative monoid and $\langle S; \cdot\rangle$ is a semigroup with the conditions*

$$z \cdot (x + y) = z \cdot x + z \cdot y \text{ and } (x + y) \cdot z = x \cdot z + y \cdot z \tag{1}$$

for all $x, y, z \in S$.

From simplicity, we denote a semiring $\langle S; +, \cdot, 0\rangle$ by the boldface $\mathbf{S}$ of its underlying set. Moreover, the multiplication $a \cdot b$ of any elements a and b will be written by ab. The element 0 of a semiring $\mathbf{S}$ is called an *absorbing zero* if $0x = 0 = x0$ for all $x \in S$. Throughout this work, we suppose that $\mathbf{S}$ is a semiring with an absorbing zero 0.

Let $\mathbf{S}$ be a semiring and $\mathbb{N} = \{1, 2, 3, \ldots\}$ the set of all positive integers. For nonempty subsets A and B of S, we let

$$A + B = \{a + b : a \in A \text{ and } b \in B\}, \quad AB = \{ab : a \in A \text{ and } b \in B\},$$

and

$$\sum A = \left\{ \sum_{i \leq k} a_i : a_i \in A \text{ and } k \in \mathbb{N} \right\}.$$

For each $x \in S$, we write $\{x\}A$ as xA and $A\{x\}$ as Ax. We also denote the n-product of $x \in S$ by x^n, where $n \in \mathbb{N}$.

Below, we present the elementary results derived from the concepts defined above.

Lemma 2.1 ([24])**.** *Let* $\mathbf{S}$ *be a semiring, and* $A, B \subseteq S$*. Then, the following conditions hold.*

(1) $\sum \sum A = \sum A$.

(2) $A \subseteq B$ *implies* $\sum A \subseteq \sum B$.

(3) $A\big(\sum B\big) \subseteq \sum AB$ *and* $\big(\sum A\big)B \subseteq \sum AB$.

(4) $\sum(A + B) \subseteq \sum A + \sum B$.

(5) $\sum A = A$ *if and only if* $A + A \subseteq A$.

Let $\mathbf{S}$ be a semiring. A nonempty subset A of S is said to be an *additive closed subset* and a *multiplicative closed subset* of $\mathbf{S}$ if $A + A \subseteq A$ and $AA \subseteq A$, respectively.

Definition 2.2 ([21])**.** *Let* $\mathbf{S}$ *be a semiring. A subset* A *of* S *satisfying additive and multiplicative closed properties is called a* subsemiring *of* $\mathbf{S}$ *if* $0 \in A$.

Let $\mathbf{S}$ be a semiring, and A a nonempty subset of S. We denote by $\langle A\rangle$ the subsemiring of $\mathbf{S}$ generated by A. In particular, if $A = \{a_1, \ldots, a_n\}$, where $a_i \in S$ for all $1 \leq i \leq n$, then we write $\langle a_1, \ldots, a_n\rangle$ instead of $\langle\{a_1, \ldots, a_n\}\rangle$. We note that if A is a subsemiring of $\mathbf{S}$, then $\langle A\rangle = A$.

We recall particular kinds of subsemiring as follows.

Definition 2.3 ([5])**.** *Let* $\mathbf{S}$ *be a semiring and* A *an additive closed subset of* S*. Then,* A *is said to be*

(1) a left ideal *of* $\mathbf{S}$ *if* $SA \subseteq A$*;*

(2) a right ideal *of* $\mathbf{S}$ *if* $AS \subseteq A$*;*

(3) a two-sided ideal *of* $\mathbf{S}$ *if* A *is both a left and a right ideal of* $\mathbf{S}$.

Another ideal in semirings that we concentrate on in the current investigation is the concept of bi-ideals. Let $\mathbf{S}$ be a semiring. A subsemiring A of S is said to be a *bi-ideal* [21] of $\mathbf{S}$ if $ASA \subseteq A$.

We note here that any semiring in our study contains an absorbing zero; hence, any left (resp., right, two-sided) ideal also contains 0.

Palakawong na Ayutthaya and Pibaljommee [24], first applied Lemma 2.1 to demonstrating the following lemma in ordered semirings. However, it is also true for semirings.

Lemma 2.2 ([24])**.** *Let* $\mathbf{S}$ *be a semiring, and* A *a nonempty subset of* S*. Then*

(1) $\sum SA$ *is a left ideal of* $\mathbf{S}$*;*
(2) $\sum AS$ *is a right ideal of* $\mathbf{S}$*;*
(3) $\sum SAS$ *is a two-sided ideal of* $\mathbf{S}$*;*
(4) $\sum ASA$ *is a bi-ideal of* $\mathbf{S}$*.*

Proof: (1) Clearly, $\sum SA + \sum SA \subseteq \sum SA$, and

$$S\left(\sum SA\right) \subseteq \sum S(SA) = \sum (SS)A \subseteq \sum SA.$$

Hence, $\sum SA$ is a left ideal of $\mathbf{S}$.

The proofs for (2), (3) and (4) can be similarly demonstrated as in the case of (1). □

Remark 2.1. *By the conditions (1) of Definition 2.1, it is not difficult to see that*

$$\sum Sa = Sa \quad \sum aS = aS \text{ and } \sum aSa = aSa$$

for any $a \in S$*.*

By Remark 2.1 and Lemma 2.2, as a consequence, the following result is obtained immediately.

