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SEVERAL TYPES OF $\mathcal{N}$-FUZZY UP (BCC)-SUBALGEBRAS OF UP (BCC)-ALGEBRAS

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Abstract. *This paper explores the role of $\mathcal{N}$-structures within BCC-algebras, particularly those induced by hyperfuzzy structures. We introduce and define the concept of $\mathcal{N}_k$-fuzzy BCC-subalgebras, for $k \in \{1, 2, 3, 4\}$, and analyze their fundamental properties. Additionally, we delve into the relationships between $\mathcal{N}_k$-fuzzy BCC-subalgebras and (i, j)-hyperfuzzy BCC-subalgebras, where $i, j, k \in \{1, 2, 3, 4\}$.*
Keywords: BCC-algebra, $\mathcal{N}$-function, Hyperfuzzy set, Hyperfuzzy structure, (i, j)-hyperfuzzy BCC-subalgebra, $\mathcal{N}_k$-fuzzy BCC-subalgebra

1. **Introduction.** The hyperstructures theory (also called multialgebras) was introduced in 1934 by Marty [1] at the 8th Congress of Scandinavian Mathematicians. Hyperstructures have many applications to several sectors of both pure and applied parts of mathematics. A good reference for the theory of hyperstructures and its applications to mathematics and computer science can be found in [2, 3].

The study of hyperstructures has seen significant progress, particularly in applications to algebraic systems like fuzzy logic and decision-making models. Changphas and Davvaz [4] introduced bi-hyperideals and quasi-hyperideals in ordered semihypergroups, providing a foundation for further exploration. Building on this, Pibaljommee and Davvaz [5] characterized simple and regular semihypergroups using fuzzy bi-hyperideals, while Tang et al. [6] extended this work by introducing fuzzy interior hyperideals. Kehayopulu [7] explored fuzzy hypersemigroups, adding depth to the understanding of their structure.

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Sanpan et al. [8] advanced the field with interval-valued Q-fuzzy Γ-hyperideals, emphasizing flexibility in handling uncertainty. Lekkoksung et al. [9] generalized hyperideal concepts with (m, n)-hyperideals and n-interior hyperideals, providing new insights into ideal elements. Recently, Tangtragoon et al. [10] introduced bi-interior hyperideals, further refining the structure of hypersemigroups. These contributions highlight the importance of hyperstructures in managing complexity and uncertainty across diverse fields, with ongoing research promising further advancements in algebraic theory and its applications.

Iampan [11] introduced a new algebraic structure, which is called UP-algebras, as a generalization of KU-algebras. He studied ideals and congruences in UP-algebras. Romano [12] applied the hyperstructures to UP-algebras and introduced the concept of hyper UP-algebras, which is a generalization of UP-algebras and investigated some related properties. They also introduced the concept of several types of hyper UP-ideals. In 2020, Iampan et al. [13] introduced the concept of UP-hyperalgebras and the concepts of UP-hypersubalgebras, UP-hyperideals of types 1 and 2, and s-UP-hyperideals of types 1 and 2 in UP-hyperalgebras. The notion of UP-algebras (see [11]) and the notion of BCC-algebras (see [14]) are the same, as shown by Jun et al. [15] in 2022. We will refer to it as BCC instead of UP in this article as a sign of respect for Komori, who first described it in 1984.

In this paper, we explore the concept of $\mathcal{N}$-structures in BCC-algebras derived from hyperfuzzy structures. Specifically, we introduce the notion of $\mathcal{N}_k$-subalgebras within BCC-algebras for $k \in \{1, 2, 3, 4\}$ and examine their fundamental properties. Additionally, we analyze the relationships between $\mathcal{N}_k$-subalgebras induced by hyperfuzzy sets and (i, j)-hyperfuzzy BCC-subalgebras within BCC-algebras, where $i, j, k \in \{1, 2, 3, 4\}$. Through this investigation, we aim to enhance the understanding of how hyperfuzzy structures interact with BCC-algebras, providing insights into their algebraic behaviour and potential applications.

2. Preliminaries.

The concept of BCC-algebras (see [14]) can be redefined without the condition (6) as follows.

Definition 2.1. [16] *An algebra $X = (X, *, 0)$ of type $(2, 0)$ is called a BCC-algebra if it satisfies the following conditions:*

$$(\forall x, y, z \in X)((y * z) * ((x * y) * (x * z)) = 0) \tag{1}$$

$$(\forall x \in X)(0 * x = x) \tag{2}$$

$$(\forall x \in X)(x * 0 = 0) \tag{3}$$

$$(\forall x, y \in X)(x * y = 0 = y * x \Rightarrow x = y) \tag{4}$$

Example 2.1. [17] *Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by*

$$(\forall x, y \in \mathbb{N}) \left(x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases} \right)$$

and

$$(\forall x, y \in \mathbb{N}) \left(x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise} \end{cases} \right).$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are BCC-algebras.

For more examples of BCC-algebras, see [18, 19, 20, 21, 22, 23, 24, 25].

After this, we assign X instead of a BCC-algebra $(X, *, 0)$ until otherwise specified.

We define a binary relation $\leq$ on X as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x * y = 0) \tag{5}$$

In X, the following assertions are valid (see [11]).

$$(\forall x \in X)(x \leq x) \tag{6}$$
$$(\forall x, y, z \in X)(x \leq y, y \leq z \Rightarrow x \leq z) \tag{7}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow z * x \leq z * y) \tag{8}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow y * z \leq x * z) \tag{9}$$
$$(\forall x, y, z \in X)(x \leq y * x, \text{ in particular, } y * z \leq x * (y * z)) \tag{10}$$
$$(\forall x, y \in X)(y * x \leq x \Leftrightarrow x = y * x) \tag{11}$$
$$(\forall x, y \in X)(x \leq y * y) \tag{12}$$
$$(\forall a, x, y, z \in X)(x * (y * z) \leq x * ((a * y) * (a * z))) \tag{13}$$
$$(\forall a, x, y, z \in X)(((a * x) * (a * y)) * z \leq (x * y) * z) \tag{14}$$
$$(\forall x, y, z \in X)((x * y) * z \leq y * z) \tag{15}$$
$$(\forall x, y, z \in X)(x \leq y \Rightarrow x \leq z * y) \tag{16}$$
$$(\forall x, y, z \in X)((x * y) * z \leq x * (y * z)) \tag{17}$$
$$(\forall a, x, y, z \in X)((x * y) * z \leq y * (a * z)) \tag{18}$$

Definition 2.2. [11] *A nonempty subset S of X is called a BCC-subalgebra of X if*

$$(\forall x, y \in S)(x * y \in S) \tag{19}$$

Definition 2.3. [26] *A negative fuzzy set (briefly, $\mathcal{N}$-fuzzy set) in a nonempty set X (or a negative fuzzy subset (briefly, $\mathcal{N}$-fuzzy subset) of X) is an arbitrary function from the set X into $[-1, 0]$, where $[-1, 0]$ is the unit segment of the real line.*

Definition 2.4. [27] *Let X be a nonempty set. A mapping $\widetilde{f} : X \to \mathcal{P}([0, 1])$ is called a hyperfuzzy set over X, where $\mathcal{P}([0, 1])$ is the family of all nonempty subsets of $[0, 1]$. An ordered pair $\left(X, \widetilde{f}\right)$ is called a hyperfuzzy structure over X. Given a hyperfuzzy structure $\left(X, \widetilde{f}\right)$ over a nonempty set X, we consider two fuzzy structures $\left(X, \widetilde{f}_{\inf}\right)$ and $\left(X, \widetilde{f}_{\sup}\right)$ over X in which*

$$\widetilde{f}_{\inf} : X \to [0, 1]; x \mapsto \inf \left\{\widetilde{f}(x)\right\},$$
$$\widetilde{f}_{\sup} : X \to [0, 1]; x \mapsto \sup \left\{\widetilde{f}(x)\right\}.$$

3. $\mathcal{N}_k$-Fuzzy BCC-Subalgebras Based on Hyperfuzzy Structures.

