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PYTHAGOREAN FUZZY SETS: A NEW PERSPECTIVE ON IUP-ALGEBRAS

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Abstract. *Zadeh introduced the concept of fuzzy sets in 1965. In 1986, Atanassov introduced intuitionistic fuzzy sets, a generalization of fuzzy sets. Pythagorean fuzzy sets are a recent extension of intuitionistic fuzzy sets introduced by Yager. The aim of this study is to apply the concept of Pythagorean fuzzy sets to IUP-algebras and introduce the notions of Pythagorean fuzzy IUP-subalgebras, Pythagorean fuzzy IUP-ideals, Pythagorean fuzzy IUP-filters, and Pythagorean fuzzy strong IUP-ideals. The study identified a relationship between four concepts, showing that Pythagorean fuzzy IUP-ideals and Pythagorean fuzzy IUP-subalgebras are generalizations of Pythagorean fuzzy strong IUP-ideals in IUP-algebras, where the latter can only be a constant Pythagorean fuzzy set. Additionally, Pythagorean fuzzy IUP-filters were found to be a further generalization of Pythagorean fuzzy IUP-ideals and IUP-subalgebras. Their properties are investigated, and the characteristic Pythagorean fuzzy sets, the upper t-(strong) level subsets, and the lower t-(strong) level subsets of the Pythagorean fuzzy set are studied.*

Keywords: IUP-algebra, Pythagorean fuzzy set, Pythagorean fuzzy IUP-subalgebra, Pythagorean fuzzy IUP-ideal, Pythagorean fuzzy IUP-filter, Pythagorean fuzzy strong IUP-ideal, Upper t-(strong) level subset, Lower t-(strong) level subset

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1. **Introduction.** Zadeh [1] commenced the concept of fuzzy sets (FSs) in 1965, an important concept. After that, Atanassov [2] introduced the notion of intuitionistic fuzzy sets (IFSs) in 1986, which is a generalization of fuzzy sets. Pythagorean fuzzy sets (PFSs) are a recent extension of IFSs. They arise when the sum of the membership level and non-membership level exceeds 1, but the sum of their squares is less than or equal to 1. For instance, Yager [3] described a scenario where someone expresses a preference for an option x_i in criterion C_j, and the level at which x_i makes C_j acceptable is $\frac{\sqrt{3}}{2}$, while the level at which x_i makes C_j unacceptable is $\frac{1}{2}$. This situation cannot be adequately captured by IFS but can be modelled using PFS. In summary, the relationship between IFS and PFS lies in PFS's ability to simulate uncertainty that occurs in real-world cases where the sum of membership and non-membership levels exceeds 1, yet the sum of their squares remains less than or equal to 1.

After that, the concept of PFSs has been studied extensively and continuously in many spaces. For example, in 2020, Chinnadurai and Selvam [4] defined the new notion of interval-valued Pythagorean fuzzy ideals in semigroups. Chinram and Panityakul [5] studied rough Pythagorean fuzzy ideals in ternary semigroups and extended them to the lower and upper approximations of Pythagorean fuzzy ideals. In 2021, Satirad et al. [6] applied the concept of PFSs to UP-algebras, introduced five types of PFSs in UP-algebras, and studied upper and lower approximations of PFSs. Hameed et al. [7] elaborated a new structure of Pythagorean fuzzy N-soft groups, which is the generalization of intuitionistic fuzzy soft groups. In 2022, Satirad et al. [8] applied the concept of rough sets to PFSs in UP-algebras, introduced fifteen types of rough PFSs in UP-algebras, and studied their generalization. Razaq and Alhamzi [9] characterized the concept of Pythagorean fuzzy ideals in classical rings. In 2023, Satirad et al. [10] introduced new types of PFSs in UP-algebras, which are referred to as Pythagorean fuzzy implicative UP-filters, Pythagorean fuzzy comparative UP-filters, and Pythagorean fuzzy shift UP-filters. Palanikumar et al. [11] explored the concept of the possibility Pythagorean neutrosophic vague soft set (PPyNSVSS) and introduced related operations, including complement, union, intersection, AND, OR, along with fundamental laws such as commutative, De Morgan's, associative, and distributive laws. They also compared PPyNSVSS with the Pythagorean neutrosophic vague soft set (PyNSVSS), particularly in the context of decision-making and similarity measures. Palanikumar et al. [12] investigated the concept of the possibility Pythagorean cubic fuzzy soft set, a generalization of soft sets, and explored its practical applications. They proposed an algorithm based on the fuzzy soft set approach to address decision-making problems and developed a similarity measure by comparing the possibility Pythagorean cubic fuzzy soft set with the Pythagorean cubic fuzzy soft set. In 2024, Meesri et al. [13] innovatively applied the concept of PFSs to clarifying complexities within KU-algebras. They introduced the notion of Pythagorean fuzzy KU-subalgebras, providing fundamental properties and exploring the intricate relationships between the image and preimage of these subalgebras under a homomorphism of KU-algebras. This groundbreaking work sheds new light on the structural dynamics of KU-algebras, paving the way for further advancements in the field. Research on PFSs is essential for advancing decision-making processes, particularly in uncertain environments and imprecise data. PFS facilitates the development of similarity measures and computational techniques that enable more accurate evaluation of choices under ambiguous conditions. For example, PFS plays a pivotal role in assessing product selection or guiding decision-making in business contexts where information is unclear. Furthermore, PFS significantly improves the management of uncertainty and vagueness within algebraic frameworks such as IUP-algebras, BCC-algebras, and Hilbert algebras. The exploration of PFS is, therefore, critical for creating sophisticated decision-making models and enhancing the practical application of algebraic systems in complex, real-world scenarios.

In 2022, Iampan et al. [14] made a significant contribution to the field by introducing the concept of IUP-algebras. They explored the foundational notions of IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals, thoroughly investigating their basic properties. This pioneering work has laid the groundwork for further advancements and applications in studying IUP-algebras. Following this, numerous researchers have delved into the notions of IUP-algebras and expanded their applications to other concepts. For instance, in 2023, Chanmanee et al. [15] introduced the concept of the direct product of an infinite family of IUP-algebras. They examined the results of the external direct product of specific subsets of IUP-algebras and also introduced the concept of the weak direct product of IUP-algebras. Furthermore, they provided several fundamental theorems concerning (anti-)IUP-homomorphisms in the context of external direct product IUP-algebras. This work significantly advances the theoretical framework and practical understanding of IUP-algebras. This year, Kuntama et al. [16] revolutionized the application of fuzzy set theory to IUP-algebras by introducing four novel concepts of fuzzy sets: fuzzy IUP-subalgebras, fuzzy IUP-ideals, fuzzy IUP-filters, and fuzzy strong IUP-ideals. Their study delved into these concepts, meticulously exploring their unique properties and intricate relationships. Suayngam et al. [17] made significant strides in the study of IUP-algebras by introducing the concepts of intuitionistic fuzzy IUP-subalgebras, intuitionistic fuzzy IUP-ideals, intuitionistic fuzzy IUP-filters, and intuitionistic fuzzy strong IUP-ideals.

Based on the previous review, it is evident that many researchers have explored the concept of the PFS. Therefore, we are interested in applying this concept to IUP-algebras. In this paper, we will introduce the notions of Pythagorean fuzzy IUP-subalgebras, Pythagorean fuzzy IUP-ideals, Pythagorean fuzzy IUP-filters, and Pythagorean fuzzy strong IUP-ideals. Their properties are investigated, and the characteristic PFSs, the upper t-(strong) level subsets, and the lower t-(strong) level subsets of the PFS are studied.

2. **Preliminaries.** Before delving into our study, let us review the foundational concepts of IUP-algebras, including their various properties and pertinent definitions crucial to this research.

Definition 2.1. [14] *An algebra $X = (X; \cdot, 0)$ of type $(2, 0)$ is called an IUP-algebra, where X is a nonempty set, $\cdot$ is a binary operation on X, and 0 is a fixed element of X if it satisfies the following axioms:*

$$(\forall x \in X)(0 \cdot x = x) \tag{IUP-1}$$

$$(\forall x \in X)(x \cdot x = 0) \tag{IUP-2}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot (x \cdot z) = y \cdot z) \tag{IUP-3}$$

For convenience, we refer to X as an IUP-algebra $X = (X; \cdot, 0)$ until otherwise specified.