Corollary 2.1 ([24])**.** *Let* $\mathbf{S}$ *be a semiring, and* $a \in S$*. Then*
(1) Sa *is a left ideal of* $\mathbf{S}$*;*
(2) aS *is a right ideal of* $\mathbf{S}$*;*
(3) $\sum SaS$ *is a two-sided ideal of* $\mathbf{S}$*;*
(4) aSa *is a bi-ideal of* $\mathbf{S}$*.*

Denote by $\mathrm{L}(\mathbf{S}), \mathrm{R}(\mathbf{S}), \mathrm{T}(\mathbf{S}), \mathrm{B}(\mathbf{S})$ and $\mathrm{Sub}(\mathbf{S})$ the set of all left ideals, right ideals, two-sided ideals, bi-ideals, and subsemirings of a semiring $\mathbf{S}$. Figure 1 illustrates the connections of such subsemirings under the conclusion.

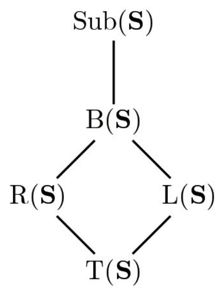

FIGURE 1. The relationships between subsemirings and ideals in semirings under the conclusion

The following concept forms the central theme of our work, and we provide a formal definition below.

Definition 2.4 ([13])**.** *Let* $\mathbf{S}$ *be a semiring, and* A *a nonempty subset of* S*. We let*

$$\sqrt{A} = \{a \in S : a^n \in A \text{ for some } n \in \mathbb{N}\}. \tag{2}$$

The set defined by above is called the radical *of* A*.*

By the above definition, it is not difficult to see that $\sqrt{A} \subseteq \sqrt{B}$ whenever $A \subseteq B$.

A semiring $\mathbf{S}$ is *commutative* if it is commute under the multiplication. It is worth noting that in some literature, the semirings under consideration are commutative semirings. Consequently, the radical of a set is defined as the intersection of all prime ideals of a semiring containing such set. Let us recall the definition of prime ideals. A two-sided ideal $A \subset S$ of a semiring $\mathbf{S}$ is said to be *prime* if $XY \subseteq A$ implies $X \subseteq A$ or $Y \subseteq A$ for any two-sided ideals X and Y of $\mathbf{S}$. This leads to the following result.

Theorem 2.1 ([22, Theorem 3.11])**.** *Let* $\mathbf{S}$ *be a commutative semiring with unity, and* I *a two-sided ideal of* $\mathbf{S}$*. Then, we obtain the following statements.*
(1) $\sqrt{I} = \bigcap\{P : I \subseteq P$ *and* P *is a prime ideal of* $\mathbf{S}\}$.
(2) $\sqrt{I}$ *is a two-sided ideal of* $\mathbf{S}$.

Based on the theorem above, we can deduce that the radical of a two-sided ideal in commutative semirings with unity remains a two-sided ideal and it can be regarded as the intersection of prime ideals with a particular setting. We can see that this result considers the preservation of two-sided ideals under radical formation by the property of prime. However, our study involves semirings where the commutative law may not apply. Moreover, we use only the concept of radicals when considering the preservation of various kinds of ideals, not only two-sided ideals. These examples illustrate the preservation of two-sided ideals under radical formation.

Example 2.1 ([25])**.** *Let* $S = \{0, 1, 2, 3, 4, 5\}$*. Define binary operations* $+$ *and* $\cdot$ *on* S *by the following tables:*

$+$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	1	1	1	1
2	2	1	1	1	1	1
3	3	1	1	1	1	1
4	4	1	1	1	1	1
5	5	1	1	1	1	1

and

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	1	1	1
2	0	1	2	1	4	1
3	0	1	5	3	3	5
4	0	1	2	4	4	2
5	0	1	5	1	3	1

By careful calculation, we have that $\mathbf{S} := \langle S; +, \cdot, 0\rangle$ *is a semiring with absorbing zero* 0*. Let* $A = \{0, 1\}$*. Then,* $\sqrt{A} = \{0, 1, 5\}$*. We can see here that* A *is a two-sided ideal of* $\mathbf{S}$*, but* $\sqrt{A}$ *is not a two-sided ideal of* $\mathbf{S}$ *since* $SA \nsubseteq A$ *and* $AS \nsubseteq A$*.*

Example 2.2 ([14])**.** *Let* $S = \{0, 1, 2\}$*. Define binary operations* $+$ *and* $\cdot$ *on* S *by the following tables:*

$+$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

By careful calculation, we have that $\mathbf{S} := \langle S; +, \cdot, 0\rangle$ *is a semiring with absorbing zero* 0*. We see that* $\{0\}$*,* $\{0, 2\}$ *and* S *are all two-sided ideals of* $\mathbf{S}$*. Moreover, we can simply calculate that their radicals are also a two-sided ideal of* $\mathbf{S}$*.*

Let $\mathbf{S}$ be a semiring. We observe that the radical concept can be regarded as a mapping from the set of all nonempty subsets $\mathcal{P}^*(S)$ of S to itself. All the types of ideals introduced here are nonempty sets. Consequently, it raises the question of whether the radical of an ideal is also an ideal. Unfortunately, the answer to this question is negative as shown by Example 2.1.

3. **Main Results.** Our primary focus in this section addresses the question raised in the preceding section: the possibility that the radical of ideals may not be an ideal in semirings. A similar scenario occurs in the case of subsemirings. The main points in this section are to 1) establish conditions under which the radical of ideals is indeed an ideal and 2) establish conditions under which the radical of a subsemiring is a subsemiring

(resp., ideals). In considering 1), since we focus on two-sided, left, right, and bi-ideals, we provide conditions under which

(1) the radical of a two-sided ideal is a two-sided (resp., left and right) ideal;
(2) the radical of a left ideal is a two-sided (resp., left and right) ideal;
(3) the radical of a right ideal is a two-sided (resp., left and right) ideal;
(4) the radical of a bi-ideal is a two-sided (resp., left and right) ideal;
(5) the radical of a two-sided (resp., left, right, and bi-) ideal is a bi-ideal.