In this section, we introduce the notions of $\mathcal{N}_1$-fuzzy BCC-subalgebras, $\mathcal{N}_2$-fuzzy BCC-subalgebras, $\mathcal{N}_3$-fuzzy BCC-subalgebras, and $\mathcal{N}_4$-fuzzy BCC-subalgebras of hyperfuzzy structure $\left(X, \widetilde{f}\right)$ over a BCC-algebra $(X, *, 0)$.

Definition 3.1. *Given a hyperfuzzy structure $\left(X, \widetilde{f}\right)$ over a BCC-algebra $(X, *, 0)$, we define an $\mathcal{N}$-function on $(X, *, 0)$ as follows:*

$$\widetilde{f}_{\mathcal{N}} : X \to [-1, 0]; x \mapsto \widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x),$$

which is called an induced $\mathcal{N}$-function from $\left(X, \widetilde{f}\right)$ on $(X, *, 0)$.

Definition 3.2. *A hyperfuzzy structure $\left(X, \widetilde{f}\right)$ over a BCC-algebra $(X, *, 0)$ is called an*

(1) $\mathcal{N}_1$*-fuzzy BCC-subalgebra of* X *if* $(\forall x, y \in X)\left(\widetilde{f}_{\mathcal{N}}(x * y) \geq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}\right)$,

(2) $\mathcal{N}_2$*-fuzzy BCC-subalgebra of* X *if* $(\forall x, y \in X)\left(\widetilde{f}_{\mathcal{N}}(x * y) \leq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}\right)$,

(3) $\mathcal{N}_3$*-fuzzy BCC-subalgebra of* X *if* $(\forall x, y \in X)\left(\widetilde{f}_{\mathcal{N}}(x * y) \geq \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}\right)$,

(4) $\mathcal{N}_4$*-fuzzy BCC-subalgebra of* X *if* $(\forall x, y \in X)\left(\widetilde{f}_{\mathcal{N}}(x * y) \leq \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}\right)$.

Proposition 3.1. *For a hyperfuzzy structure* $\left(X, \widetilde{f}\right)$ *over a BCC-algebra* $(X, *, 0)$*, we have the following:*

(1) if $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_1$*-fuzzy BCC-subalgebra of* X*, then*

$$(\forall x \in X)\left(\widetilde{f}_{\mathcal{N}}(0) \geq \widetilde{f}_{\mathcal{N}}(x)\right),$$

(2) if $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_2$*-fuzzy BCC-subalgebra of* X*, then*

$$(\forall x \in X)\left(\widetilde{f}_{\mathcal{N}}(0) \leq \widetilde{f}_{\mathcal{N}}(x)\right),$$

(3) if $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_3$*-fuzzy BCC-subalgebra of* X*, then*

$$(\forall x \in X)\left(\widetilde{f}_{\mathcal{N}}(0) \geq \widetilde{f}_{\mathcal{N}}(x)\right),$$

(4) if $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_4$*-fuzzy BCC-subalgebra of* X*, then*

$$(\forall x \in X)\left(\widetilde{f}_{\mathcal{N}}(0) \leq \widetilde{f}_{\mathcal{N}}(x)\right).$$

Proof: (1) If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X, then for all $x \in X$, $\widetilde{f}_{\mathcal{N}}(0) = \widetilde{f}_{\mathcal{N}}(x * x) \geq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(x)\right\} = \widetilde{f}_{\mathcal{N}}(x)$.

(2) If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X, then for all $x \in X$, $\widetilde{f}_{\mathcal{N}}(0) = \widetilde{f}_{\mathcal{N}}(x * x) \leq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(x)\right\} = \widetilde{f}_{\mathcal{N}}(x)$.

(3) If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X, then for all $x \in X$, $\widetilde{f}_{\mathcal{N}}(0) = \widetilde{f}_{\mathcal{N}}(x * x) \geq \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(x)\right\} = \widetilde{f}_{\mathcal{N}}(x)$.

(4) If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X, then for all $x \in X$, $\widetilde{f}_{\mathcal{N}}(0) = \widetilde{f}_{\mathcal{N}}(x * x) \leq \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(x)\right\} = \widetilde{f}_{\mathcal{N}}(x)$. □

Theorem 3.1. *For a hyperfuzzy structure* $\left(X, \widetilde{f}\right)$ *over a BCC-algebra* $(X, *, 0)$*, we have the following:*

(1) every $\mathcal{N}_3$*-fuzzy BCC-subalgebra of* X *is an* $\mathcal{N}_1$*-fuzzy BCC-subalgebra,*

(2) every $\mathcal{N}_2$*-fuzzy BCC-subalgebra of* X *is an* $\mathcal{N}_4$*-fuzzy BCC-subalgebra.*

Proof: (1) Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X. Let $x, y \in X$. Then $\widetilde{f}_{\mathcal{N}}(x * y) \geq \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \geq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Hence, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X.

(2) Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X. Let $x, y \in X$. Then $\widetilde{f}_{\mathcal{N}}(x * y) \leq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \leq \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Hence, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X. □

The following example shows that the converse of Theorem 3.1 is not true in general.

Example 3.1. *Consider a BCC-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given as follows:*

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	0
2	0	2	0	3	0
3	0	2	2	0	0
4	0	2	2	3	0

(1) Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over $(X, *, 0)$ in which $\widetilde{f}$ is given as follows:

$$\widetilde{f} : X \to \mathcal{P}([0,1]); x \mapsto \begin{cases} [0.2, 0.7] & \text{if } x \in \{0, 1, 2\} \\ (0.1, 0.9] & \text{if } x \in \{3, 4\}. \end{cases}$$

Then the induced $\mathcal{N}$-function from $\left(X, \widetilde{f}\right)$ is given as follows:

$$\widetilde{f}_{\mathcal{N}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -0.5 & -0.5 & -0.5 & -0.8 & -0.8 \end{pmatrix}.$$

Then $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X. We see that $\widetilde{f}_{\mathcal{N}}(0 * 3) = -0.8 \not\geq -0.5 = \max\{-0.5, -0.8\} = \max\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(3)\right\}$. Thus, $\left(X, \widetilde{f}\right)$ is not an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X.