Example 2.1. [14] *Let $(G, \bullet, e)$ be a group such that all elements self-inverse. Then $(G, \bullet, e)$ is an IUP-algebra.*

Example 2.2. [14] *Let X be a set and $\mathcal{P}(X)$ means the power set of X. It follows from Example 2.1 that $(\mathcal{P}(X), \triangle, \emptyset)$ is an IUP-algebra where the binary operation $\triangle$ is defined as the symmetric difference of any two sets.*

Example 2.3. [14] *Let $(G, \bullet, e)$ be a group with the identity element e. Define a binary operation $\bullet$ on G by*

$$(\forall x, y \in G)\left(x \bullet y = yx^{-1}\right) \tag{1}$$

Then $(G, \bullet, e)$ *is an IUP-algebra.*

Proposition 2.1. [14] *In* X*, the following assertions are valid (see* [14]*).*

$$(\forall x, y \in X)((x \cdot 0) \cdot (x \cdot y) = y) \tag{2}$$

$$(\forall x \in X)((x \cdot 0) \cdot (x \cdot 0) = 0) \tag{3}$$

$$(\forall x, y \in X)((x \cdot y) \cdot 0 = y \cdot x) \tag{4}$$

$$(\forall x \in X)((x \cdot 0) \cdot 0 = x) \tag{5}$$

$$(\forall x, y \in X)(x \cdot ((x \cdot 0) \cdot y) = y) \tag{6}$$

$$(\forall x, y \in X)(((x \cdot 0) \cdot y) \cdot x = y \cdot 0) \tag{7}$$

$$(\forall x, y, z \in X)(x \cdot y = x \cdot z \Leftrightarrow y = z) \tag{8}$$

$$(\forall x, y \in X)(x \cdot y = 0 \Leftrightarrow x = y) \tag{9}$$

$$(\forall x \in X)(x \cdot 0 = 0 \Leftrightarrow x = 0) \tag{10}$$

$$(\forall x, y, z \in X)(y \cdot x = z \cdot x \Leftrightarrow y = z) \tag{11}$$

$$(\forall x, y \in X)(x \cdot y = y \Rightarrow x = 0) \tag{12}$$

$$(\forall x, y, z \in X)((x \cdot y) \cdot 0 = (z \cdot y) \cdot (z \cdot x)) \tag{13}$$

$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (z \cdot x) \cdot (z \cdot y) = 0) \tag{14}$$

$$(\forall x, y, z \in X)(x \cdot y = 0 \Leftrightarrow (x \cdot z) \cdot (y \cdot z) = 0) \tag{15}$$

$$\text{the right and the left cancellation laws hold} \tag{16}$$

In the realm of IUP-algebras, four distinct types of subsets play pivotal roles: IUP-subalgebras, IUP-filters, IUP-ideals, and strong IUP-ideals. These subsets are integral to understanding the structure and properties of IUP-algebras. These subsets collectively form a nuanced framework that facilitates the study and application of IUP-algebras in various mathematical contexts.

Definition 2.2. [14] *A nonempty subset* S *of* X *is called*

(*i*) *an IUP-subalgebra of* X *if it satisfies the following condition:*

$$(\forall x, y \in S)(x \cdot y \in S) \tag{17}$$

(*ii*) *an IUP-filter of* X *if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } X \text{ is in } S \tag{18}$$

$$(\forall x, y \in X)(x \cdot y \in S, x \in S \Rightarrow y \in S) \tag{19}$$

(*iii*) *an IUP-ideal of* X *if it satisfies the condition* (18) *and the following condition:*

$$(\forall x, y, z \in X)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S) \tag{20}$$

(*iv*) *a strong IUP-ideal of* X *if it satisfies the following condition:*

$$(\forall x, y \in X)(y \in S \Rightarrow x \cdot y \in S) \tag{21}$$

According to [14], the concept of IUP-filters serves as a generalization encompassing IUP-ideals and IUP-subalgebras. Both IUP-ideals and IUP-subalgebras, in turn, generalize strong IUP-ideals. In an IUP-algebra X, it is observed that strong IUP-ideals coincide with X itself. This relationship is illustrated in the diagram of special subsets of IUP-algebras, depicted in Figure 1.

3. **Main Results.** PFSs offer a comprehensive framework for managing uncertainty across various fields, improving decision-making processes and broadening the application

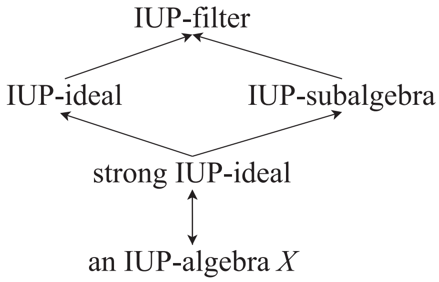

Figure 1. Special subsets of IUP-algebras

of fuzzy logic in algebraic systems. Understanding their foundational principles is crucial for grasping their significance and effectiveness in complex real-world scenarios. Before defining PFSs, it is important to examine these foundational concepts.

Definition 3.1. [3] *A Pythagorean fuzzy set (briefly, PFS) P in a nonempty set X is described by their membership function μ_P and non-membership function γ_P. To every point $x \in X$, these functions associate real numbers $\mu_P(x)$ and $\gamma_P(x)$ in the closed interval $[0,1]$, with the following assertion:*

$$(\forall x \in X)\left(0 \leq \mu_P(x)^2 + \gamma_P(x)^2 \leq 1\right) \tag{22}$$

The real numbers $\mu_P(x)$ and $\gamma_P(x)$ are interpreted for the point as a degree of membership and non-membership of an object $x \in X$, respectively, to the PFS P, that is, $P = \{(x, \mu_P(x), \gamma_P(x)) | x \in X\}$. For the sake of simplicity, a PFS P is denoted by $P = (\mu_P, \gamma_P)$. A PFS P in X is constant if its membership function μ_P and non-membership function γ_P are constant.

For a subset G of a nonempty set X, the characteristic functions μ_{P_G} and γ_{P_G} are functions of X into $\{0,1\}$ defined as follows:

$$\mu_{P_G}(x) = \begin{cases} 1 & \text{if } x \in G \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_{P_G}(x) = \begin{cases} 0 & \text{if } x \in G \\ 1 & \text{otherwise} \end{cases}$$

By the definition of the characteristic function, μ_{P_G} and γ_{P_G} are functions of X into $\{0,1\} \subset [0,1]$. Therefore, the PFS $P_G = (\mu_{P_G}, \gamma_{P_G})$ is defined as the characteristic PFS of G in X.

Consider a fuzzy set f in a nonempty set X. The fuzzy set $\overline{f}$, defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in X$, is known as the *complement* of f in X. This complement captures the degree to which an element does not belong to the fuzzy set f, providing a useful tool for analyzing and manipulating fuzzy sets.

We extend the concept of PFSs to IUP-algebras, introducing four innovative types: Pythagorean fuzzy IUP-subalgebras, Pythagorean fuzzy IUP-ideals, Pythagorean fuzzy IUP-filters, and Pythagorean fuzzy strong IUP-ideals. This application opens new dimensions in the study of IUP-algebras, enriching their theoretical and practical frameworks.

Definition 3.2. *A PFS $P = (\mu_P, \gamma_P)$ in X is called a Pythagorean fuzzy IUP-subalgebra of X if it satisfies the following properties:*

$$(\forall x, y \in X)(\mu_P(x \cdot y) \geq \min\{\mu_P(x), \mu_P(y)\}) \tag{23}$$

$$(\forall x, y \in X)(\gamma_P(x \cdot y) \leq \max\{\gamma_P(x), \gamma_P(y)\}) \tag{24}$$

Definition 3.3. *A PFS $P = (\mu_P, \gamma_P)$ in X is called a Pythagorean fuzzy IUP-ideal of X if it satisfies the following properties:*

$$(\forall x \in X)(\mu_P(0) \geq \mu_P(x)) \tag{25}$$

$$(\forall x \in X)(\gamma_P(0) \leq \gamma_P(x)) \tag{26}$$

$$(\forall x, y, z \in X)(\mu_P(x \cdot z) \geq \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\}) \tag{27}$$

$$(\forall x, y, z \in X)(\gamma_P(x \cdot z) \leq \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\}) \tag{28}$$

Definition 3.4. *A PFS $P = (\mu_P, \gamma_P)$ in X is called a Pythagorean fuzzy IUP-filter of X if it satisfies* (25), (26) *and the following properties:*

$$(\forall x, y \in X)(\mu_P(y) \geq \min\{\mu_P(x \cdot y), \mu_P(x)\}) \tag{29}$$

$$(\forall x, y \in X)(\gamma_P(y) \leq \max\{\gamma_P(x \cdot y), \gamma_P(x)\}) \tag{30}$$

Definition 3.5. *A PFS $P = (\mu_P, \gamma_P)$ in X is called a Pythagorean fuzzy strong IUP-ideal of X if it satisfies the following properties:*

$$(\forall x, y \in X)(\mu_P(x \cdot y) \geq \mu_P(y)) \tag{31}$$

$$(\forall x, y \in X)(\gamma_P(x \cdot y) \leq \gamma_P(y)) \tag{32}$$

Lemma 3.1. *Every Pythagorean fuzzy IUP-subalgebra of X satisfies* (25) *and* (26)*.*

Proof: Assume that P is a Pythagorean fuzzy IUP-subalgebra of X. Let $x \in X$. Then

$$\begin{aligned}
\mu_P(0) &= \mu_P(x \cdot x) && \text{(by (IUP-2))} \\
&\geq \min\{\mu_P(x), \mu_P(x)\} && \text{(by (23))} \\
&= \mu_P(x), \\
\gamma_P(0) &= \gamma_P(x \cdot x) && \text{(by (IUP-2))} \\
&\leq \max\{\gamma_P(x), \gamma_P(x)\} && \text{(by (24))} \\
&= \gamma_P(x).
\end{aligned}$$

Hence, it is satisfies (25) and (26). □

Theorem 3.1. *Every Pythagorean fuzzy strong IUP-ideal of X satisfies* (25) *and* (26)*.*

Proof: Assume that P is a Pythagorean fuzzy strong IUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned}
\mu_P(0) &= \mu_P(x \cdot x) && \text{(by (IUP-2))} \\
&\geq \mu_P(x), && \text{(by (31))} \\
\gamma_P(0) &= \gamma_P(x \cdot x) && \text{(by (IUP-2))} \\
&\leq \gamma_P(x). && \text{(by (32))}
\end{aligned}$$

Hence, it is satisfies (25) and (26). □

Theorem 3.2. *A Pythagorean fuzzy strong IUP-ideal and a constant PFS coincide.*

Proof: Assume that P is a Pythagorean fuzzy strong IUP-ideal of X. Let $x \in X$. Then

$$\begin{aligned}
\mu_P(x) &= \mu_P((x \cdot 0) \cdot 0) && \text{(by (5))} \\
&\geq \mu_P(0), && \text{(by (31))} \\
\gamma_P(x) &= \gamma_P((x \cdot 0) \cdot 0) && \text{(by (5))} \\
&\leq \gamma_P(0). && \text{(by (32))}
\end{aligned}$$

Hence, P is a constant of X.