In describing 2), we give conditions under which

(1) the radical of a subsemiring is a two-sided (resp., left and right) ideal;
(2) the radical of a subsemiring is a bi-ideal.

Building on the above considerations, we will demonstrate that the investigation of the preservation of radicals of ideals leads to the induction of a binary relation on semirings. This binary relation plays a pivotal role in our final study, where we use it to characterize the preservation of radicals of left and right ideals.

To achieve these characterizations, we introduce the essential tools necessary for addressing this question. Let $\mathbf{S}$ be a semiring. For any element $a \in S$, and A a nonempty subset of S. We define

$$A \xrightarrow{\mathrm{l}} a \Longleftrightarrow a \in \sqrt{\sum SA}, \qquad A \xrightarrow{\mathrm{r}} a \Longleftrightarrow a \in \sqrt{\sum AS},$$
$$A \xrightarrow{\mathrm{i}} a \Longleftrightarrow a \in \sqrt{\sum SAS}, \quad A \xrightarrow{\mathrm{b}} a \Longleftrightarrow a \in \sqrt{\sum ASA},$$
$$A \xrightarrow{\mathrm{s}} a \Longleftrightarrow a \in \sqrt{\langle A\rangle}.$$

By the definition of the radical of sets, we observe that if $A \xrightarrow{\mathrm{b}} a$, then $A \xrightarrow{\mathrm{l}} a$ and $A \xrightarrow{\mathrm{r}} a$. The set $\sqrt{\sum SA}$, $\sqrt{\sum AS}$, $\sqrt{\sum SAS}$, $\sqrt{\sum ASA}$ and $\sqrt{\langle A\rangle}$ is called the *radical part of A with respect to* $\xrightarrow{\mathrm{l}}$, $\xrightarrow{\mathrm{r}}$, $\xrightarrow{\mathrm{i}}$, $\xrightarrow{\mathrm{b}}$ and $\xrightarrow{\mathrm{s}}$, respectively. If $A = \{x\}$ is singleton, we write $x \xrightarrow{h} a$ instead of $\{x\} \xrightarrow{h} a$, where h is l, r, i, b or s.

The following consequences are useful for proving our main results.

Lemma 3.1. *Let $\mathbf{S}$ be a semiring, and A a nonempty subset of S. Suppose that $a, b, c \in S$, and h is one of* l, r, i, b, *or* s. *Then, the following statements hold.*

(1) If $A \xrightarrow{h} a$ and the radical part of A respect to $\xrightarrow{h}$ is a two-sided ideal of $\mathbf{S}$, then $A \xrightarrow{h} ab$ and $A \xrightarrow{h} ba$.
(2) If $A \xrightarrow{h} a$ and the radical part of A respect to $\xrightarrow{h}$ is a left (resp., right) ideal of $\mathbf{S}$, then $A \xrightarrow{h} ba$ (resp., $A \xrightarrow{h} ab$).
(3) If $A \xrightarrow{h} a$, $A \xrightarrow{h} c$ and the radical part of A respect to $\xrightarrow{h}$ is a bi-ideal of $\mathbf{S}$, then $A \xrightarrow{h} abc$.

Proof: For all the proof in this lemma we let $h =$ i. The other cases can be done similarly.

(1). Assume that $A \xrightarrow{\mathrm{i}} a$ and $\sqrt{\sum SAS}$ is a two-sided ideal of $\mathbf{S}$. By the definition of $\xrightarrow{\mathrm{i}}$, we have that $a \in \sqrt{\sum SAS}$. Then,

$$ab \in \sqrt{\sum SAS}S \subseteq \sqrt{\sum SAS} \text{ and } ba \in S\sqrt{\sum SAS} \subseteq \sqrt{\sum SAS}.$$

Thus, $A \xrightarrow{\mathrm{i}} ab$ and $A \xrightarrow{\mathrm{i}} ba$.

(2). Assume that $A \xrightarrow{\mathrm{i}} a$ and $\sqrt{\sum SAS}$ is a left ideal of $\mathbf{S}$. By the definition of $\xrightarrow{\mathrm{i}}$, we have that $a \in \sqrt{\sum SAS}$. Then,

$$ba \in S\sqrt{\sum SAS} \subseteq \sqrt{\sum SAS}.$$

Thus, $A \xrightarrow{\mathrm{i}} ba$.

(3). Assume that $A \xrightarrow{\mathrm{i}} a$, $A \xrightarrow{\mathrm{i}} c$ and $\sqrt{\sum SAS}$ is a bi-ideal of $\mathbf{S}$. By the definition of $\xrightarrow{\mathrm{i}}$, we have that $a, c \in \sqrt{\sum SAS}$. Then,

$$abc \in \sqrt{\sum SAS}S\sqrt{\sum SAS} \subseteq \sqrt{\sum SAS}.$$

Thus, $A \xrightarrow{\mathrm{i}} abc$. □

Lemma 3.2. *Let* $\mathbf{S}$ *be a semiring, and* A*,* X *nonempty subsets of* S *such that* $X \subseteq A$*. Suppose that* $a \in S$*, and* $h = \mathrm{i}$ (*resp.,* $h = \mathrm{l}$, $h = \mathrm{r}$, $h = \mathrm{b}$, $h = \mathrm{s}$)*. Then, if* $X \xrightarrow{h} a$ *and* A *is a two-sided ideal* (*resp., left ideal, right ideal, bi-ideal, subsemiring*) *of* $\mathbf{S}$*, then* $a \in \sqrt{A}$*.*