(2) Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over $(X, *, 0)$ in which $\widetilde{f}$ is given as follows:

$$\widetilde{f} : X \to \mathcal{P}([0,1]); x \mapsto \begin{cases} (0.1, 0.8] & \text{if } x = 0 \\ (0.1, 0.3) & \text{if } x = 1 \\ [0.3, 0.5] & \text{if } x = 2 \\ (0.2, 0.4] & \text{if } x = 3 \\ (0.1, 0.9] & \text{if } x = 4. \end{cases}$$

Then the induced $\mathcal{N}$-function from $\left(X, \widetilde{f}\right)$ is given as follows:

$$\widetilde{f}_{\mathcal{N}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -0.7 & -0.2 & -0.2 & -0.2 & -0.8 \end{pmatrix}.$$

Then $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X. We see that $\widetilde{f}_{\mathcal{N}}(0 * 2) = -0.2 \not\leq -0.7 = \min\{-0.7, -0.2\} = \min\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(2)\right\}$. Thus, $\left(X, \widetilde{f}\right)$ is not an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X.

Theorem 3.2. *A hyperfuzzy structure $\left(X, \widetilde{f}\right)$ over a BCC-algebra $(X, *, 0)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X if and only if it is constant.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X. Then $\widetilde{f}_{\mathcal{N}}(0) \leq \widetilde{f}_{\mathcal{N}}(x)$ for all $x \in X$. By (2), we have $\widetilde{f}_{\mathcal{N}}(x) = \widetilde{f}_{\mathcal{N}}(0 * x) \leq \min\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(x)\right\} = \widetilde{f}_{\mathcal{N}}(0)$ for all $x \in X$. Thus, $\widetilde{f}_{\mathcal{N}}(x) = \widetilde{f}_{\mathcal{N}}(0)$ for all $x \in X$, so $\widetilde{f}$ is constant. Hence, $\left(X, \widetilde{f}\right)$ is constant.

Conversely, assume that $\left(X, \widetilde{f}\right)$ is constant. Then $\widetilde{f}_{\mathcal{N}}(x) = \widetilde{f}_{\mathcal{N}}(0)$ for all $x \in X$. Let $x, y \in X$. Then $\widetilde{f}_{\mathcal{N}}(x*y) = \widetilde{f}_{\mathcal{N}}(0) \leq \widetilde{f}_{\mathcal{N}}(0) = \min\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(0)\right\} = \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X. □

Theorem 3.3. *A fuzzy structure $\left(X, \widetilde{f}\right)$ over a BCC-algebra $(X, *, 0)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X if and only if it is constant.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X. Then $\widetilde{f}_{\mathcal{N}}(0) \geq \widetilde{f}_{\mathcal{N}}(x)$ for all $x \in X$. By (2), we have $\widetilde{f}_{\mathcal{N}}(x) = \widetilde{f}_{\mathcal{N}}(0 * x) \geq \max\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(x)\right\} = \widetilde{f}_{\mathcal{N}}(0)$. Thus, $\widetilde{f}_{\mathcal{N}}(x) = \widetilde{f}_{\mathcal{N}}(0)$ for all $x \in X$, so $\widetilde{f}$ is constant. Hence, $\left(X, \widetilde{f}\right)$ is constant.

Conversely, assume that $\left(X, \widetilde{f}\right)$ is constant. Then $\widetilde{f}_{\mathcal{N}}(0) = \widetilde{f}_{\mathcal{N}}(x)$ for all $x \in X$. Let x, $y \in X$. Then $\widetilde{f}_{\mathcal{N}}(x * y) = \widetilde{f}_{\mathcal{N}}(0) \geq \widetilde{f}_{\mathcal{N}}(0) = \max\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(0)\right\} = \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X. □

By Theorems 3.2 and 3.3, we obtain that $\mathcal{N}_2$-fuzzy BCC-subalgebras, $\mathcal{N}_3$-fuzzy BCC-subalgebras, and constant fuzzy structures coincide. The relationships among the $\mathcal{N}_k$-subalgebras can be summarized in the following diagram.

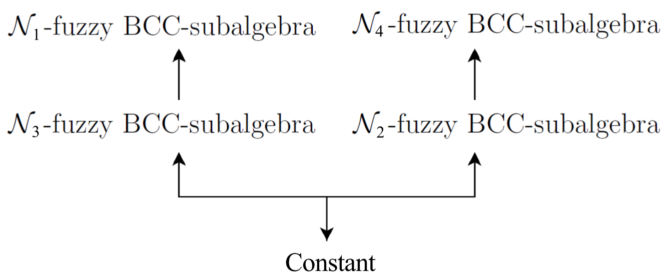

FIGURE 1. $\mathcal{N}_k$-fuzzy BCC-subalgebras

Definition 3.3. *For any $i, j \in \{1, 2, 3, 4\}$, a hyperfuzzy structure $\left(X, \widetilde{f}\right)$ over a BCC-algebra $(X, *, 0)$ is called an (i, j)-hyperfuzzy BCC-subalgebra of X if a fuzzy structure $\left(X, \widetilde{f}_{\inf}\right)$ is an i-fuzzy BCC-subalgebra of X and a fuzzy structure $\left(X, \widetilde{f}_{\sup}\right)$ is a j-fuzzy BCC-subalgebra of X.*

Theorem 3.4. *Given a BCC-subalgebra S of a BCC-algebra $(X, *, 0)$ and $B_1, B_2 \in \mathcal{P}([0, 1])$, let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over X given by*

$$\widetilde{f} : X \to \mathcal{P}([0, 1]); x \mapsto \begin{cases} B_2 & \text{if } x \in S \\ B_1 & \text{otherwise.} \end{cases}$$

If $B_1 \subseteq B_2$, then $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X. Also, if $B_2 \subseteq B_1$, then $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X.

Proof: If $x \in S$, then $\widetilde{f}(x) = B_2$ and so $\widetilde{f}_{\mathcal{N}}(x) = \widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x) = \inf\left\{\widetilde{f}(x)\right\} - \sup\left\{\widetilde{f}(x)\right\} = \inf B_2 - \sup B_2$.

If $x \notin S$, then $\widetilde{f}(x) = B_1$ and so $\widetilde{f}_{\mathcal{N}}(x) = \widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x) = \inf\left\{\widetilde{f}(x)\right\} - \sup\left\{\widetilde{f}(x)\right\} = \inf B_1 - \sup B_1$.

Assume that $B_1 \subseteq B_2$. Then $\inf B_2 - \sup B_2 \leq \inf B_1 - \sup B_1$.

Case 1: Let $x, y \in S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_2 - \sup B_2$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_2 - \sup B_2$. Thus, $\max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_2 - \sup B_2$. Since S is a BCC-subalgebra of X, we have $x * y \in S$ and so $\widetilde{f}_{\mathcal{N}}(x * y) = \inf B_2 - \sup B_2$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) = \inf B_2 - \sup B_2 \leq \inf B_2 - \sup B_2 = \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Case 2: Let $x, y \notin S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_1 - \sup B_1$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_1 - \sup B_1$, so $\max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_1 - \sup B_1$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) \leq \inf B_1 - \sup B_1 = \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Case 3: Let $x \notin S$ and $y \in S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_1 - \sup B_1$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_2 - \sup B_2$, so $\max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_1 - \sup B_1$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) \leq \inf B_1 - \sup B_1 = \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Case 4: Let $x \in S$ and $y \notin S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_2 - \sup B_2$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_1 - \sup B_1$, so $\max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_1 - \sup B_1$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) \leq \inf B_1 - \sup B_1 = \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Hence, $\widetilde{f}_{\mathcal{N}}$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X.