Conversely, it is obvious that every constant PFS is a Pythagorean fuzzy strong IUP-ideal. □

Theorem 3.3. *Every Pythagorean fuzzy strong IUP-ideal of X is a Pythagorean fuzzy IUP-subalgebra of X.*

Proof: It is straightforward by Theorem 3.2. □

Example 3.1. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	4	0	3	5	1	2
2	2	5	0	4	3	1
3	5	3	1	0	2	4
4	1	4	5	2	0	3
5	3	2	4	1	5	0

Then X is an IUP-algebra. We define a PFS P in X as follows:

$$\mu_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.8 & 0.2 & 0.5 & 0.2 & 0.2 & 0.2 \end{pmatrix}$$

$$\gamma_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.1 & 0.6 & 0.3 & 0.6 & 0.6 & 0.6 \end{pmatrix}$$

Then P is a Pythagorean fuzzy IUP-subalgebra of X. Since $\mu_P(2 \cdot 0) = \mu_P(2) = 0.5 \not\geq 0.8 = \mu_P(0)$ and $\gamma_P(4 \cdot 0) = \gamma_P(1) = 0.6 \not\leq 0.1 = \gamma_P(0)$, P is not a Pythagorean fuzzy strong IUP-ideal of X.

Theorem 3.4. *Every Pythagorean fuzzy strong IUP-ideal of X is a Pythagorean fuzzy IUP-ideal of X.*

Proof: It is straightforward by Theorem 3.2. □

Example 3.2. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	3	0	5	1	2	4
2	5	2	0	4	1	3
3	1	3	4	0	5	2
4	4	5	3	2	0	1
5	2	4	1	5	3	0

Then X is an IUP-algebra. We define a PFS P in X as follows:

$$\mu_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.9 & 0.5 & 0.3 & 0.5 & 0.3 & 0.3 \end{pmatrix}$$

$$\gamma_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0.1 & 0.4 & 0.1 & 0.4 & 0.4 \end{pmatrix}$$

Then P is a Pythagorean fuzzy IUP-ideal of X. Since $\mu_P(5 \cdot 1) = \mu_P(4) = 0.3 \not\geq 0.5 = \mu_P(1)$ and $\gamma_P(4 \cdot 3) = \gamma_P(2) = 0.4 \not\leq 0.1 = \gamma_P(3)$. Hence, P is not a Pythagorean fuzzy strong IUP-ideal of X.

Theorem 3.5. *Every Pythagorean fuzzy IUP-ideal of X is a Pythagorean fuzzy IUP-filter of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-ideal of X. By the assumption, it satisfies (25) and (26). Let $x, y \in X$. Then

$$\mu_P(y) = \mu_P(0 \cdot y) \qquad \text{(by (IUP-1))}$$

$$\begin{aligned} &\geq \min\{\mu_P(0\cdot(x\cdot y)),\mu_P(x)\} && \text{(by (27))}\\ &= \min\{\mu_P(x\cdot y),\mu_P(x)\}, && \text{(by (IUP-1))}\\ \gamma_P(y) &= \gamma_P(0\cdot y) && \text{(by (IUP-1))}\\ &\leq \max\{\gamma_P(0\cdot(x\cdot y)),\gamma_P(x)\} && \text{(by (28))}\\ &= \max\{\gamma_P(x\cdot y),\gamma_P(x)\}. && \text{(by (IUP-1))} \end{aligned}$$

Hence, P is a Pythagorean fuzzy IUP-filter of X. □

Example 3.3. *Let $X = \{0, 1, 2, 3, 4, 5\}$ with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	3	0	4	1	5	2
2	2	4	0	5	1	3
3	1	3	5	0	2	4
4	4	5	3	2	0	1
5	5	2	1	4	3	0

Then X is an IUP-algebra. We define a PFS P in X as follows:

$$\mu_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5\\ 0.7 & 0.3 & 0.3 & 0.3 & 0.6 & 0.3 \end{pmatrix}$$

$$\gamma_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5\\ 0.1 & 0.8 & 0.8 & 0.8 & 0.5 & 0.8 \end{pmatrix}$$

Then P is a Pythagorean fuzzy IUP-filter of X. Since $\mu_P(2\cdot 3) = \mu_P(5) = 0.3 \ngeq 0.6 = \min\{0.7, 0.6\} = \min\{\mu_P(0), \mu_P(4)\} = \min\{\mu_P(2\cdot 2), \mu_P(4)\} = \min\{\mu_P(2\cdot(4\cdot 3)), \mu_P(4)\}$ and $\gamma_P(2\cdot 5) = \gamma_P(3) = 0.8 \nleq 0.5 = \max\{0.5, 0.5\} = \max\{\gamma_P(4), \gamma_P(4)\} = \max\{\gamma_P(2\cdot 1), \gamma_P(4)\} = \max\{\gamma_P(2\cdot(4\cdot 5)), \gamma_P(4)\}$, P is not a Pythagorean fuzzy IUP-ideal of X.

Theorem 3.6. *Every Pythagorean fuzzy IUP-subalgebra of X is a Pythagorean fuzzy IUP-filter of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-subalgebra of X. By Lemma 3.1, it satisfies (25) and (26). Let $x, y \in X$. Then

$$\begin{aligned} \mu_P(y) &= \mu_P(0\cdot y) && \text{(by (IUP-1))}\\ &= \mu_P((x\cdot 0)\cdot(x\cdot y)) && \text{(by (IUP-3))}\\ &\geq \min\{\mu_P(x\cdot 0),\mu_P(x\cdot y)\} && \text{(by (23))}\\ &\geq \min\{\min\{\mu_P(x),\mu_P(0)\},\mu_P(x\cdot y)\} && \text{(by (23))}\\ &= \min\{\mu_P(x),\mu_P(x\cdot y)\}, && \text{(by (25))}\\ \gamma_P(y) &= \gamma_P(0\cdot y) && \text{(by (IUP-1))}\\ &= \gamma_P((x\cdot 0)\cdot(x\cdot y)) && \text{(by (IUP-3))}\\ &\leq \max\{\gamma_P(x\cdot 0),\gamma_P(x\cdot y)\} && \text{(by (24))}\\ &\leq \max\{\max\{\gamma_P(x),\gamma_P(0)\},\gamma_P(x\cdot y)\} && \text{(by (24))}\\ &= \max\{\gamma_P(x),\gamma_P(x\cdot y)\}. && \text{(by (26))} \end{aligned}$$

Hence, P is a Pythagorean fuzzy IUP-filter of X. □

Example 3.4. [14] *Let $\mathbb{R}^*$ be the set of all nonzero real numbers. Define a binary operation $\cdot$ on $\mathbb{R}^*$ by*

$$(\forall x, y \in \mathbb{R}^*)\left(x\cdot y = \frac{y}{x}\right)$$

Then $(\mathbb{R}^*, \cdot, 1)$ *is an IUP-algebra.*

Example 3.5. *From Example 3.4, let* $H = \{x \in \mathbb{R}^* | x \geq 1\}$. *Then* $1 \in H$. *Next, let* $x, y, z \in \mathbb{R}^*$ *be such that* $x \cdot (y \cdot z) \geq 1$ *and* $y \geq 1$. *Then* $\frac{z}{yx} \geq 1$. *Thus,* $x \cdot z = \frac{z}{x} = \left(\frac{z}{yx}\right) y \geq 1$, *that is,* $x \cdot z \in H$. *Hence,* H *is an IUP-ideal of* $\mathbb{R}^*$. *Then* H *is an IUP-filter of* $\mathbb{R}^*$. *By Theorems 3.10 and 3.11, we have the characteristic PFS* P_H *are a Pythagorean fuzzy IUP-ideal and a Pythagorean fuzzy IUP-filter of* $\mathbb{R}^*$. *It implies that the characteristic PFS* P_H *are a Pythagorean fuzzy IUP-ideal and a Pythagorean fuzzy IUP-filter of* $\mathbb{R}^*$. *Since* $1, 3 \in H$ *but* $3 \cdot 1 = \frac{1}{3} \in H$, *we have* H *is not an IUP-subalgebra of* $\mathbb{R}^*$. *By Theorem 3.9, we have the characteristic PFS* P_H *is not a Pythagorean fuzzy IUP-subalgebra of* $\mathbb{R}^*$.

Example 3.6. *Let* $X = \{0, 1, 2, 3, 4, 5\}$ *with the following Cayley table:*

$\cdot$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	5	4	3	2
2	2	3	0	1	5	4
3	5	4	1	0	2	3
4	4	5	3	2	0	1
5	3	2	4	5	1	0

Then X *is an IUP-algebra. We define a PFS* P *in* X *as follows:*

$$\mu_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.7 & 0.4 & 0.5 & 0.4 & 0.4 & 0.4 \end{pmatrix}$$

$$\gamma_P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0.5 & 0.9 & 0.6 & 0.9 & 0.9 & 0.9 \end{pmatrix}$$

Then P *is a Pythagorean fuzzy IUP-subalgebra of* X. *Since* $\mu_P(5 \cdot 4) = \mu_P(1) = 0.4 \not\geq 0.5 = \min\{0.7, 0.5\} = \min\{\mu_P(0), \mu_P(2)\} = \min\{\mu_P(5 \cdot (2 \cdot 4)), \mu_P(2)\}$ *and* $\gamma_P(5 \cdot 3) = \gamma_P(5) = 0.9 \not\leq 0.6 = \max\{0.6, 0.6\} = \max\{\gamma_P(2), \gamma_P(2)\} = \max\{\gamma_P(5 \cdot (2 \cdot 3)), \gamma_P(2)\}$, P *is not a Pythagorean fuzzy IUP-ideal of* X.