Proof: We provide only the case that $h = \mathrm{i}$. The other cases can be done similarly. Assume that $X \xrightarrow{\mathrm{i}} a$ and A is a two-sided ideal of $\mathbf{S}$. Then, $a \in \sqrt{\sum SXS}$. That is,

$$a^m \in \sum SXS \subseteq \sum SAS \subseteq \sum SA \subseteq \sum A = A$$

for some $m \in \mathbb{N}$. Hence, $a \in \sqrt{A}$. □

The following conditions are labeled for compacting our results' statements.

$$(\forall x_1, x_2 \in S, \forall k_1, k_2 \in \mathbb{N}) \quad \left\{x_1^{k_1}, x_2^{k_2}\right\} \xrightarrow{h} x_1 + x_2. \tag{ha}$$

$$(\forall x_1, x_2 \in S, \forall k_1, k_2 \in \mathbb{N}) \quad \left\{x_1^{k_1}, x_2^{k_2}\right\} \xrightarrow{h} x_1 x_2. \tag{hm}$$

The condition (ha) will be (la), (ra), (ia), (ba), and (sa) if $h = \mathrm{l}$, $h = \mathrm{r}$, $h = \mathrm{i}$, $h = \mathrm{b}$, and $h = \mathrm{s}$, respectively. Similarly, the condition (hm) will be (lm), (rm), (im), (bm), and (sm) if $h = \mathrm{l}$, $h = \mathrm{r}$, $h = \mathrm{i}$, $h = \mathrm{b}$, and $h = \mathrm{s}$, respectively.

Lemma 3.3. *Let* $\mathbf{S}$ *be a semiring. Then, we obtain the following statements.*

(1) $\sqrt{A}$ *is additive closed for every two-sided* (*resp., left, right, bi-*) *ideal* A *of* $\mathbf{S}$ *if and only if the condition* (ha) *holds, where* $h = \mathrm{i}$ (*resp.,* $h = \mathrm{l}$, $h = \mathrm{r}$, $h = \mathrm{b}$).

(2) $\sqrt{A}$ *is multiplicative closed for every two-sided* (*resp., left, right, bi-*) *ideal* A *of* $\mathbf{S}$ *if and only if the condition* (hm) *holds, where* $h = \mathrm{i}$ (*resp.,* $h = \mathrm{l}$, $h = \mathrm{r}$, $h = \mathrm{b}$).

(3) $\sqrt{A}$ *is both additive and multiplicative closed for every two-sided* (*resp., left, right, bi-*) *ideal* A *of* $\mathbf{S}$ *if and only if the conditions* (ha) *and* (hm) *hold, where* $h = \mathrm{i}$ (*resp.,* $h = \mathrm{l}$, $h = \mathrm{r}$, $h = \mathrm{b}$).

Proof: We prove (3). The proof of (1) and (2) is a part of (3).

(⇒). Let $x, y \in S$ and $k, l \in \mathbb{N}$. Then, $x^m, y^n \in \sum S\left\{x^k, y^l\right\}S$ for some $m, n \in \mathbb{N}$. That is, $x, y \in \sqrt{\sum S\{x^k, y^l\}S}$. By our assumption, $\sqrt{\sum S\{x^k, y^l\}S}$ is additive and multiplicative closed. Thus, $x + y, xy \in \sqrt{\sum S\{x^k, y^l\}S}$. That is, $\left\{x^k, y^l\right\} \xrightarrow{\mathrm{i}} x + y$ and $\left\{x^k, y^l\right\} \xrightarrow{\mathrm{i}} xy$. Therefore, (ia) and (im) hold.

(⇐). Suppose that A is a two-sided ideal of $\mathbf{S}$. Let $x, y \in \sqrt{A}$. Then, $x^m, y^n \in A$ for some $m, n \in \mathbb{N}$. By (ia), we have $\{x^m, y^n\} \xrightarrow{\mathrm{i}} x + y$. This implies, by Lemma 3.2, that $x + y \in \sqrt{A}$. By (im), we have that $\{x^m, y^n\} \xrightarrow{\mathrm{i}} xy$. By Lemma 3.2, we obtain that $xy \in \sqrt{A}$. Therefore, $\sqrt{A}$ is both additive and multiplicative closed. □

Since any of the ideals we considered in the current paper contain absorbing zero, their radicals also contain absorbing zero. This fact and the above lemma lead us to the following corollary.

Corollary 3.1. *Let* $\mathbf{S}$ *be a semiring, and* A *a two-sided (resp., left, right, bi-) ideal of* $\mathbf{S}$*. Then,* $\sqrt{A}$ *is a subsemiring of* $\mathbf{S}$ *if and only if the conditions* (ha) *and* (hm) *hold, where* $h = \mathrm{i}$ *(resp.,* $h = \mathrm{l}$*,* $h = \mathrm{r}$*,* $h = \mathrm{b}$*).*

We begin by characterizing the conditions under which the radical of two-sided ideals constitutes an ideal in semirings. Other ideals can be proved by using similar arguments. Thus, we provide only the case of two-sided ideals.

To shorten our paper, let us provide the essential conditions as follows.

$$(\forall x_1, x_2 \in S, \forall k_1 \in \mathbb{N}) \qquad \left(x_1^{k_1} \xrightarrow{h} x_1x_2\right) \wedge \left(x_1^{k_1} \xrightarrow{h} x_2x_1\right). \tag{hi}$$

$$(\forall x_1, x_2 \in S, \forall k_1 \in \mathbb{N}) \qquad x_1^{k_1} \xrightarrow{h} x_2x_1. \tag{hl}$$

$$(\forall x_1, x_2 \in S, \forall k_1 \in \mathbb{N}) \qquad x_1^{k_1} \xrightarrow{h} x_1x_2. \tag{hr}$$

$$(\forall x_1, x_2, x_3 \in S, \forall k_1, k_2 \in \mathbb{N}) \qquad \left\{x_1^{k_1}, x_3^{k_2}\right\} \xrightarrow{h} x_1x_2x_3. \tag{hb}$$

We note that the relation $\xrightarrow{h}$ depends on the use of h, where h is l, r, i, b, or s.