Assume that $B_2 \subseteq B_1$. Then $\sup B_2 - \inf B_2 \leq \sup B_1 - \inf B_1$.

Case 1: Let $x, y \in S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_2 - \sup B_2$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_2 - \sup B_2$. Thus, $\min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_2 - \sup B_2$. Since S is a BCC-subalgebra of X, we have $x * y \in S$ and so $\widetilde{f}_{\mathcal{N}}(x * y) = \inf B_2 - \sup B_2$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) = \inf B_2 - \sup B_2 \geq \inf B_2 - \sup B_2 = \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Case 2: Let $x, y \notin S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_1 - \sup B_1$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_1 - \sup B_1$, so $\min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_1 - \sup B_1$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) \geq \inf B_1 - \sup B_1 = \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Case 3: Let $x \notin S$ and $y \in S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_1 - \sup B_1$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_2 - \sup B_2$, so $\min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_1 - \sup B_1$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) \geq \inf B_1 - \sup B_1 = \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Case 4: Let $x \in S$ and $y \notin S$. Then $\widetilde{f}_{\mathcal{N}}(x) = \inf B_2 - \sup B_2$ and $\widetilde{f}_{\mathcal{N}}(y) = \inf B_1 - \sup B_1$, so $\min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} = \inf B_1 - \sup B_1$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) \geq \inf B_1 - \sup B_1 = \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$.

Hence, $\widetilde{f}_{\mathcal{N}}$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X. □

Remark 3.1. *The hyperfuzzy structure* $\left(X, \widetilde{f}\right)$ *in Theorem 3.4 is neither* $\mathcal{N}_2$*-fuzzy BCC-subalgebra nor* $\mathcal{N}_3$*-fuzzy BCC-subalgebra of* $(X, *, 0)$*. Consider a BCC-algebra* $X = \{0, 1, 2,$

3, 4} in Example 3.1, and take a BCC-subalgebra $S = \{0, 1, 2\}$ *of* X*. Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over* X *given by*

$$\widetilde{f} : X \to \mathcal{P}([0,1]); x \mapsto \begin{cases} (0.2, 0.7) & \text{if } x \in S \\ [0.3, 0.7] & \text{otherwise.} \end{cases}$$

Then $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_4$*-fuzzy BCC-subalgebra of* X *by Theorem 3.4. However, it is not an* $\mathcal{N}_2$*-fuzzy BCC-subalgebra of* X *since* $f_{\mathcal{N}}(3 * 1) = f_{\mathcal{N}}(2) = -0.5 > -0.4 = \min\{f_{\mathcal{N}}(3), f_{\mathcal{N}}(1)\}$*. Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over* X *given by*

$$\widetilde{f} : X \to \mathcal{P}([0,1]); x \mapsto \begin{cases} (0.4, 0.6) & \text{if } x \in S \\ [0.2, 0.9] & \text{otherwise.} \end{cases}$$

Then $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_1$*-fuzzy BCC-subalgebra of* X *by Theorem 3.4. However, it is not an* $\mathcal{N}_3$*-fuzzy BCC-subalgebra of* $(X, *, 0)$ *since* $f_{\mathcal{N}}(2 * 3) = f_{\mathcal{N}}(3) = -0.7 < -0.2 = \max\{f_{\mathcal{N}}(2), f_{\mathcal{N}}(3)\}$*.*

Definition 3.4. *Given a hyperfuzzy structure* $\left(X, \widetilde{f}\right)$ *over* $(X, *, 0)$ *and* $t \in [-1, 0]$*, consider the following sets:*

$$U_{\mathcal{N}}\left(\widetilde{f}, t\right) = \left\{x \in X \,\middle|\, \widetilde{f}_{\mathcal{N}}(x) \geq t\right\},$$
$$L_{\mathcal{N}}\left(\widetilde{f}, t\right) = \left\{x \in X \,\middle|\, \widetilde{f}_{\mathcal{N}}(x) \leq t\right\}.$$

Theorem 3.5. *A hyperfuzzy structure* $\left(X, \widetilde{f}\right)$ *over a BCC-algebra* $(X, *, 0)$ *is an* $\mathcal{N}_1$*-fuzzy BCC-subalgebra of* X *if and only if the set* $U_{\mathcal{N}}\left(\widetilde{f}, t\right)$ *is a BCC-subalgebra of* X *for all* $t \in [-1, 0]$ *with* $U_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$*.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X. Let $t \in [-1, 0]$ be such that $U_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$ and let $x, y \in U_{\mathcal{N}}\left(\widetilde{f}, t\right)$. Then $\widetilde{f}_{\mathcal{N}}(x) \geq t$ and $\widetilde{f}_{\mathcal{N}}(y) \geq t$. Since $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X, we have $\widetilde{f}_{\mathcal{N}}(x*y) \geq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \geq t$. Thus, $x * y \in U_{\mathcal{N}}\left(\widetilde{f}, t\right)$. Hence, $U_{\mathcal{N}}\left(\widetilde{f}, t\right)$ is a BCC-subalgebra of X.

Conversely, assume that for all $t \in [-1, 0]$, the set $U_{\mathcal{N}}\left(\widetilde{f}, t\right)$ is a BCC-subalgebra of X if $U_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$. Let $x, y \in X$. Then $\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y) \in [-1, 0]$. Choose $t = \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Thus, $\widetilde{f}_{\mathcal{N}}(x) \geq t$ and $\widetilde{f}_{\mathcal{N}}(y) \geq t$ and so $x, y \in U_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$. By the assumption, we have $U_{\mathcal{N}}\left(\widetilde{f}, t\right)$ is a BCC-subalgebra of X and so $x * y \in U_{\mathcal{N}}\left(\widetilde{f}, t\right)$. Thus, $\widetilde{f}_{\mathcal{N}}(x * y) \geq t = \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Hence, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X. □

Corollary 3.1. *If* $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_3$*-fuzzy BCC-subalgebra of a BCC-algebra* $(X, *, 0)$*, then the set* $U_{\mathcal{N}}\left(\widetilde{f}, t\right)$ *is a BCC-subalgebra of* X *for all* $t \in [-1, 0]$ *with* $U_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$*.*

Proof: It is straightforward by Theorems 3.1 (1) and 3.5. □

The following example shows that the converse of Corollary 3.1 is not true in general.