The study revealed a relationship between the four concepts: Pythagorean fuzzy IUP-ideals and Pythagorean fuzzy IUP-subalgebras are generalizations of Pythagorean fuzzy strong IUP-ideals of IUP-algebras, where Pythagorean fuzzy strong IUP-ideals of IUP-algebras can only be a constant PFS. Pythagorean fuzzy IUP-filters are a generalization of Pythagorean fuzzy IUP-ideals and Pythagorean fuzzy IUP-subalgebras. We summarize the relationship between these four concepts, shown in Figure 2.

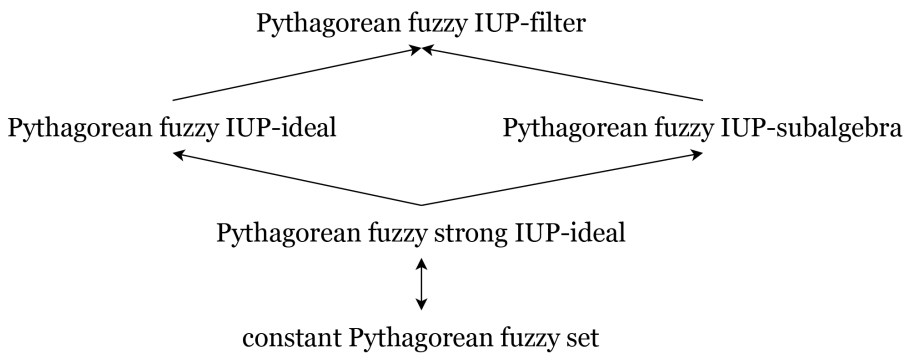

Figure 2. PFSs in IUP-algebras

Theorem 3.7. *If P is a Pythagorean fuzzy IUP-subalgebra of X satisfying the following condition:*

$$(\forall x, y \in X)\left(x \cdot y \neq 0 \Rightarrow \left\{\begin{array}{l} \mu_P(x) \geq \mu_P(y) \\ \gamma_P(x) \leq \gamma_P(y) \end{array}\right.\right), \tag{33}$$

then P is a Pythagorean fuzzy strong IUP-ideal of X.

Proof: Assume that P is a Pythagorean fuzzy IUP-subalgebra of X satisfying (33). Let $x, y \in X$.

Case 1: Suppose $x \cdot y = 0$. Thus,

$$\begin{aligned} \mu_P(x \cdot y) &= \mu_P(0) \\ &\geq \mu_P(y), && \text{(by (25))} \\ \gamma_P(x \cdot y) &= \gamma_P(0) \\ &\leq \gamma_P(y). && \text{(by (26))} \end{aligned}$$

Case 2: Suppose $x \cdot y \neq 0$. Thus,

$$\begin{aligned} \mu_P(x \cdot y) &\geq \min\{\mu_P(x), \mu_P(y)\} && \text{(by (23))} \\ &= \mu_P(y), \\ \gamma_P(x \cdot y) &\leq \max\{\gamma_P(x), \gamma_P(y)\} && \text{(by (24))} \\ &= \gamma_P(y). \end{aligned}$$

Hence, P is a Pythagorean fuzzy strong IUP-ideal of X. □

Theorem 3.8. *If P is a Pythagorean fuzzy IUP-filter of X satisfying the following condition:*

$$(\forall x, y, z \in X)\begin{pmatrix} \mu_P(y \cdot (x \cdot z)) = \mu_P(x \cdot (y \cdot z)) \\ \gamma_P(y \cdot (x \cdot z)) = \gamma_P(x \cdot (y \cdot z)) \end{pmatrix} \tag{34}$$

then P is a Pythagorean fuzzy IUP-ideal of X.

Proof: Assume that P is a Pythagorean fuzzy IUP-filter of X satisfying (34). By the assumption, it satisfies (25) and (26). Let $x, y \in X$. Then

$$\begin{aligned} \mu_P(x \cdot z) &\geq \min\{\mu_P(y \cdot (x \cdot z)), \mu_P(y)\} && \text{(by (29))} \\ &= \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\}, \\ \gamma_P(x \cdot z) &\leq \max\{\gamma_P(y \cdot (x \cdot z)), \gamma_P(y)\} && \text{(by (30))} \\ &= \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\}. \end{aligned}$$

Hence, P is a Pythagorean fuzzy IUP-ideal of X. □

Lemma 3.2. *Let G be a nonempty subset of X. Then the constant 0 of X is in G if and only if the characteristic PFS P_G satisfies (25) and (26).*

Proof: Assume that the constant 0 of X is in G. Then $\mu_{P_G}(0) = 1$ and $\gamma_{P_G}(0) = 0$. Thus, $\mu_{P_G}(0) = 1 \geq \mu_{P_G}(x)$ and $\gamma_{P_G}(0) = 0 \leq \gamma_{P_G}(x)$ for all $x \in X$, that is, P_G satisfies (25) and (26).

Conversely, assume that P_G satisfies (25) and (26). Then $\mu_{P_G}(0) \geq \mu_{P_G}(x)$ for all $x \in X$. Since G is a nonempty subset of X, we let $a \in G$. Then $\mu_{P_G}(0) \geq \mu_{P_G}(a) = 1$, so $\mu_{P_G}(0) = 1$. Hence, the constant 0 of X is in G. □

Theorem 3.9. *A nonempty subset G is an IUP-subalgebra of X if and only if the characteristic PFS P_G is a Pythagorean fuzzy IUP-subalgebra of X.*

Proof: Assume that G is an IUP-subalgebra of X. Let $x, y \in X$.

Case 1: Suppose $x, y \in G$. Then $\mu_{P_G}(x) = 1$ and $\mu_{P_G}(y) = 1$. Since G is an IUP-subalgebra of X, we have $x \cdot y \in G$. Thus, $\mu_{P_G}(x \cdot y) = 1 \geq \min\{1, 1\} = \min\{\mu_{P_G}(x), \mu_{P_G}(y)\}$.

Case 2: Suppose $x \notin G$ or $y \notin G$. Then $\mu_{P_G}(x) = 0$ or $\mu_{P_G}(y) = 0$. Thus, $\mu_{P_G}(x \cdot y) \geq 0 = \min\{\mu_{P_G}(x), \mu_{P_G}(y)\}$.

Case 1': Suppose $x, y \in G$. Then $\gamma_{P_G}(x) = 0$ and $\gamma_{P_G}(y) = 0$. Since G is an IUP-subalgebra of X, we have $x \cdot y \in G$. Thus, $\gamma_{P_G}(x \cdot y) = 0 \leq 0 = \max\{\gamma_{P_G}(x), \gamma_{P_G}(y)\}$.

Case 2': Suppose $x \notin G$ or $y \notin G$. Then $\gamma_{P_G}(x) = 1$ or $\gamma_{P_G}(y) = 1$. Thus, $\gamma_{P_G}(x \cdot y) \leq 1 = \max\{\gamma_{P_G}(x), \gamma_{P_G}(y)\}$.

Hence, the characteristic PFS P_G is a Pythagorean fuzzy IUP-subalgebra of X.

Conversely, assume that the characteristic PFS P_G is a Pythagorean fuzzy IUP-subalgebra of X. Let $x, y \in G$. Then $\mu_{P_G}(x) = 1$ and $\mu_{P_G}(y) = 1$. By (23), we have $\mu_{P_G}(x \cdot y) \geq \min\{\mu_{P_G}(x), \mu_{P_G}(y)\} = \min\{1, 1\} = 1$. Thus, $\mu_{P_G}(x \cdot y) = 1$, that is, $x \cdot y \in G$. Hence, G is an IUP-subalgebra of X. □

Theorem 3.10. *A nonempty subset G is an IUP-ideal of X if and only if the characteristic PFS P_G is a Pythagorean fuzzy IUP-ideal of X.*

Proof: Assume that G is an IUP-ideal of X. Since $0 \in G$, it follows from Lemma 3.2 that μ_{P_G} and γ_{P_G} satisfy (25) and (26), respectively. Next, let $x, y, z \in X$.

Case 1: Suppose $x \cdot (y \cdot z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \cdot z \in G$. Thus, $\mu_{P_G}(x \cdot z) = 1 \geq 1 = \min\{1, 1\} = \min\{\mu_{P_G}(x \cdot (y \cdot z)), \mu_{P_G}(y)\}$.

Case 2: Suppose $x \cdot (y \cdot z) \notin G$ or $y \notin G$. Then $\mu_{P_G}(x \cdot (y \cdot z)) = 0$ or $\mu_{P_G}(y) = 0$. Thus, $\mu_{P_G}(x \cdot z) \geq 0 = \min\{\mu_{P_G}(x \cdot (y \cdot z)), \mu_{P_G}(y)\}$.