Theorem 3.1. *Let* $\mathbf{S}$ *be a semiring. Then, the radical of every two-sided ideal of* $\mathbf{S}$ *is a two-sided (resp., left, right) ideal of* $\mathbf{S}$ *if and only if* (ia) *and* (ii) *(resp.,* (il), (ir)) *hold.*

Proof: ($\Rightarrow$). The condition (ia) holds by Lemma 3.3. Now, let $a, b \in S$ and $k \in \mathbb{N}$. Then, $a^m \in \sum Sa^kS$ for some $m \in \mathbb{N}$. That is, $a \in \sqrt{\sum Sa^kS}$. Hence, $a^k \xrightarrow{\mathrm{i}} a$. Since the radical part of $\left\{a^k\right\}$ with respect to $\xrightarrow{\mathrm{i}}$ is a two-sided ideal of $\mathbf{S}$, by Lemma 3.1, $a^k \xrightarrow{\mathrm{i}} ab$ and $a^k \xrightarrow{\mathrm{i}} ba$.

($\Leftarrow$). Let A be a two-sided ideal of $\mathbf{S}$. By Lemma 3.3, we have that $\sqrt{A}$ is additive closed. Now, let $a \in \sqrt{A}$ and $b \in S$. Then, $a^m \in A$ for some $m \in \mathbb{N}$. The condition (ii) implies that $a^m \xrightarrow{\mathrm{i}} ab$ and $a^m \xrightarrow{\mathrm{i}} ba$. By Lemma 3.2, $ab \in \sqrt{A}$ and $ba \in \sqrt{A}$. Therefore, we have that $\sqrt{A}$ is a two-sided ideal of $\mathbf{S}$. □

Similarly, we derive analogous results regarding preserving radicals for left (resp., right, bi-) ideals.

Theorem 3.2. *Let* $\mathbf{S}$ *be a semiring. Then, the radical of every left ideal of* $\mathbf{S}$ *is a two-sided (resp., left, right) ideal of* $\mathbf{S}$ *if and only if* (la) *and* (li) *(resp.,* (ll), (lr)) *hold.*

Theorem 3.3. *Let* $\mathbf{S}$ *be a semiring. Then, the radical of every right ideal of* $\mathbf{S}$ *is a two-sided (resp., left, right) ideal of* $\mathbf{S}$ *if and only if* (ra) *and* (ri) *(resp.,* (rl), (rr)) *hold.*

Theorem 3.4. *Let* $\mathbf{S}$ *be a semiring. Then, the radical of every bi-ideal of* $\mathbf{S}$ *is a two-sided (resp., left, right) ideal of* $\mathbf{S}$ *if and only if* (ba) *and* (bi) *(resp.,* (bl), (br)) *hold.*

Now, we characterize the radical of ideals being a bi-ideal in semirings.

Theorem 3.5. *Let* $\mathbf{S}$ *be a semiring. Then, the radical of every two-sided (resp., left, right, bi-) ideal of* $\mathbf{S}$ *is a bi-ideal of* $\mathbf{S}$ *if and only if* (ia) *(resp.,* (la), (ra), (ba)), (im) *(resp.,* (lm), (rm), (bm)), *and* (ib) *(resp.,* (lb), (rb), (bb)) *hold.*

Proof: ($\Rightarrow$). The conditions (ia) and (im) hold by Lemma 3.3. Now, let $a, b, c \in S$ and $k, l \in \mathbb{N}$. Then, $a^m, c^n \in \sum S\left\{a^k, c^l\right\}S$ for some $m, n \in \mathbb{N}$. That is, $a, c \in \sqrt{\sum S\left\{a^k, c^l\right\}S}$.

Hence, $\left\{a^k, c^l\right\} \xrightarrow{\text{i}} a$ and $\left\{a^k, c^l\right\} \xrightarrow{\text{i}} c$. Since the radical part of $\left\{a^k, c^l\right\}$ with respect to $\xrightarrow{\text{i}}$ is a bi-ideal of $\mathbf{S}$, by Lemma 3.1, $\left\{a^k, c^l\right\} \xrightarrow{\text{i}} abc$.

($\Leftarrow$). Assume that (ia), (im), and (ib) hold. Let A be a two-sided ideal of $\mathbf{S}$. By Corollary 3.1, we have that $\sqrt{A}$ is a subsemiring of $\mathbf{S}$. Now, let $a, c \in \sqrt{A}$ and $b \in S$. Then, $a^m, c^n \in A$ for some $m, n \in \mathbb{N}$. The condition (ib) implies that $\{a^m, c^n\} \xrightarrow{\text{i}} abc$. Since the radical part of $\{a^m, c^n\}$ with respect to $\xrightarrow{\text{i}}$ is a two-sided of $\mathbf{S}$, by Lemma 3.2, $abc \in \sqrt{A}$. Therefore, $\sqrt{A}$ is a bi-ideal of $\mathbf{S}$. □

Lemma 3.3 and Corollary 3.1 are also valid for relation $\xrightarrow{\text{s}}$, as shown below, and its proof can be analogous, so we skip the proof.