Example 3.2. *Let $X = \{0, 1, 2, 3, 4\}$ be a BCC-algebra with the binary operation $*$ which is given in the following table:*

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	0	0
2	0	0	0	0	0
3	0	1	2	0	4
4	0	1	2	3	0

Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over $(X, *, 0)$ in which $\widetilde{f}$ is given as follows:

$$\widetilde{f} : X \to \mathcal{P}([0, 1]); x \mapsto \begin{cases} [0.2, 0.4] & \text{if } x = 0 \\ (0.4, 0.6] & \text{if } x = 1 \\ [0.1, 0.7] & \text{if } x = 2 \\ [0.2, 0.9) & \text{if } x = 3 \\ (0.1, 0.8] & \text{if } x = 4. \end{cases}$$

Then the induced $\mathcal{N}$-function from $\left(X, \widetilde{f}\right)$ is given as follows:

$$\widetilde{f}_{\mathcal{N}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -0.2 & -0.2 & -0.6 & -0.7 & -0.7 \end{pmatrix}.$$

Thus,

$$U_{\mathcal{N}}\left(\widetilde{f}, t\right) = \begin{cases} \emptyset & \text{if } t \in (-0.1, 0] \\ \{0\} & \text{if } t \in (-0.2, -0.1] \\ \{0, 2\} & \text{if } t \in (-0.6, -0.2] \\ \{0, 1, 2\} & \text{if } t \in (-0.7, -0.6] \\ X & \text{if } t \in [-1, -0.7]. \end{cases}$$

Then $U_{\mathcal{N}}\left(\widetilde{f}, t\right)$ is a BCC-subalgebra of X for all $t \in [-1, 0]$ with $U_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$. However, $\left(X, \widetilde{f}\right)$ is not an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X, that is, $\left(X, \widetilde{f}\right)$ is not an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X, since $\widetilde{f}_{\mathcal{N}}(0 * 4) = -0.7 \ngeq -0.2 = \max\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(4)\right\}$.

Theorem 3.6. *A hyperfuzzy structure $\left(X, \widetilde{f}\right)$ over a BCC-algebra $(X, *, 0)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X if and only if the set $L_{\mathcal{N}}\left(\widetilde{f}, t\right)$ is a BCC-subalgebra of X for all $t \in [-1, 0]$ with $L_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X. Let $t \in [-1, 0]$ be such that $L_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$ and let $x, y \in L_{\mathcal{N}}\left(\widetilde{f}, t\right)$. Then $\widetilde{f}_{\mathcal{N}}(x) \leq t$ and $\widetilde{f}_{\mathcal{N}}(y) \leq t$. Since $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X, we have $\widetilde{f}_{\mathcal{N}}(x*y) \leq \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \leq t$. Thus, $x * y \in L_{\mathcal{N}}\left(\widetilde{f}, t\right)$. Hence, $L_{\mathcal{N}}\left(\widetilde{f}, t\right)$ is a BCC-subalgebra of X.

Conversely, assume that for all $t \in [-1, 0]$, the set $L_{\mathcal{N}}\left(\widetilde{f}, t\right)$ is a BCC-subalgebra of X if $L_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$. Let $x, y \in X$. Then $\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y) \in [-1, 0]$. Choose $t = \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Thus, $\widetilde{f}_{\mathcal{N}}(x) \leq t$ and $\widetilde{f}_{\mathcal{N}}(y) \leq t$ and so $x, y \in L_{\mathcal{N}}\left(\widetilde{f}, t\right) \neq \emptyset$. By the assumption, we have

$L_{\mathcal{N}}\left(\widetilde{f},t\right)$ is a BCC-subalgebra of X and so $x*y \in L_{\mathcal{N}}\left(\widetilde{f},t\right)$. Thus, $\widetilde{f}_{\mathcal{N}}(x*y) \leq t = \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$. Hence, $\left(X,\widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X. □

Corollary 3.2. *If $\left(X,\widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of a BCC-algebra $(X,*,0)$, then the set $L_{\mathcal{N}}\left(\widetilde{f},t\right)$ is a BCC-subalgebra of X for all $t \in [-1,0]$ with $L_{\mathcal{N}}\left(\widetilde{f},t\right) \neq \emptyset$.*

Proof: It is straightforward by Theorems 3.1 (2) and 3.6. □

The following example shows that the converse of Corollary 3.2 is not true in general.

Example 3.3. *Consider a BCC-algebra $X = \{0,1,2,3,4\}$ in Example 3.1. Let $\left(X,\widetilde{f}\right)$ be a hyperfuzzy structure over X in which $\widetilde{f}$ is given as follows:*

$$\widetilde{f}: X \to \mathcal{P}([0,1]), x \mapsto \begin{cases} [0.1,0.7] & \text{if } x=0 \\ (0.2,0.5] & \text{if } x=1 \\ [0.1,0.5] & \text{if } x=2 \\ [0.1,0.3) & \text{if } x=3 \\ (0.1,0.3] & \text{if } x=4. \end{cases}$$

Then the induced $\mathcal{N}$-function from $\left(X,\widetilde{f}\right)$ is given as follows:

$$\widetilde{f}_{\mathcal{N}} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ -0.6 & -0.3 & -0.4 & -0.2 & -0.2 \end{pmatrix}.$$

Thus,

$$L_{\mathcal{N}}\left(\widetilde{f},t\right) = \begin{cases} \emptyset & \text{if } t \in [-1,-0.6) \\ \{0\} & \text{if } t \in [-0.6,-0.4) \\ \{0,2\} & \text{if } t \in [-0.4,-0.3) \\ \{0,1,2\} & \text{if } t \in [-0.3,-0.2) \\ X & \text{if } t \in [-0.2,0]. \end{cases}$$

Then $L_{\mathcal{N}}\left(\widetilde{f},t\right)$ is a BCC-subalgebra of X for all $t \in [-1,0]$ with $L_{\mathcal{N}}\left(\widetilde{f},t\right) \neq \emptyset$. However, $\left(X,\widetilde{f}\right)$ is not an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X, that is, $\left(X,\widetilde{f}\right)$ is not an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X, since $\widetilde{f}_{\mathcal{N}}(0*3) = -0.2 \not\leq -0.6 = \min\left\{\widetilde{f}_{\mathcal{N}}(0), \widetilde{f}_{\mathcal{N}}(3)\right\}$.

Theorem 3.7. *Let $\left(X,\widetilde{f}\right)$ be a hyperfuzzy structure over a BCC-algebra $(X,*,0)$ in which $\left(X,\widetilde{f}_{\inf}\right)$ satisfies the following condition:*

$$(\forall x,y \in X)(f_{\inf}(x*y) \leq \min\{f_{\inf}(x), f_{\inf}(y)\}).$$

If $\left(X,\widetilde{f}\right)$ is a $(k,1)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1,2,3,4\}$, then it is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X.

Proof: Assume that $\left(X,\widetilde{f}\right)$ is a $(k,1)$-hyperfuzzy BCC-subalgebra of $(X,*,0)$ for $k \in \{1,2,3,4\}$ in which $\left(X,\widetilde{f}_{\inf}\right)$ satisfies the condition

$$(\forall x,y \in X)(f_{\inf}(x*y) \leq \min\{f_{\inf}(x), f_{\inf}(y)\}).$$

Then $\widetilde{f}_{\inf}(x*y) \le \widetilde{f}_{\inf}(x)$ and $\widetilde{f}_{\inf}(x*y) \le \widetilde{f}_{\inf}(y)$ for all $x, y \in X$. Since $\left(X, \widetilde{f}_{\inf}\right)$ is a 1-fuzzy BCC-subalgebra of X, we have

$$\begin{aligned}\widetilde{f}_{\mathcal{N}}(x*y) &= \widetilde{f}_{\inf}(x*y) - \widetilde{f}_{\sup}(x*y)\\ &\le \widetilde{f}_{\inf}(x*y) - \min\left\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\right\}\\ &= \max\left\{\widetilde{f}_{\inf}(x*y) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(x*y) - \widetilde{f}_{\sup}(y)\right\}\\ &\le \max\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(y)\right\}\\ &= \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}\end{aligned}$$

for all $x, y \in X$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X. □

Corollary 3.3. *Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over a BCC-algebra $(X, *, 0)$ in which $\left(X, \widetilde{f}_{\inf}\right)$ satisfies the following condition:*

$$(\forall x, y \in X)(f_{\inf}(x*y) \le \min\{f_{\inf}(x), f_{\inf}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ is a $(k, 3)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X.