Case 1': Suppose $x \cdot (y \cdot z) \in G$ and $y \in G$. Since G is an IUP-ideal of X, we have $x \cdot z \in G$. Thus, $\gamma_{P_G}(x \cdot z) = 0 \leq 0 = \max\{0, 0\} = \max\{\gamma_{P_G}(x \cdot (y \cdot z)), \gamma_{P_G}(y)\}$.

Case 2': Suppose $x \cdot (y \cdot z) \notin G$ or $y \notin G$. Then $\gamma_{P_G}(x \cdot (y \cdot z)) = 1$ or $\gamma_{P_G}(y) = 1$. Thus, $\gamma_{P_G}(x \cdot z) \leq 1 = \max\{\gamma_{P_G}(x \cdot (y \cdot z)), \gamma_{P_G}(y)\}$.

Hence, the characteristic PFS P_G is a Pythagorean fuzzy IUP-ideal of X.

Conversely, assume that the characteristic PFS P_G is a Pythagorean fuzzy IUP-ideal of X. Since μ_{P_G} satisfies (25), it follows from Lemma 3.2 that $0 \in G$. Next, let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in G$ and $y \in G$. Then $\mu_{P_G}(x \cdot (y \cdot z)) = 1$ and $\mu_{P_G}(y) = 1$. Thus, $\min\{\mu_{P_G}(x \cdot (y \cdot z)), \mu_{P_G}(y)\} = 1$. By (27), we have $\mu_{P_G}(x \cdot z) \geq \min\{\mu_{P_G}(x \cdot (y \cdot z)), \mu_{P_G}(y)\} = 1$, that is, $\mu_{P_G}(x \cdot z) = 1$. Hence, $x \cdot z \in G$, so G is an IUP-ideal of X. □

Theorem 3.11. *A nonempty subset G is an IUP-filter of X if and only if the characteristic PFS P_G is a Pythagorean fuzzy IUP-filter of X.*

Proof: Assume that G is an IUP-filter of X. Since $0 \in G$, it follows from Lemma 3.2 that μ_{P_G} and γ_{P_G} satisfy (25) and (26), respectively. Next, let $x, y \in X$.

Case 1: Suppose $x \cdot y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\mu_{P_G}(y) = 1 \geq 1 = \min\{1, 1\} = \min\{\mu_{P_G}(x \cdot y), \mu_{P_G}(x)\}$.

Case 2: Suppose $x \cdot y \notin G$ or $x \notin G$. Then $\mu_{P_G}(x \cdot y) = 0$ or $\mu_{P_G}(x) = 0$. Thus, $\mu_{P_G}(y) \geq 0 = \min\{\mu_{P_G}(x \cdot y), \mu_{P_G}(x)\}$.

Case 1': Suppose $x \cdot y \in G$ and $x \in G$. Since G is an IUP-filter of X, we have $y \in G$. Thus, $\gamma_{P_G}(y) = 0 \leq 0 = \max\{0, 0\} = \max\{\gamma_{P_G}(x \cdot y), \gamma_{P_G}(x)\}$.

Case 2': Suppose $x \cdot y \notin G$ or $x \notin G$. Then $\gamma_{P_G}(x \cdot y) = 1$ or $\gamma_{P_G}(x) = 1$. Thus, $\gamma_{P_G}(y) \leq 1 = \max\{\gamma_{P_G}(x \cdot y), \gamma_{P_G}(x)\}$.

Hence, the characteristic PFS P_G is a Pythagorean fuzzy IUP-filter of X.

Conversely, assume that the characteristic PFS P_G is a Pythagorean fuzzy IUP-filter of X. Since μ_{P_G} satisfies (25), it follow from Lemma 3.2 that $0 \in G$. Next, let $x, y \in$

G be such that $x \cdot y \in G$ and $x \in G$. Then $\mu_{P_G}(x \cdot y) = 1$ and $\mu_{P_G}(x) = 1$. Thus, $\min\{\mu_{P_G}(x \cdot y), \mu_{P_G}(x)\} = 1$. By (29), we have $\mu_{P_G}(y) = \min\{\mu_{P_G}(x \cdot y), \mu_{P_G}(x)\} = 1$, that is, $\mu_{P_G}(y) = 1$. Hence, $y \in G$, so G is an IUP-filter of X. □

Theorem 3.12. *A nonempty subset G is a strong IUP-ideal of X if and only if the characteristic PFS P_G is a Pythagorean fuzzy strong IUP-ideal of X.*

Proof: It is straightforward by Theorem 3.2. □

The following two lemmas are easy to prove, so we will omit them.

Lemma 3.3. *Let f be a fuzzy set in a nonempty set X. Then the following statements hold:*

$$(\forall x, y \in X)\, (1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}) \tag{35}$$

$$(\forall x, y \in X)\, (1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}) \tag{36}$$

Lemma 3.4. *Let f be a fuzzy set in a nonempty set X. Then the following statements hold:*

$$(\forall x, y, z \in X)\, \big(f(z) \geq \min\{f(x), f(y)\} \Leftrightarrow \overline{f}(z) \leq \max\{\overline{f}(x), \overline{f}(y)\}\big) \tag{37}$$

$$(\forall x, y, z \in X)\, \big(f(z) \leq \max\{f(x), f(y)\} \Leftrightarrow \overline{f}(z) \geq \min\{\overline{f}(x), \overline{f}(y)\}\big) \tag{38}$$

Before presenting the theorems on the relationship between Pythagorean fuzzy sets and their complements, it is important to understand their basic concept. Pythagorean fuzzy sets extend traditional fuzzy sets by including hesitation degrees. The following theorem highlights the key relationship between these sets and their complements.

Theorem 3.13. *A PFS P is a Pythagorean fuzzy IUP-subalgebra of X if and only if the FSs μ_P and $\overline{\gamma}_P$ satisfy* (23)*, and the FSs $\overline{\mu}_P$ and γ_P satisfy* (24)*.*

Proof: Assume that P is a Pythagorean fuzzy IUP-subalgebra of X. Then

$$\mu_P(x \cdot y) \geq \min\{\mu_P(x), \mu_P(y)\},$$
$$\gamma_P(x \cdot y) \leq \max\{\gamma_P(x), \gamma_P(y)\}.$$

Thus,

$$\overline{\mu}_P(x \cdot y) \leq \max\{\overline{\mu}_P(x), \overline{\mu}_P(y)\}, \quad \text{(by (37))}$$
$$\overline{\gamma}_P(x \cdot y) \geq \min\{\overline{\gamma}_P(x), \overline{\gamma}_P(y)\}. \quad \text{(by (38))}$$

Hence, the FSs μ_P and $\overline{\gamma}_P$ satisfy (23), and the FSs $\overline{\mu}_P$ and γ_P satisfy (24).

Conversely, assume that the FSs μ_P and $\overline{\gamma}_P$ satisfy (23), and the FSs $\overline{\mu}_P$ and γ_P satisfy (24). Then we have μ_P satisfies (23) and γ_P satisfies (24). Thus, P is a Pythagorean fuzzy IUP-subalgebra of X. □

Theorem 3.14. *A PFS P is a Pythagorean fuzzy IUP-ideal of X if and only if the FSs μ_P and $\overline{\gamma}_P$ satisfy* (25) *and* (27)*, and the FSs $\overline{\mu}_P$ and γ_P satisfy* (26) *and* (28)*.*

Proof: Assume that P is a Pythagorean fuzzy IUP-ideal of X. Then

$$\mu_P(0) \geq \mu_P(x),$$
$$\gamma_P(0) \leq \gamma_P(x),$$
$$\mu_P(x \cdot z) \geq \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\},$$
$$\gamma_P(x \cdot z) \leq \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\}.$$

Thus,

$$\overline{\mu}_P(0) \leq \overline{\mu}_P(x),$$

$$\begin{aligned}\overline{\gamma}_P(0) &\geq \overline{\gamma}_P(x),\\ \overline{\mu}_P(x \cdot z) &\leq \max\{\overline{\mu}_P(x \cdot (y \cdot z)), \overline{\mu}_P(y)\},\\ \overline{\gamma}_P(x \cdot z) &\geq \min\{\overline{\gamma}_P(x \cdot (y \cdot z)), \overline{\gamma}_P(y)\}.\end{aligned}$$

Hence, the FSs μ_P and $\overline{\gamma}_P$ satisfy (25) and (27), and the FSs $\overline{\mu}_P$ and γ_P satisfy (26) and (28).

Conversely, assume that the FSs μ_P and $\overline{\gamma}_P$ satisfy (25) and (27), and the FSs $\overline{\mu}_P$ and γ_P satisfy (26) and (28). Then we have μ_P satisfies (25) and (27) and γ_P satisfies (26) and (28). Hence, P is a Pythagorean fuzzy IUP-ideal of X. □

Theorem 3.15. *A PFS P is a Pythagorean fuzzy IUP-filter of X if and only if the FSs μ_P and $\overline{\gamma}_P$ satisfies* (25) *and* (29)*, and the FSs $\overline{\mu}_P$ and γ_P satisfies* (26) *and* (30)*.*

Proof: Assume that P is a Pythagorean fuzzy IUP-filter of X. Then

$$\begin{aligned}\mu_P(0) &\geq \mu_P(x),\\ \gamma_P(0) &\leq \gamma_P(x),\\ \mu_P(y) &\geq \min\{\mu_P(x \cdot y), \mu_P(x)\},\\ \gamma_P(y) &\leq \max\{\gamma_P(x \cdot y), \gamma_P(x)\}.\end{aligned}$$

Thus,

$$\begin{aligned}\overline{\mu}_P(0) &\leq \overline{\mu}_P(x),\\ \overline{\gamma}_P(0) &\geq \overline{\gamma}_P(x),\\ \overline{\mu}_P(y) &\leq \max\{\overline{\mu}_P(x \cdot y), \overline{\mu}_P(x)\},\\ \overline{\gamma}_P(y) &\geq \min\{\overline{\gamma}_P(x \cdot y), \overline{\gamma}_P(x)\}.\end{aligned}$$

Hence, the FSs μ_P and $\overline{\gamma}_P$ satisfy (25) and (29), and the FSs $\overline{\mu}_P$ and γ_P satisfy (26) and (30).