Lemma 3.4. *Let* $\mathbf{S}$ *be a semiring, and* T *a subsemiring of* $\mathbf{S}$. *Then, we obtain the following statements.*

(1) $\sqrt{T}$ *is additive closed if and only if the condition* (sa) *holds.*
(2) $\sqrt{T}$ *is multiplicative closed if and only if the condition* (sm) *holds.*
(3) $\sqrt{T}$ *is a subsemiring of* $\mathbf{S}$ *if and only if the conditions* (sa) *and* (sm) *hold.*

Proof: The proof is similar to Lemma 3.3 by considering the subsemiring generating form. □

We characterize the radical of subsemiring being an ideal as follows. We prove only the case that the radical of subsemirings is a two-sided ideal. The case of left and right ideals can be proved similarly.

Theorem 3.6. *Let* $\mathbf{S}$ *be a semiring. Then, the radical of every subsemiring* A *of* $\mathbf{S}$ *is a two-sided* (*resp., left, right*) *ideal of* $\mathbf{S}$ *if and only if* (sa) *and* (si) (*resp.,* (sl), (sr)) *hold.*

Proof: ($\Rightarrow$). The condition (sa) holds by Lemma 3.4. Now, let $a, b \in S$ and $k \in \mathbb{N}$. Then, $a^m \in \langle a^k \rangle$ for some $m \in \mathbb{N}$. That is, $a \in \sqrt{\langle a^k \rangle}$. Hence, $a^k \xrightarrow{\text{s}} a$. Since the radical part of $\left\{a^k\right\}$ with respect to $\xrightarrow{\text{s}}$ is a two-sided ideal of $\mathbf{S}$, by Lemma 3.1, $a^k \xrightarrow{\text{s}} ab$ and $a^k \xrightarrow{\text{s}} ba$.

($\Leftarrow$). Let A be a subsemiring of $\mathbf{S}$. By Lemma 3.4, we have that $\sqrt{A}$ is additive closed. Now, let $a \in \sqrt{A}$ and $b \in S$. Then, $a^m \in A$ for some $m \in \mathbb{N}$. The condition (si) implies that $a^m \xrightarrow{\text{s}} ab$ and $a^m \xrightarrow{\text{s}} ba$. By Lemma 3.2, $ab \in \sqrt{A}$ and $ba \in \sqrt{A}$. Therefore, we have that $\sqrt{A}$ is a two-sided ideal of $\mathbf{S}$. □

Next, we delve into characterizing the conditions under which the radical of subsemirings forms a bi-ideal.

Theorem 3.7. *Let* $\mathbf{S}$ *be a semiring. Then, the radical of every subsemiring* A *of* $\mathbf{S}$ *is a bi-ideal of* $\mathbf{S}$ *if and only if* (sa), (sm), *and* (sb) *hold.*

Proof: ($\Rightarrow$). The conditions (sa) and (sm) hold by Lemma 3.4. Now, let $a, b, c \in S$ and $k, l \in \mathbb{N}$. Then, $a^m, c^n \in \langle a^k, c^l \rangle$ for some $m, n \in \mathbb{N}$. That is, $a, c \in \sqrt{\langle a^k, c^l \rangle}$. Hence, $\left\{a^k, c^l\right\} \xrightarrow{\text{s}} a$ and $\left\{a^k, c^l\right\} \xrightarrow{\text{s}} c$. Since the radical part of $\left\{a^k, c^l\right\}$ with respect to $\xrightarrow{\text{s}}$ is a bi-ideal of $\mathbf{S}$, by Lemma 3.1, $\left\{a^k, c^l\right\} \xrightarrow{\text{s}} abc$.

($\Leftarrow$). Let A be a subsemiring of $\mathbf{S}$. By Lemma 3.4, we have that $\sqrt{A}$ is a semiring of $\mathbf{S}$. Now, let $a, c \in \sqrt{A}$ and $b \in S$. Then, $a^m, c^n \in A$ for some $m, n \in \mathbb{N}$. The condition (sb) implies that $\{a^m, c^n\} \xrightarrow{\text{s}} abc$. By Lemma 3.2, $abc \in \sqrt{A}$. Therefore, we have that $\sqrt{A}$ is a bi-ideal of $\mathbf{S}$. □

The following result presents a characterization of being a left ideal or a right ideal of the radical of ideals by the implication form.

Let $\mathbf{S}$ be a semiring. We define $|_{\mathrm{l}}$ and $|_{\mathrm{r}}$ by

$$a \mid_{\mathrm{l}} b \Leftrightarrow b \in Sa \text{ and } a \mid_{\mathrm{r}} b \Leftrightarrow b \in aS,$$

for all $a, b \in S$.

We note here that semirings that we consider do not contain the unity; it may not be the case that $a \mid_h a$ for any element a of a semiring, where h is l or r. We obtain the characterization of the radical of ideals as follows.

Theorem 3.8. *Let $\mathbf{S}$ be a semiring in which the radical of any left (resp., right) ideal is additive closed. Then, the following conditions are equivalent.*

(1) The radical of any left (resp., right) ideal of $\mathbf{S}$ is a left (resp., right) ideal of $\mathbf{S}$.

(2) For any $a \in S$, $\sqrt{Sa}$ (resp., $\sqrt{aS}$) is a left (resp., right) ideal of $\mathbf{S}$.

(3) For any $a, b \in S$, $a \mid_{\mathrm{l}} b \Rightarrow a^2 \xrightarrow{\mathrm{l}} b$ (resp., $a \mid_{\mathrm{r}} b \Rightarrow a^2 \xrightarrow{\mathrm{r}} b$).

(4) For any $a, b \in S$, $a \mid_{\mathrm{l}} b \Rightarrow a^k \xrightarrow{\mathrm{l}} b$ (resp., $a \mid_{\mathrm{r}} b \Rightarrow a^l \xrightarrow{\mathrm{r}} b$ for all $k, l \in \mathbb{N}$).