In general, any $\mathcal{N}_4$-fuzzy BCC-subalgebra may not be a $(k, 1)$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ as seen in the following example.

Example 3.4. *In Example 3.1, the $\mathcal{N}_4$-fuzzy BCC-subalgebra $\left(X, \widetilde{f}\right)$ of X is not a $(k, 1)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$ since*

$$\begin{aligned}&\widetilde{f}_{\inf}(2*2) = \widetilde{f}_{\inf}(0) = 0.1 < 0.3 = \min\left\{\widetilde{f}_{\inf}(2), \widetilde{f}_{\inf}(2)\right\},\\ &\widetilde{f}_{\inf}(3*1) = \widetilde{f}_{\inf}(2) = 0.3 > 0.1 = \min\left\{\widetilde{f}_{\inf}(3), \widetilde{f}_{\inf}(1)\right\},\\ &\widetilde{f}_{\inf}(2*2) = \widetilde{f}_{\inf}(0) = 0.1 < 0.3 = \max\left\{\widetilde{f}_{\inf}(2), \widetilde{f}_{\inf}(2)\right\},\\ &\widetilde{f}_{\inf}(3*1) = \widetilde{f}_{\inf}(2) = 0.3 > 0.2 = \max\left\{\widetilde{f}_{\inf}(3), \widetilde{f}_{\inf}(1)\right\}.\end{aligned}$$

We consider a condition for an $\mathcal{N}_4$-fuzzy BCC-subalgebra to be a $(k, 1)$-hyperfuzzy BCC-subalgebra for $k \in \{1, 2, 3, 4\}$.

Theorem 3.8. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\inf}$ is constant on X, then it is a $(k, 1)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ in which $\widetilde{f}_{\inf}$ is constant. It is clear that $\left(X, \widetilde{f}_{\inf}\right)$ is a k-fuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. Let $\widetilde{f}_{\inf}(x) = t$ for all $x \in X$. Then

$$\begin{aligned}\widetilde{f}_{\mathcal{N}}(x*y) &= \widetilde{f}_{\inf}(x*y) - \widetilde{f}_{\sup}(x*y)\\ &= t - \widetilde{f}_{\sup}(x*y)\end{aligned}$$

$$\begin{aligned}
&\geq t - \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \\
&= \min\left\{t - \widetilde{f}_{\mathcal{N}}(x), t - \widetilde{f}_{\mathcal{N}}(y)\right\} \\
&= \min\left\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\right\}
\end{aligned}$$

for all $x, y \in X$. Thus, $\left(X, \widetilde{f}_{\inf}\right)$ is a 1-fuzzy BCC-subalgebra of X. Therefore, $\left(X, \widetilde{f}\right)$ is a $(k, 1)$-hyperfuzzy BCC-subalgebra of X. □

Corollary 3.4. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\inf}$ is constant on X, then it is a $(k, 1)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Theorem 3.9. *Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over a BCC-algebra $(X, *, 0)$ in which $\left(X, \widetilde{f}_{\sup}\right)$ satisfies the following condition:*

$$(\forall x, y \in X)(f_{\sup}(x * y) \geq \max\{f_{\sup}(x), f_{\sup}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ is a $(4, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X.

Proof: Assume that $\left(X, \widetilde{f}\right)$ is a $(4, k)$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $\left(X, \widetilde{f}_{\sup}\right)$ satisfies the condition

$$(\forall x, y \in X)(f_{\inf}(x * y) \geq \max\{f_{\inf}(x), f_{\inf}(y)\}).$$

Then $\widetilde{f}_{\sup}(x * y) \geq \widetilde{f}_{\sup}(x)$ and $\widetilde{f}_{\sup}(x * y) \geq \widetilde{f}_{\sup}(y)$ for all $x, y \in X$. Since $\left(X, \widetilde{f}_{\sup}\right)$ is a 4-fuzzy BCC-subalgebra of X, we have

$$\begin{aligned}
\widetilde{f}_{\mathcal{N}}(x * y) &= \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x * y) \\
&\leq \widetilde{f}_{\inf}(x * y) - \min\left\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\right\} \\
&= \max\left\{\widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(y)\right\} \\
&\leq \max\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(y)\right\} \\
&= \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}
\end{aligned}$$

for all $x, y \in X$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X. □

Corollary 3.5. *Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over a BCC-algebra $(X, *, 0)$ in which $\left(X, \widetilde{f}_{\sup}\right)$ satisfies the following condition.*

$$(\forall x, y \in X)\left(f_{\sup}(x * y) \geq \max\left\{f_{\sup}(x), f_{\sup}(y)\right\}\right).$$

If $\left(X, \widetilde{f}\right)$ is a $(2, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of X.

Theorem 3.10. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant on X, then it is a $(4, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_4$-fuzzy BCC-subalgebra of $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant. It is clear that $\left(X, \widetilde{f}_{\sup}\right)$ is a k-fuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. Let $\widetilde{f}_{\sup}(x) = t$ for all $x \in X$. Then

$$\begin{aligned}
\widetilde{f}_{\inf}(x * y) &= \widetilde{f}_{\mathcal{N}}(x * y) + \widetilde{f}_{\sup}(x * y) \\
&\le \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} + t \\
&= \max\left\{\widetilde{f}_{\mathcal{N}}(x) + t, \widetilde{f}_{\mathcal{N}}(y) + t\right\} \\
&= \max\left\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\right\}
\end{aligned}$$

for all $x, y \in X$. Thus, $\left(X, \widetilde{f}_{\inf}\right)$ is a 4-fuzzy BCC-subalgebra of X. Therefore, $\left(X, \widetilde{f}\right)$ is a $(4, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. □

Corollary 3.6. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant on X, then it is a $(4, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Theorem 3.11. *Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over a BCC-algebra $(X, *, 0)$ in which $\left(X, \widetilde{f}_{\inf}\right)$ satisfies the following condition:*

$$(\forall x, y \in X)(f_{\inf}(x * y) \ge \max\{f_{\inf}(x), f_{\inf}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ is a $(k, 4)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X.