Conversely, assume that the FSs μ_P and $\overline{\gamma}_P$ satisfy (25) and (29), and the FSs $\overline{\mu}_P$ and γ_P satisfy (26) and (30). Then we have μ_P satisfies (25) and (29) and γ_P satisfies (26) and (30). Hence, P is a Pythagorean fuzzy IUP-filter of X. □

Theorem 3.16. *A PFS P is a Pythagorean fuzzy strong IUP-ideal of X if and only if the FSs μ_P and $\overline{\gamma}_P$ satisfy* (31)*, and the FSs $\overline{\mu}_P$ and γ_P satisfy* (32)*.*

Proof: It is straightforward by Theorem 3.2. □

Theorem 3.17. *A PFS P is a Pythagorean fuzzy IUP-subalgebra of X if and only if PFS $\blacksquare P = (\mu_P, \overline{\mu}_P)$ and $\blacktriangle P = (\overline{\gamma}_P, \gamma_P)$ are Pythagorean fuzzy IUP-subalgebras of X.*

Proof: It is straightforward by Theorem 3.13. □

Theorem 3.18. *A PFS P is a Pythagorean fuzzy IUP-ideal of X if and only if PFS $\blacksquare P = (\mu_P, \overline{\mu}_P)$ and $\blacktriangle P = (\overline{\gamma}_P, \gamma_P)$ are Pythagorean fuzzy IUP-ideals of X.*

Proof: It is straightforward by Theorem 3.14. □

Theorem 3.19. *A PFS P is a Pythagorean fuzzy IUP-filter of X if and only if PFS $\blacksquare P = (\mu_P, \overline{\mu}_P)$ and $\blacktriangle P = (\overline{\gamma}_P, \gamma_P)$ are Pythagorean fuzzy IUP-filters of X.*

Proof: It is straightforward by Theorem 3.15. □

Theorem 3.20. *A PFS P is a Pythagorean fuzzy strong IUP-ideal of X if and only if PFS $\blacksquare P = (\mu_P, \overline{\mu}_P)$ and $\blacktriangle P = (\overline{\gamma}_P, \gamma_P)$ are Pythagorean fuzzy strong IUP-ideals of X.*

Proof: It is straightforward by Theorem 3.16. □

Definition 3.6. *Let f be a fuzzy set in a nonempty set X. For any $t \in [0,1]$, the sets*

$$U(f;t) = \{x \in X | f(x) \geq t\}, \tag{39}$$

$$L(f;t) = \{x \in X | f(x) \leq t\} \tag{40}$$

are called an upper t-level subset and a lower t-level subset of f, respectively. The sets

$$U^{+}(f;t) = \{x \in X | f(x) > t\}, \tag{41}$$

$$L^{-}(f;t) = \{x \in X | f(x) < t\} \tag{42}$$

are called an upper t-strong level subset and a lower t-strong level subset of f, respectively.

Before presenting the theorems that show the relationship between level subsets and their corresponding Pythagorean fuzzy sets, it is important to grasp the key concepts. Level subsets help characterize Pythagorean fuzzy sets by detailing the distribution of membership degrees. The following theorem formalizes this relationship, offering insights into the structure of Pythagorean fuzzy sets.

Theorem 3.21. *A PFS P is a Pythagorean fuzzy IUP-subalgebra of X if and only if for all $t, s \in [0,1]$, the sets $U(\mu_P;t)$ and $L(\gamma_P;s)$ are either empty or IUP-subalgebras of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-subalgebra of X. Let $t \in [0,1]$ be such that $U(\mu_P;t) \neq \emptyset$. Let $x, y \in U(\mu_P;t)$. Then $\mu_P(x) \geq t$ and $\mu_P(y) \geq t$. Thus, $\min\{\mu_P(x), \mu_P(y)\} \geq t$. By (23), we have $\mu_P(x \cdot y) \geq \min\{\mu_P(x), \mu_P(y)\} \geq t$, that is, $\mu_P(x \cdot y) \geq t$. Thus, $x \cdot y \in U(\mu_P;t)$. Hence, $U(\mu_P;t)$ is an IUP-subalgebra of X.

Let $s \in [0,1]$ be such that $L(\gamma_P;s) \neq \emptyset$. Let $x, y \in L(\gamma_P;s)$. Then $\gamma_P(x) \leq s$ and $\gamma_P(y) \leq s$. Thus, $\max\{\gamma_P(x), \gamma_P(y)\} \leq s$. By (24), we have $\gamma_P(x \cdot y) \leq \max\{\gamma_P(x), \gamma_P(y)\} \leq s$, that is, $\gamma_P(x \cdot y) \leq s$. Thus, $x \cdot y \in L(\gamma_P;s)$. Hence, $L(\gamma_P;s)$ is an IUP-subalgebra of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U(\mu_P;t)$ and $L(\gamma_P;s)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Let $t = \min\{\mu_P(x), \mu_P(y)\}$. Then $\mu_P(x) \geq t$ and $\mu_P(y) \geq t$. Thus, $x, y \in U(\mu_P;t) \neq \emptyset$. By the assumption, we have $U(\mu_P;t)$ is an IUP-subalgebra of X. By (17), we have $x \cdot y \in U(\mu_P;t)$. Thus, $\mu_P(x \cdot y) \geq t = \min\{\mu_P(x), \mu_P(y)\}$.

Let $x, y \in X$. Let $s = \max\{\gamma_P(x), \gamma_P(y)\}$. Then $\gamma_P(x) \leq s$ and $\gamma_P(y) \leq s$. Thus, $x, y \in L(\gamma_P;s) \neq \emptyset$. By the assumption, we have $L(\gamma_P;s)$ is an IUP-subalgebra of X. By (17), we have $x \cdot y \in L(\gamma_P;s)$. Thus, $\gamma_P(x \cdot y) \leq s = \max\{\gamma_P(x), \gamma_P(y)\}$.

Hence, P is a Pythagorean fuzzy IUP-subalgebra of X. □

Theorem 3.22. *A PFS P in X is a Pythagorean fuzzy IUP-ideal of X if and only if for all $t, s \in [0,1]$, the sets $U(\mu_P;t)$ and $L(\gamma_P;s)$ are either empty or IUP-ideals of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-ideal of X. Let $t \in [0,1]$ be such that $U(\mu_P;t) \neq \emptyset$. Let $a \in U(\mu_P;t)$. Then $\mu_P(a) \geq t$. By (23), we have $\mu_P(0) \geq \mu_P(a) \geq t$. Thus, $0 \in U(\mu_P;t)$. Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in U(\mu_P;t)$ and $y \in U(\mu_P;t)$. Then $\mu_P(x \cdot (y \cdot z)) \geq t$ and $\mu_P(y) \geq t$. Thus, $\min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\} \geq t$. By (27), we have $\mu_P(x \cdot z) \geq \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\} \geq t$. Thus, $x \cdot z \in U(\mu_P;t)$. Hence, $U(\mu_P;t)$ is an IUP-ideal of X.

Let $s \in [0,1]$ be such that $L(\gamma_P;s) \neq \emptyset$. Let $b \in L(\gamma_P;s)$. Then $\gamma_P(b) \leq s$. By (26), we have $\gamma_P(0) \leq \gamma_P(b) \leq s$. Thus, $0 \in L(\gamma_P;s)$. Let $x, y, z \in X$ be such that $x \cdot (y \cdot z) \in L(\gamma_P;s)$ and $y \in L(\gamma_P;s)$. Then $\gamma_P(x \cdot (y \cdot z)) \leq s$ and $\gamma_P(y) \leq s$. Thus, $\max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\} \leq s$. By (28), we have $\gamma_P(x \cdot z) \leq \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\} \leq s$. Thus, $x \cdot z \in L(\gamma_P;s)$. Hence, $L(\gamma_P;s)$ is an IUP-ideal of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U(\mu_P; t)$ and $L(\gamma_P; s)$ are either empty or IUP-ideals of X. Let $t = \mu_P(x)$. Then $\mu_P(x) \geq t$. Thus, $x \in U(\mu_P; t) \neq \emptyset$. By the assumption, we have $U(\mu_P; t)$ is an IUP-ideal of X. By (18), we have $0 \in U(\mu_P; t)$. Then $\mu_P(0) \geq t = \mu_P(x)$. Let $x, y, z \in X$. Let $t = \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\}$. Then $\mu_P(x \cdot (y \cdot z)) \geq t$ and $\mu_P(y) \geq t$. Thus, $x \cdot (y \cdot z), y \in U(\mu_P; t) \neq \emptyset$. By the assumption, we have $U(\mu_P; t)$ is an IUP-ideal of X. By (20), we have $x \cdot z \in U(\mu_P; t)$. Thus, $\mu_P(x \cdot z) \geq t = \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\}$.