Proof: (1) $\Rightarrow$ (2). Since Sa is a left ideal of $\mathbf{S}$ for any $a \in S$, by our presumption, we complete the proof.

(2) $\Rightarrow$ (3). Let $a, b \in S$. Assume that $a \mid_{\mathrm{l}} b$. Then, we have $b \in Sa$. It is not difficult to see that $a \in \sqrt{Sa^2}$. By our presumption, $\sqrt{Sa^2}$ is left ideal of $\mathbf{S}$. Thus,

$$b \in Sa \subseteq S\sqrt{Sa^2} \subseteq \sqrt{Sa^2} = \sqrt{Sa^2}.$$

This means that $a^2 \xrightarrow{\mathrm{l}} b$.

(3) $\Rightarrow$ (4). Let $a, b \in S$ and $k \in \mathbb{N}$. Assume that $a \mid_{\mathrm{l}} b$. We prove by mathematical induction. By (3), we have $a^2 \xrightarrow{\mathrm{l}} b$. That is, $b \in \sqrt{Sa^2} \subseteq \sqrt{Sa}$. Thus, $a \xrightarrow{\mathrm{l}} b$. Now, suppose that $a^k \xrightarrow{\mathrm{l}} b$, where $k > 1$. This means that $a^k \mid_{\mathrm{l}} b^m$ for some $m \in \mathbb{N}$. By our assumption, $a^{2k} \xrightarrow{\mathrm{l}} b^m$. That is, there exists $n \in \mathbb{N}$ such that

$$(b^m)^l \in Sa^{2k} \subseteq Sa^{k+1}.$$

Therefore, $a^{k+1} \xrightarrow{\mathrm{l}} b$. By mathematical induction, we complete the proof.

(4) $\Rightarrow$ (1). Let A be a left ideal of $\mathbf{S}$. Suppose that $a \in \sqrt{A}$ and $b \in S$. Then, $a^m \in A$ for some $m \in \mathbb{N}$. Since $ba \in Sa$, we have that $a \mid_{\mathrm{l}} ba$. By our presumption, we have $a^k \xrightarrow{\mathrm{l}} ba$ for any $k \in \mathbb{N}$. This implies that

$$(ba)^n \in Sa^m \subseteq \sum SA \subseteq \sum A = A$$

for some $n \in \mathbb{N}$. Thus, $ba \in \sqrt{A}$. Therefore, $\sqrt{A}$ is a left ideal of $\mathbf{S}$.

In the case of right ideals can be proved similarly. □

4. **An Application of the Investigation.** The application of a result can be ascribed through the following approach.

From Example 2.1, we see that there is no $m \in \mathbb{N}$ such that

$$(5 \cdot 4)^m \in \sum S \cdot 5^2 \cdot S.$$

This means that the semiring in Example 2.1 does not satisfy the condition (ii). By applying Theorem 3.1, there is a two-sided ideal in which its radical is not a two-sided ideal.

Now, let us focus on Example 2.2 by considering the following cases.

$$0 = (0 \cdot x)^1 \in \sum S \cdot 0^k \cdot S = \{0\} \text{ for any } x \in S \text{ and } k \in \mathbb{N}, \text{ and}$$

$$0 = (x \cdot 0)^1 \in \sum S \cdot 0^k \cdot S = \{0\} \text{ for any } x \in S \text{ and } k \in \mathbb{N}$$

$$x = (1 \cdot x)^1 \in \sum S \cdot 1^k \cdot S = S \text{ for any } x \in S \text{ and } k \in \mathbb{N}, \text{ and}$$
$$x = (x \cdot 1)^1 \in \sum S \cdot 1^k \cdot S = S \text{ for any } x \in S \text{ and } k \in \mathbb{N}$$
$$(2 \cdot x)^1 \in \sum S \cdot 2^k \cdot S = \{0, 2\} \text{ for any } x \in S \text{ and } k \in \mathbb{N}, \text{ and}$$
$$(x \cdot 2)^1 \in \sum S \cdot 2^k \cdot S = \{0, 2\} \text{ for any } x \in S \text{ and } k \in \mathbb{N}.$$

This means that the semiring in Example 2.2 satisfies (ia) and (ii) conditions. By applying Theorem 3.1, every two-sided ideal is a two-sided ideal. However, the semiring presented in Example 2.2 is a commutative semiring with unity. This implies, by Theorem 2.1, any two-sided ideal is a two-sided ideal under radical formation.

Since Theorem 2.1 confirms that the radical of any two-sided ideal is a two-sided ideal in commutative semirings with unity, we can ask if there is a noncommutative semiring without unity that also holds such property. Fortunately, the address is positive. We provide an example illustrating this question and confirm our results.

Let $S = \{0, 1, 2\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:

$+$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

By careful calculation, we have that $\mathbf{S} := \langle S; +, \cdot, 0\rangle$ is a noncommutative semiring with absorbing zero 0. Let us verify that the condition (ia) holds. It is clear for any pair $(x_1, x_2) \in S \times S \smallsetminus \{(0, 0)\}$ that

$$x_1 + x_2 \in \sqrt{\sum S \left\{x_1^{k_1}, x_2^{k_2}\right\} S} = \sqrt{S}$$

for all $k_1, k_2 \in \mathbb{N}$. For the last case, we have

$$0 = 0 + 0 \in \sqrt{\sum S \left\{0^{k_1}, 0^{k_2}\right\} S} = \sqrt{0}$$

for all $k_1, k_2 \in \mathbb{N}$. This means that $\mathbf{S}$ satisfies (ia). Now, we verify the condition (ii). For any pair $(x_1, x_2) \in S \times S \smallsetminus \{(0, 0), (0, 1), (0, 2)\}$, we have

$$x_1x_2 \in \sqrt{\sum S \left\{x_1^{k_1}\right\} S} = \sqrt{S} \text{ and } x_2x_1 \in \sqrt{\sum S \left\{x_1^{k_1}\right\} S} = \sqrt{S}$$

for all $k_1 \in \mathbb{N}$. For the pair $(x_1, x_2) \in \{(0, 0), (0, 1), (0, 2)\}$, we have

$$0 \in \sqrt{\sum S \left\{0^{k_1}\right\} S} = \sqrt{0}$$

for all $k_1 \in \mathbb{N}$. This implies that $\mathbf{S}$ satisfies the condition (ii). By applying Theorem 3.1, we have that the radical of any two-sided ideal of $\mathbf{S}$ is a two-sided ideal.