Proof: Assume that $\left(X, \widetilde{f}\right)$ is a $(k, 4)$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $\left(X, \widetilde{f}_{\inf}\right)$ satisfies the condition

$$(\forall x, y \in X)(f_{\inf}(x * y) \ge \max\{f_{\inf}(x), f_{\inf}(y)\}).$$

Then $\widetilde{f}_{\inf}(x * y) \ge \widetilde{f}_{\inf}(x)$ and $\widetilde{f}_{\inf}(x * y) \ge \widetilde{f}_{\inf}(y)$ for all $x, y \in X$. Since $\left(X, \widetilde{f}_{\sup}\right)$ is a 4-fuzzy BCC-subalgebra of X, we have

$$\begin{aligned}
\widetilde{f}_{\mathcal{N}}(x * y) &= \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x * y) \\
&\ge \widetilde{f}_{\inf}(x * y) - \max\left\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\right\} \\
&= \min\left\{\widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(y)\right\} \\
&\ge \min\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(y)\right\} \\
&= \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}
\end{aligned}$$

for all $x, y \in X$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X. □

Corollary 3.7. *Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over a BCC-algebra* $(X, *, 0)$ *in which* $\left(X, \widetilde{f}_{\inf}\right)$ *satisfies the following condition:*

$$(\forall x, y \in X)(f_{\inf}(x * y) \geq \max\{f_{\inf}(x), f_{\inf}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ *is a* $(k, 2)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*, then it is an* $\mathcal{N}_1$*-fuzzy BCC-subalgebra of* X*.*

Theorem 3.12. *If* $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_1$*-hyperfuzzy BCC-subalgebra of a BCC-algebra* $(X, *, 0)$ *in which* $\widetilde{f}_{\inf}$ *is constant on* X*, then it is a* $(k, 4)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ in which $\widetilde{f}_{\inf}$ is constant. It is clear that $\left(X, \widetilde{f}_{\inf}\right)$ is a k-fuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. Let $\widetilde{f}_{\inf}(x) = t$ for all $x \in X$. Then

$$\begin{aligned}
\widetilde{f}_{\sup}(x * y) &= \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\mathcal{N}}(x * y) \\
&\leq t - \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \\
&= \max\left\{t - \widetilde{f}_{\mathcal{N}}(x), t - \widetilde{f}_{\mathcal{N}}(y)\right\} \\
&= \max\left\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\right\}
\end{aligned}$$

for all $x, y \in X$. Thus, $\left(X, \widetilde{f}_{\sup}\right)$ is a 4-fuzzy BCC-subalgebra of X. Therefore, $\left(X, \widetilde{f}\right)$ is a $(k, 4)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. □

Corollary 3.8. *If* $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_3$*-hyperfuzzy BCC-subalgebra of a BCC-algebra* $(X, *, 0)$ *in which* $\widetilde{f}_{\inf}$ *is constant on* X*, then it is a* $(k, 4)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*.*

Theorem 3.13. *Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over a BCC-algebra* $(X, *, 0)$ *in which* $\left(X, \widetilde{f}_{\sup}\right)$ *satisfies the following condition:*

$$(\forall x, y \in X)(f_{\inf}(x * y) \leq \min\{f_{\inf}(x), f_{\inf}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ *is a* $(1, k)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*, then it is an* $\mathcal{N}_1$*-fuzzy BCC-subalgebra of* X*.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is a $(1, k)$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$ in which $\left(X, \widetilde{f}_{\sup}\right)$ satisfies the condition

$$(\forall x, y \in X)(f_{\inf}(x * y) \leq \min\{f_{\inf}(x), f_{\inf}(y)\}).$$

Then $\widetilde{f}_{\sup}(x * y) \leq \widetilde{f}_{\sup}(x)$ and $\widetilde{f}_{\sup}(x * y) \leq \widetilde{f}_{\sup}(y)$ for all $x, y \in X$. Since $\left(X, \widetilde{f}_{\inf}\right)$ is a 1-fuzzy BCC-subalgebra of X, we have

$$\begin{aligned}
\widetilde{f}_{\mathcal{N}}(x * y) &= \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x * y) \\
&\geq \min\left\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\right\} - \widetilde{f}_{\sup}(x * y)
\end{aligned}$$

$$= \min\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x*y), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(x*y)\right\}$$
$$\geq \min\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(y)\right\}$$
$$= \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$$

for all $x, y \in X$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X. □

Corollary 3.9. *Let $\left(X, \widetilde{f}\right)$ be a hyperfuzzy structure over a BCC-algebra $(X, *, 0)$ in which $\left(X, \widetilde{f}_{\sup}\right)$ satisfies the following condition:*

$$(\forall x, y \in X)(f_{\sup}(x*y) \leq \min\{f_{\sup}(x), f_{\sup}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ is a $(3, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$, then it is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of X.

Theorem 3.14. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant on X, then it is a $(1, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_1$-fuzzy BCC-subalgebra of $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant. It is clear that $\left(X, \widetilde{f}_{\sup}\right)$ is a k-fuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. Let $\widetilde{f}_{\sup}(x) = t$ for all $x \in X$. Then

$$\begin{aligned} \widetilde{f}_{\inf}(x*y) &= \widetilde{f}_{\mathcal{N}}(x*y) + \widetilde{f}_{\sup}(x*y) \\ &\geq t + \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \\ &= \min\left\{t + \widetilde{f}_{\mathcal{N}}(x), t + \widetilde{f}_{\mathcal{N}}(y)\right\} \\ &= \min\left\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\right\} \end{aligned}$$

for all $x, y \in X$. Thus, $\left(X, \widetilde{f}_{\inf}\right)$ is a 1-fuzzy BCC-subalgebra of X. Therefore, $\left(X, \widetilde{f}\right)$ is a $(1, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. □

Corollary 3.10. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant on X, then it is a $(1, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Theorem 3.15. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\inf}$ is constant on X, then it is a $(k, 3)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of $(X, *, 0)$ in which $\widetilde{f}_{\inf}$ is constant. It is clear that $\left(X, \widetilde{f}_{\inf}\right)$ is a k-fuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. Let $\widetilde{f}_{\inf}(x) = t$ for all $x \in X$. Then

$$\widetilde{f}_{\sup}(x*y) = \widetilde{f}_{\inf} - \widetilde{f}_{\mathcal{N}}(x*y)$$

$$\geq t - \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}$$
$$= \max\left\{t - \widetilde{f}_{\mathcal{N}}(x), t - \widetilde{f}_{\mathcal{N}}(y)\right\}$$
$$= \max\left\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\right\}$$

for all $x, y \in X$. Thus, $\left(X, \widetilde{f}_{\sup}\right)$ is a 3-fuzzy BCC-subalgebra of X. Therefore, $\left(X, \widetilde{f}\right)$ is a $(k, 3)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. □

Theorem 3.16. *If* $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_3$*-fuzzy BCC-subalgebra of a BCC-algebra* $(X, *, 0)$ *in which* $\widetilde{f}_{\inf}$ *is constant on* X*, then it is a* $(k, 2)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$.

Proof: Similar to the proof of Theorem 3.15. □

Theorem 3.17. *Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over a BCC-algebra* $(X, *, 0)$ *in which* $\left(X, \widetilde{f}_{\inf}\right)$ *satisfies the following condition:*

$$(\forall x, y \in X)(f_{\inf}(x * y) \geq \max\{f_{\inf}(x), f_{\inf}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ *is a* $(k, 2)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*, then it is an* $\mathcal{N}_3$*-fuzzy BCC-subalgebra of* X.