Let $x \in X$. Let $s = \gamma_P(x)$. Then $\gamma_P(x) \leq s$. Thus, $x \in L(\gamma_P; s) \neq \emptyset$. By the assumption, we have $L(\gamma_P; s)$ is an IUP-ideal of X. By (18), we have $0 \in L(\gamma_P; s)$. Then $\gamma_P(0) \leq s = \gamma_P(x)$. Let $x, y, z \in X$. Let $s = \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\}$. Then $\gamma_P(x \cdot (y \cdot z)) \leq s$ and $\gamma_P(y) \leq s$. Thus, $x \cdot (y \cdot z), y \in L(\gamma_P; s) \neq \emptyset$. By the assumption, we have $U(\gamma_P; s)$ is an IUP-ideal of X. By (20), we have $x \cdot z \in L(\gamma_P; s)$. Thus, $\gamma_P(x \cdot z) \leq s = \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\}$.

Hence, P is a Pythagorean fuzzy IUP-ideal of X. □

Theorem 3.23. *A PFS P in X is a Pythagorean fuzzy IUP-filter of X if and only if for all $t, s \in [0,1]$, the sets $U(\mu_P; t)$ and $L(\gamma_P; s)$ are either empty or IUP-filters of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-filter of X. Let $t \in [0,1]$ be such that $U(\mu_P; t) \neq \emptyset$. Let $a \in U(\mu_P; t)$. Then $\mu_P(a) \geq t$. By (25), we have $\mu_P(0) \geq \mu_P(a) \geq t$. Thus, $0 \in U(\mu_P; t)$. Let $x, y \in X$ be such that $x \cdot y \in U(\mu_P; t)$ and $x \in U(\mu_P; t)$. Then $\mu_P(x \cdot y) \geq t$ and $\mu_P(x) \geq t$. Thus, $\min\{\mu_P(x \cdot y), \mu_P(x)\} \geq t$. By (29), we have $\mu_P(y) \geq \min\{\mu_P(x \cdot y), \mu_P(x)\} \geq t$. Thus, $y \in U(\mu_P; t)$. Hence, $U(\mu_P; t)$ is an IUP-filter of X.

Let $s \in [0,1]$ be such that $L(\gamma_P; s) \neq \emptyset$. Let $b \in L(\gamma_P; s)$. Then $\gamma_P(b) \leq s$. By (26), we have $\gamma_P(0) \leq \gamma_P(b) \leq s$. Thus, $0 \in L(\gamma_P; s)$. Let $x, y \in X$ be such that $x \cdot y \in L(\gamma_P; s)$ and $x \in L(\gamma_P; s)$. Then $\gamma_P(x \cdot y) \leq s$ and $\gamma_P(x) \leq s$. Thus, $\max\{\gamma_P(x \cdot y), \gamma_P(x)\} \leq s$. By (30), we have $\gamma_P(y) \leq \max\{\gamma_P(x \cdot y), \gamma_P(x)\} \leq s$. Thus, $y \in L(\gamma_P; s)$. Hence, $L(\gamma_P; s)$ is an IUP-filter of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U(\mu_P; t)$ and $L(\gamma_P; s)$ are either empty or IUP-filters of X. Let $x \in X$. Let $t = \mu_P(x)$. Then $\mu_P(x) \geq t$. Thus, $x \in U(\mu_P; t) \neq \emptyset$. By the assumption, we have $U(\mu_P; t)$ is an IUP-filter of X. By (18), we have $0 \in U(\mu_P; t)$. Then $\mu_P(0) \geq t = \mu_P(x)$. Let $x, y \in X$. Let $t = \min\{\mu_P(x \cdot y), \mu_P(x)\}$. Then $\mu_P(x \cdot y) \geq t$ and $\mu_P(x) \geq t$. Thus, $x \cdot y, x \in U(\mu_P; t) \neq \emptyset$. By the assumption, we have $U(\mu_P; t)$ is an IUP-filter of X. By (19), we have $y \in U(\mu_P; t)$. Thus, $\mu_P(y) \geq t = \min\{\mu_P(x \cdot y), \mu_P(x)\}$.

Let $x \in X$. Let $s = \gamma_P(x)$. Then $\gamma_P(x) \leq s$. Thus, $x \in L(\gamma_P; s) \neq \emptyset$. By the assumption, we have $L(\gamma_P; s)$ is an IUP-filter of X. By (18), we have $0 \in L(\gamma_P; s)$. Then $\gamma_P(0) \leq s = \gamma_P(x)$. Let $x, y \in X$. Let $s = \max\{\gamma_P(x \cdot y), \gamma_P(x)\}$. Then $\gamma_P(x \cdot y) \leq s$ and $\gamma_P(x) \leq s$. Thus, $x \cdot y, x \in L(\gamma_P; s) \neq \emptyset$. By the assumption, we have $L(\gamma_P; s)$ is an IUP-filter of X. By (19), we have $y \in L(\gamma_P; s)$. Thus, $\gamma_P(y) \leq s = \max\{\gamma_P(x \cdot y), \gamma_P(x)\}$.

Hence, P is a Pythagorean fuzzy IUP-filter of X. □

Theorem 3.24. *A PFS P in X is a Pythagorean fuzzy strong IUP-ideal of X if and only if for all $t, s \in [0,1]$, the sets $U(\mu_P; t)$ and $L(\gamma_P; s)$ are either empty or strong IUP-ideals of X.*

Proof: It is straightforward by Theorem 3.2. □

Theorem 3.25. *A PFS P in X is a Pythagorean fuzzy IUP-subalgebra of X if and only if for all $t, s \in [0,1]$, the sets $U^+(\mu_P; t)$ and $L^-(\gamma_P; s)$ are either empty or IUP-subalgebras of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-subalgebra of X. Let $t \in [0,1]$ be such that $U^{+}(\mu_P;t) \neq \emptyset$. Let $x, y \in U^{+}(\mu_P;t)$. Then $\mu_P(x) > t$ and $\mu_P(y) > t$. Thus, $\min\{\mu_P(x), \mu_P(y)\} > t$. By (23), we have $\mu_P(x \cdot y) \geq \min\{\mu_P(x), \mu_P(y)\} > t$. Thus, $x \cdot y \in U^{+}(\mu_P;t)$. Hence, $U^{+}(\mu_P;t)$ is an IUP-subalgebra of X.

Let $s \in [0,1]$ be such that $L^{-}(\gamma_P;s) \neq \emptyset$. Let $x, y \in L^{-}(\gamma_P;s)$. Then $\gamma_P(x) < s$ and $\gamma_P(y) < s$. Thus, $\max\{\gamma_P(x), \gamma_P(y)\} < s$. By (24), we have $\gamma_P(x\cdot y) \leq \max\{\gamma_P(x), \gamma_P(y)\}$ $< s$. Thus, $x \cdot y \in L^{-}(\gamma_P;s)$. Hence, $L^{-}(\gamma_P;s)$ is an IUP-subalgebra of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U^{+}(\mu_P;t)$ and $L^{-}(\gamma_P;s)$ are either empty or IUP-subalgebras of X. Let $x, y \in X$. Assume that $\mu_P(x \cdot y) < \min\{\mu_P(x),$ $\mu_P(y)\}$. Let $t = \mu_P(x\cdot y)$. Then $\mu_P(x) > t$ and $\mu_P(y) > t$. Thus, $x, y \in U^{+}(\mu_P;t)$. By the assumption, we have $U^{+}(\mu_P;t)$ is an IUP-subalgebra. By (17), we have $x \cdot y \in U^{+}(\mu_P;t)$. So $\mu_P(x\cdot y) > t = \mu_P(x\cdot y)$, which is a contradiction. Thus, $\mu_P(x\cdot y) \geq \min\{\mu_P(x), \mu_P(y)\}$.

Let $x, y \in X$. Assume that $\gamma_P(x \cdot y) > \max\{\gamma_P(x), \gamma_P(y)\}$. Let $s = \gamma_P(x \cdot y)$. Then $\gamma_P(x) < s$ and $\gamma_P(y) < s$. Thus, $x, y \in L^{-}(\gamma_P;s)$. By the assumption, we have $L^{-}(\gamma_P;s)$ is an IUP-subalgebra. By (17), we have $x \cdot y \in L^{-}(\gamma_P;s)$. So $\gamma_P(x \cdot y) < s = \gamma_P(x \cdot y)$, which is a contradiction. Thus, $\gamma_P(x \cdot y) \leq \max\{\gamma_P(x), \gamma_P(y)\}$.

Hence, P is a Pythagorean fuzzy IUP-subalgebra of X. □

Theorem 3.26. *A PFS P in X is a Pythagorean fuzzy IUP-ideal of X if and only if for all $t, s \in [0,1]$, the sets $U^{+}(\mu_P;t)$ and $L^{-}(\gamma_P;s)$ are either empty or IUP-ideals of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-ideal of X. Let $t \in [0,1]$ be such that $U^{+}(\mu_P;t) \neq \emptyset$. Let $a \in U^{+}(\mu_P;t)$. Then $\mu_P(a) > t$. By (25), we have $\mu_P(0) \geq \mu_P(a) > t$. Thus, $0 \in U^{+}(\mu_P;t)$. Let $x, y, z \in U^{+}(\mu_P;t)$ be such that $x \cdot (y \cdot z), y \in U^{+}(\mu_P;t)$. Then $\mu_P(x \cdot (y \cdot z)) > t$ and $\mu_P(y) > t$. Thus, $\min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\} > t$. By (27), we have $\mu_P(x \cdot z) \geq \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\} > t$. Thus, $x \cdot z \in U^{+}(\mu_P;t)$. Hence, $U^{+}(\mu_P;t)$ is an IUP-ideal of X.