5. **Conclusion and Discussion.** In this paper, we delineate semirings in which the radical of every ideal forms an ideal and present various conditions that classify such semirings. These conditions are involved by the relation $\xrightarrow{h}$, where h is l, r, i, b or s. These conditions are also formed in ways similar to ideal and subsemiring conditions. In Theorems 3.1 to 3.4 (Theorem 3.6), we characterize semirings in which the radical of an ideal (subsemiring) is a two-sided ideal. We see that the condition that we use is the two-sided ideal form. Theorem 3.5 (Theorem 3.7) characterizes the radical of an ideal (subsemiring) being a bi-ideal. The condition that is used in describing is in the bi-ideal form. Our results demonstrate the patterns of conditions used in characterizing semirings, preserving ideal properties upon radical formation. Additionally, we offer a

characterization of semirings where the radical of one-sided ideals is itself a one-sided ideal, presented in an implication form. In the case that $\mathbf{S}$ is a commutative semiring with unity, for any $a \in S$ and nonempty subset A of S, we observe that

$$A \overset{\mathrm{l}}{\to} a \text{ if and only if } A \overset{\mathrm{r}}{\to} a.$$

Moreover, we have that

(1) if $A \overset{\mathrm{b}}{\to} a$, then $A \overset{\mathrm{i}}{\to} a$;
(2) if $A \overset{\mathrm{b}}{\to} a$, then $A \overset{\mathrm{l}}{\to} a$;
(3) if $A \overset{\mathrm{b}}{\to} a$, then $A \overset{\mathrm{r}}{\to} a$;
(4) if $A \overset{\mathrm{i}}{\to} a$, then $A \overset{\mathrm{l}}{\to} a$;
(5) if $A \overset{\mathrm{i}}{\to} a$, then $A \overset{\mathrm{r}}{\to} a$.

This is true by the fact that $\sqrt{\sum ASA} \subseteq \sqrt{\sum SA} = \sqrt{\sum AS}$. The relations discussed above can be visualized in the following diagram for noncommutative and commutative semirings.

(a) Noncommutative semirings (b) Commutative semirings

FIGURE 2. The relations of the radical parts in noncommutative and commutative semirings

We note that, even if the radicals of sets in commutative and noncommutative semirings are the same set, these radicals may impact computing the ideal's properties.

Section 4 applies our results to illustrating determining semirings that preserve ideal conditions under radical formation. For noncommutative semirings focusing on two-sided ideals, we provide all situations where the radicals of two-sided ideals are two-sided ideals and not two-sided ideals.

In future work, we will use the results to classify Archimedean semirings. For the another future study, let us consider the following example. Let $S = \{0, 1, 2, 3\}$. Define binary operations $+$ and $\cdot$ on S by the following tables:

$+$	0	1	2	3
0	0	1	2	3
1	1	0	2	3
2	2	2	2	2
3	3	3	2	3

and

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	2	2
3	0	0	3	3

By careful calculation, we have that $\mathbf{S} := \langle S; +, \cdot, 0\rangle$ is a noncommutative semiring with absorbing zero 0. We see that $\{\{0\}, \{0, 1\}, \{0, 2, 3\}, S\}$ is the set of all two-sided ideals of $\mathbf{S}$. Consider the radicals of these two-sided ideals as follows.

- $\sqrt{\{0\}} = \{0, 1\} \neq \{0\}$
- $\sqrt{\{0, 1\}} = \{0, 1\}$
- $\sqrt{\{0, 2, 3\}} = S \neq \{0, 2, 3\}$
- $\sqrt{S} = S$

We can see that the radicals of these two-sided ideals are two-sided ideals. Furthermore, $\{0\}$ and $\{0, 2, 3\}$ are the only two-sided ideals of $\mathbf{S}$ that do not equal themselves after taking radical. Therefore, we can ask for the conditions under which the radical of an ideal A is A, namely, $A = \sqrt{A}$. This problem will advance the decomposing of semirings into classes. The investigation's limitation is the example that satisfies and does not satisfy our conditions. Some of the examples provided in this paper involve computer programming. We encourage the readers to do computer programming and apply the results obtained by the theories.

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Author Biography

Jatuporn Sanborisoot is currently an Assistant Professor in Mathematics, at the Faculty of Science, Mahasarakham University, Thailand. He received his B.S., M.S., and Ph.D. degrees in Mathematics from Khon Kaen University, under the supervision of Professor Thawhat Changphas. His areas of interest include the fuzzy algebraic structures.

Nareupanat Lekkoksung obtained his B.S., M.S., and Ph.D. degrees from Khon Kaen University, under the supervision of Assistant Professor Prakit Jampachon. He is currently an Associate Professor of Rajamangala University of Technology Isan, Khon Kaen Campus, since 2020. His research interests are hypersubstitution theory and various algebraic structures theories such as ordered semigroups and ordered semihypergroups.