Proof: Assume that $\left(X, \widetilde{f}\right)$ is a $(k, 2)$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ in which $\left(X, \widetilde{f}_{\inf}\right)$ satisfies the condition

$$(\forall x, y \in X)\left(f_{\inf}(x * y) \geq \max\left\{f_{\inf}(x), f_{\inf}(y)\right\}\right).$$

Then $\widetilde{f}_{\inf}(x * y) \geq \widetilde{f}_{\inf}(x)$ and $\widetilde{f}_{\inf}(x * y) \geq \widetilde{f}_{\inf}(y)$ for all $x, y \in X$. Since $\left(X, \widetilde{f}_{\sup}\right)$ is a 2-fuzzy BCC-subalgebra of X, we have

$$\begin{aligned}
\widetilde{f}_{\mathcal{N}}(x * y) &= \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x * y) \\
&\geq \widetilde{f}_{\inf}(x * y) - \min\left\{\widetilde{f}_{\sup}(x), \widetilde{f}_{\sup}(y)\right\} \\
&= \max\left\{\widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(y)\right\} \\
&\geq \max\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(y)\right\} \\
&= \max\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\}
\end{aligned}$$

for all $x, y \in X$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of X. □

Theorem 3.18. *Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over a BCC-algebra* $(X, *, 0)$ *in which* $\left(X, \widetilde{f}_{\inf}\right)$ *satisfies the following condition.*

$$(\forall x, y \in X)(f_{\inf}(x * y) \leq \min\{f_{\inf}(x), f_{\inf}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ *is a* $(k, 3)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*, then it is an* $\mathcal{N}_2$*-fuzzy BCC-subalgebra of* X.

Proof: Similar to the proof of Theorem 3.17. □

Theorem 3.19. *Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over a BCC-algebra* $(X, *, 0)$ *in which* $\left(X, \widetilde{f}_{\sup}\right)$ *satisfies the following condition:*

$$(\forall x, y \in X)(f_{\sup}(x * y) \geq \max\{f_{\sup}(x), f_{\sup}(y)\}).$$

If $\left(X, \widetilde{f}\right)$ *is a* $(2, k)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*, then it is an* $\mathcal{N}_2$*-fuzzy BCC-subalgebra of* X*.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is a $(2, k)$-hyperfuzzy BCC-subalgebra of $(X, *, 0)$ in which $\left(X, \widetilde{f}_{\sup}\right)$ satisfies the condition

$$(\forall x, y \in X)(f_{\sup}(x * y) \geq \max\{f_{\sup}(x), f_{\sup}(y)\}).$$

Then $\widetilde{f}_{\sup}(x * y) \geq \widetilde{f}_{\sup}(x)$ and $\widetilde{f}_{\sup}(x * y) \geq \widetilde{f}_{\sup}(y)$ for all $x, y \in X$. Since $\left(X, \widetilde{f}_{\inf}\right)$ is a 2-fuzzy BCC-subalgebra of X, we have

$$\begin{aligned} \widetilde{f}_{\mathcal{N}}(x * y) &= \widetilde{f}_{\inf}(x * y) - \widetilde{f}_{\sup}(x * y) \\ &\leq \min\left\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\right\} - \widetilde{f}_{\sup}(x * y) \\ &= \min\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x * y), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(x * y)\right\} \\ &\leq \min\left\{\widetilde{f}_{\inf}(x) - \widetilde{f}_{\sup}(x), \widetilde{f}_{\inf}(y) - \widetilde{f}_{\sup}(y)\right\} \\ &= \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} \end{aligned}$$

for all $x, y \in X$. Therefore, $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of X. □

Theorem 3.20. *Let* $\left(X, \widetilde{f}\right)$ *be a hyperfuzzy structure over a BCC-algebra* $(X, *, 0)$ *in which* $\left(X, \widetilde{f}_{\sup}\right)$ *satisfies the following condition:*

$$(\forall x, y \in X)\left(f_{\sup}(x * y) \leq \min\left\{f_{\sup}(x), f_{\sup}(y)\right\}\right).$$

If $\left(X, \widetilde{f}\right)$ *is a* $(3, k)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*, then it is an* $\mathcal{N}_3$*-fuzzy BCC-subalgebra of* X*.*

Proof: Similar to the proof of Theorem 3.19. □

Theorem 3.21. *If* $\left(X, \widetilde{f}\right)$ *is an* $\mathcal{N}_2$*-fuzzy BCC-subalgebra of a BCC-algebra* $(X, *, 0)$ *in which* $\widetilde{f}_{\sup}$ *is constant on* X*, then it is a* $(2, k)$*-hyperfuzzy BCC-subalgebra of* X *for* $k \in \{1, 2, 3, 4\}$*.*

Proof: Assume that $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_2$-fuzzy BCC-subalgebra of $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant. It is clear that $\left(X, \widetilde{f}_{\sup}\right)$ is a k-fuzzy BCC-subalgebra of $(X, *, 0)$ for $k \in \{1, 2, 3, 4\}$. Let $\widetilde{f}_{\sup}(x) = t$ for all $x \in X$. Then

$$\begin{aligned} \widetilde{f}_{\inf}(x * y) &= \widetilde{f}_{\mathcal{N}}(x * y) + \widetilde{f}_{\sup}(x * y) \\ &= \widetilde{f}_{\mathcal{N}}(x * y) + t \\ &\leq \min\left\{\widetilde{f}_{\mathcal{N}}(x), \widetilde{f}_{\mathcal{N}}(y)\right\} + t \end{aligned}$$

$$= \min\left\{\widetilde{f}_{\mathcal{N}}(x) + t, \widetilde{f}_{\mathcal{N}}(y) + t\right\}$$
$$= \min\left\{\widetilde{f}_{\inf}(x), \widetilde{f}_{\inf}(y)\right\}$$

for all $x, y \in X$. Thus, $\left(X, \widetilde{f}_{\inf}\right)$ is a 2-fuzzy BCC-subalgebra of X. Therefore, $\left(X, \widetilde{f}\right)$ is a $(2, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$. □

Theorem 3.22. *If $\left(X, \widetilde{f}\right)$ is an $\mathcal{N}_3$-fuzzy BCC-subalgebra of a BCC-algebra $(X, *, 0)$ in which $\widetilde{f}_{\sup}$ is constant on X, then it is a $(3, k)$-hyperfuzzy BCC-subalgebra of X for $k \in \{1, 2, 3, 4\}$.*

Proof: Similar to the proof of Theorem 3.21. □

4. **Conclusions.** In this paper, we introduced the concept of $\mathcal{N}$-structures in BCC-algebras derived from hyperfuzzy structures, along with the notion of $\mathcal{N}_k$-fuzzy BCC-subalgebras for $k \in \{1, 2, 3, 4\}$. We investigated several key properties of these subalgebras and provided concrete examples to illustrate how these structures function in practice. Additionally, we examined the relationships between $\mathcal{N}_k$-fuzzy BCC-subalgebras and (i, j)-hyperfuzzy BCC-subalgebras in BCC-algebras for $i, j, k \in \{1, 2, 3, 4\}$, demonstrating how these relationships help refine the algebraic structure. By simplifying complex descriptions and offering examples, we aim to make the theoretical results more accessible and applicable to readers interested in fuzzy logic and algebraic structures.

To further advance this field, future research could focus on expanding the applications of $\mathcal{N}_k$-fuzzy BCC-subalgebras in real-world problems, such as optimization in fuzzy systems, machine learning algorithms for handling uncertain data or modelling decision processes under uncertainty. Additionally, investigating more complex relationships between higher-dimensional fuzzy sets and their algebraic counterparts could lead to new insights into multi-level fuzzy logic systems. Another promising area of exploration is the extension of these concepts to other algebraic structures, such as lattice theory or non-commutative algebras, to broaden their applicability across various domains.

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