Let $s \in [0,1]$ be such that $L^{-}(\gamma_P;s) \neq \emptyset$. Let $b \in L^{-}(\gamma_P;s)$. Then $\gamma_P(b) < s$. By (26), we have $\gamma_P(0) \leq \gamma_P(b) < s$. Thus, $0 \in L^{-}(\gamma_P;s)$. Let $x, y, z \in L^{-}(\gamma_P;s)$ be such that $x \cdot (y \cdot z), y \in L^{-}(\gamma_P;s)$. Then $\gamma_P(x \cdot (y \cdot z)) < s$ and $\gamma_P(y) < s$. Thus, $\max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\} < s$. By (28), we have $\gamma_P(x \cdot z) \leq \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\} < s$. Thus, $x \cdot z \in L^{-}(\gamma_P;s)$. Hence, $L^{-}(\gamma_P;s)$ is an IUP-ideal of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U^{+}(\mu_P;t)$ and $L^{-}(\gamma_P;s)$ are either empty or IUP-ideals of X. Let $x \in X$. Assume that $\mu_P(0) < \mu_P(x)$. Let $t = \mu_P(0)$. Then $x \in U^{+}(\mu_P;t) \neq \emptyset$. By the assumption, we have $U^{+}(\mu_P;t)$ is an IUP-ideal of X. By (18), we have $0 \in U^{+}(\mu_P;t)$. So $\mu_P(0) > t = \mu_P(0)$, which is a contradiction. Thus, $\mu_P(0) \geq \mu_P(x)$. Let $x, y, z \in X$. Assume that $\mu_P(x \cdot z) < \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\}$. Let $t = \mu_P(x \cdot z)$. Then $x \cdot (y \cdot z), y \in U^{+}(\mu_P;t) \neq \emptyset$. By the assumption, we have $U^{+}(\mu_P;t)$ is an IUP-ideal of X. By (20), we have $x \cdot z \in U^{+}(\mu_P;t)$. So $\mu_P(x \cdot z) > t = \mu_P(x \cdot z)$, which is a contradiction. Thus, $\mu_P(x \cdot z) \geq \min\{\mu_P(x \cdot (y \cdot z)), \mu_P(y)\}$.

Let $x \in X$. Assume that $\gamma_P(0) > \gamma_P(x)$. Let $s = \gamma_P(0)$. Then $x \in L^{-}(\gamma_P;s) \neq \emptyset$. By the assumption, we have $L^{-}(\gamma_P;s)$ is an IUP-ideal of X. By (18), we have $0 \in L^{-}(\gamma_P;s)$. So $\gamma_P(0) < s = \gamma_P(0)$, which is a contradiction. Thus, $\gamma_P(0) \leq \gamma_P(x)$. Let $x, y, z \in X$. Assume that $\gamma_P(x \cdot z) > \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\}$. Let $s = \gamma_P(x \cdot z)$. Then $x \cdot (y \cdot z), y \in L^{-}(\gamma_P;s) \neq \emptyset$. By the assumption, we have $L^{-}(\gamma_P;s)$ is an IUP-ideal of X. By (20), we have $x \cdot z \in L^{-}(\gamma_P;s)$. So $\gamma_P(x \cdot z) < s = \gamma_P(x \cdot z)$, which is a contradiction. Thus, $\gamma_P(x \cdot z) \leq \max\{\gamma_P(x \cdot (y \cdot z)), \gamma_P(y)\}$.

Hence, P is a Pythagorean fuzzy IUP-ideal of X. □

Theorem 3.27. *A PFS P in X is a Pythagorean fuzzy IUP-filter of X if and only if for all $t, s \in [0,1]$, the sets $U^+(\mu_P;t)$ and $L^-(\gamma_P;s)$ are either empty or IUP-filters of X.*

Proof: Assume that P is a Pythagorean fuzzy IUP-filter of X. Let $t \in [0,1]$ be such that $U^+(\mu_P;t) \neq \emptyset$. Let $a \in U^+(\mu_P;t)$. Then $\mu_P(a) > t$. By (25), we have $\mu_P(0) \geq \mu_P(a) > t$. Thus, $0 \in U^+(\mu_P;t)$. Let $x, y \in U^+(\mu_P;t)$ be such that $x \cdot y, x \in U^+(\mu_P;t)$. Then $\mu_P(x \cdot y) > t$ and $\mu_P(x) > t$. Thus, $\min\{\mu_P(x \cdot y), \mu_P(x)\} > t$. By (29), we have $\mu_P(y) \geq \min\{\mu_P(x \cdot y), \mu_P(x)\} > t$. Thus, $y \in U^+(\mu_P;t)$. Hence, $U^+(\mu_P;t)$ is an IUP-filter of X.

Let $s \in [0,1]$ be such that $L^-(\gamma_P;s) \neq \emptyset$. Let $b \in L^-(\gamma_P;s)$. Then $\gamma_P(b) < s$. By (26), we have $\gamma_P(0) \leq \gamma_P(b) < s$. Thus, $0 \in L^-(\gamma_P;s)$. Let $x, y \in L^-(\gamma_P;s)$ be such that $x \cdot y, x \in L^-(\gamma_P;s)$. Then $\gamma_P(x \cdot y) < s$ and $\gamma_P(x) < s$. Thus, $\max\{\gamma_P(x \cdot y), \gamma_P(x)\} < s$. By (30), we have $\gamma_P(y) \leq \max\{\gamma_P(x \cdot y), \gamma_P(x)\} < s$. Thus, $y \in L^-(\gamma_P;s)$. Hence, $L^-(\gamma_P;s)$ is an IUP-filter of X.

Conversely, assume that for all $t, s \in [0,1]$, the sets $U^+(\mu_P;t)$ and $L^-(\gamma_P;s)$ are either empty or IUP-filters of X. Let $x \in X$. Assume that $\mu_P(0) < \mu_P(x)$. Let $t = \mu_P(0)$. Then $x \in U^+(\mu_P;t) \neq \emptyset$. By the assumption, we have $U^+(\mu_P;t)$ is an IUP-ideal of X. By (18), we have $0 \in U^+(\mu_P;t)$. So $\mu_P(0) > t = \mu_P(0)$, which is a contradiction. Thus, $\mu_P(0) \geq \mu_P(x)$. Let $x, y \in X$. Assume that $\mu_P(y) < \min\{\mu_P(x \cdot y), \mu_P(x)\}$. Let $t = \mu_P(y)$. Then $x \cdot y, x \in U^+(\mu_P;t) \neq \emptyset$. By the assumption, we have $U^+(\mu_P;t)$ is an IUP-filter of X. By (19), we have $y \in U^+(\mu_P;t)$. So $\mu_P(y) > t = \mu_P(y)$, which is a contradiction. Thus, $\mu_P(y) \geq \min\{\mu_P(x \cdot y), \mu_P(x)\}$.

Let $x \in X$. Assume that $\gamma_P(0) > \gamma_P(x)$. Let $s = \gamma_P(0)$. Then $x \in L^-(\gamma_P;s) \neq \emptyset$. By the assumption, we have $L^-(\gamma_P;s)$ is an IUP-filter of X. By (18), we have $0 \in L^-(\gamma_P;s)$. So $\gamma_P(0) < s = \gamma_P(0)$, which is a contradiction. Thus, $\gamma_P(0) \leq \gamma_P(x)$. Let $x, y \in X$. Assume that $\gamma_P(y) > \max\{\gamma_P(x \cdot y), \gamma_P(x)\}$. Let $s = \gamma_P(y)$. Then $x \cdot y, x \in L^-(\gamma_P;s) \neq \emptyset$. By the assumption, we have $L^-(\gamma_P;s)$ is an IUP-filter of X. By (19), we have $y \in L^-(\gamma_P;s)$. So $\gamma_P(y) < s = \gamma_P(y)$, which is a contradiction. Thus, $\gamma_P(y) \leq \max\{\gamma_P(x \cdot y), \gamma_P(x)\}$.

Hence, P is a Pythagorean fuzzy IUP-filter of X. □

Theorem 3.28. *A PFS P in X is a Pythagorean fuzzy strong IUP-ideal of X if and only if for all $t, s \in [0,1]$, the sets $U^+(\mu_P;t)$ and $L^-(\gamma_P;s)$ are either empty or strong IUP-ideals of X.*

Proof: It is straightforward by Theorem 3.2. □

4. **Conclusions.** In our paper, we introduce pioneering concepts such as Pythagorean fuzzy IUP-subalgebras, Pythagorean fuzzy IUP-ideals, Pythagorean fuzzy IUP-filters, and Pythagorean fuzzy strong IUP-ideals. Our research delves into their crucial properties and uncovers how these notions relate to complement, characteristic functions, and level subsets. These insights illuminate the complex relationships within the structure of PFSs and IUP-algebras, offering fresh perspectives and practical applications in mathematical and real-world problem-solving.

The ongoing study of PFSs represents a dynamic evolution in fuzzy set theory, inspiring researchers to explore their potential in conjunction with IUP-algebras. This convergence is poised to drive innovations that shape the future of fuzzy mathematics and its diverse applications. In particular, PFSs are likely to enhance decision-making in artificial intelligence and machine learning by providing nuanced interpretations of uncertainty, improving healthcare decision support systems for personalized treatments, and refining economic and financial models for better risk assessment. Additionally, PFSs could assist in environmental management by modelling uncertain factors and enhance smart city initiatives by processing ambiguous data from Internet of Things (IoT) devices. As research progresses, the application of PFSs is expected to expand significantly across various interdisciplinary fields.

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