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INPUT-TO-STATE STABILITY FOR NONLINEAR STOCHASTIC TIME-VARYING SWITCHED DELAY SYSTEMS

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Abstract. *In this paper, our focus is on a class of nonlinear time-varying switched stochastic delay systems. By using one integral Halanay inequality, Lyapunov function and dwell time, sufficient conditions are provided, respectively, to ensure the input-to-state stability and the stochastic input-to-state stability for nonlinear time-varying switched stochastic delay systems. The feasibility of the result derived is checked by one example.*
Keywords: Nonlinear stochastic switched systems, Time-varying delay, Input-to-state stability

1. **Introduction.** Nonlinear stochastic systems (SS) have been widely used in many fields such as financial systems and population dynamical model [1], and the stochastic stability has been extensively investigated over the past few decades [2, 3, 10, 11, 12]. In [2], Kusher has systematically introduced some methodologies, which were further developed by Khasminskii in [3] to investigate the stochastic stability of nonlinear SS. Additionally, there exists time delay in some dynamical models such as chemical reaction-diffusion processes, aerodynamic vehicles and communication networks, which is an important source of systems' instability [4, 5, 6, 7, 8, 9]. In [10], Mao has provided some results to analyze the stochastic stability of nonlinear stochastic delay systems (SDS). Some results on the stochastic stability of nonlinear SDS have been successively presented, see [11, 12] and the references therein.

The continuous-time switched systems (CTSS) [13] can be seen as one of the hybrid continuous-time dynamical systems. The CTSS consist of a family of continuous-time subsystems and a rule orchestrating the switching between the subsystems. Some results on the stability analysis of CTSS have been presented in [14, 15] and the references therein. When the time delay and the stochastic perturbation are also considered in CTSS, the stability has been extensively analyzed in recent few years in [16, 17]. For example, in [16], by using the Razumikhin theorem, the stability of delay CTSS was analyzed. Three techniques have been provided to investigate the stability of delay CTSS

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with time delay, which are the dwell time in [18], the average dwell time in [19], and the mode-dependent average dwell time in [20]. The multiple Lyapunov functions mixed with these three techniques can obtain some less conservative stability criteria to some extent, respectively.

The input-to-state stability (ISS) and the integral input-to-state stability (iISS) were initially formulated by Sontag in [21]. Usually, the initial value and the input external disturbance can be fully reflected in these two stability concepts [22]. Some results have been presented on the ISS and iISS of nonlinear delay CTSS in [23, 24] and the references therein. The CTSS is a special case of non-autonomous systems [13]. Compared with autonomous systems, analyzing the stability of non-autonomous systems is more complicated [25]. The most obvious difference is that for autonomous linear systems, the eigenvalue analysis of its system matrix can establish its stability and give a complete answer. However, for non-autonomous linear systems, the stability is not guaranteed by the eigenvalue of its system matrix. Time-varying CTSS have considerable practical significance in many engineering applications such as robotic arm manipulations, spacecraft navigation system, and precision positioning of servo system of swing cylinder [26]. Thus, analyzing the stability of time-varying CTSS is a more challenging work. In [27], by using the Rauzmikhin theorem and the average impulsive interval, the ISS/iISS of impulsive time-varying CTSS with time delay have been studied. Note that although the stability analysis of non-autonomous delay systems has been discussed in [28], the results obtained are suitable for small time-varying delay. In [29], the problems on the ISS/iISS of delayed stochastic CTSS of neutral-type have been considered by using the Lyapunov function and the algebraic integral Halanay inequality, when the Lyapunov monotonicity does not have a sign-changed and unbounded time-varying coefficient. Under such situation, some results on stability analysis have been given in [30, 31]. To our knowledge, when the Lyapunov monotonicity does not have a sign-changed and unbounded time-varying coefficient, there are few papers which investigate the problems on the ISS/iISS of delayed stochastic CTSS.

This paper will consider the problems regarding the ISS/SISS of nonlinear switched SDS, when the external disturbance is absent. Note that in this paper, the Lyapunov monotonicity condition has a sign-changed and unbounded time-varying coefficient. One integral Halanay inequality will be established in Lemma 3.1, which can pave the way for studying the ISS/SISS. Finally, the feasibility of the result derived is checked by one example.

The following content of this paper is organized. The problem formulation and some preliminaries are given in Section 2; Section 3 presents the main results, in which the input-to-state stability for nonlinear stochastic time-varying switched delay systems is analyzed; Section 4 gives one example to show the feasibility of the theoretical result obtained; Finally, Section 5 gives conclusion.

Notation: Throughout this paper, unless otherwise specified, the notation used in this paper is the same as the one given in [30].

2. **Problem Statement and Preliminaries.** In this paper, we consider the following highly nonlinear stochastic switched time-varying systems with time-varying delay:

$$dx(t) = f_{\chi(t)}(t, x(t), x(t-\tau(t)), \varpi(t))dt + g_{\chi(t)}(t, x(t), x(t-\tau(t)))d\omega(t), \quad t \geq t_0, \tag{1}$$

with the initial value $\{x(\theta) : t_0 - \tau \leq \theta \leq t_0\} = \xi \in \mathcal{PC}([t_0 - \tau, t_0]; R^{n_x})$ and $\chi(t_0) \in \mathcal{N}$, where $\varpi(t) \in \mathcal{L}_\infty$ denotes the bounded disturbance input, $x(t) = [x_1(t), x_2(t), \ldots, x_{n_x}(t)]^T \in R^{n_x}$ and $x(t-\tau(t)) = [x_1(t-\tau(t)), x_2(t-\tau(t)), \ldots, x_{n_x}(t-\tau(t))]^T \in R^{n_x}$ are the state vector and the delayed state vector, respectively. $\mathcal{N} = \{1, 2, \ldots, \mathbb{N}\}$ denotes the index set, in which $\mathbb{N}$ is a positive integer denoting the number of modes. $\chi(\cdot):[t_0,+\infty)\to\mathcal{N}$ represents a non-random switching signal, which is assumed to be piecewise-continuous constant function from the right hand. For $\chi(t)$, the switching sequence is denoted as $\{(\iota_0,t_0),(\iota_1,t_1),\ldots,(\iota_k,t_k),\ldots|\iota_k\in\mathcal{N},k\in\mathbb{N}\}$, in which the ι_kth subsystem is only active when $t\in[t_k,t_{k+1})$ $(k\in\mathbb{N})$. $\tau(t)$ is the time-varying delay, which is a bounded function with $0\le\tau(t)\le\tau$. For any $\chi(t)=i\in\mathcal{N}$, $f_i(\cdot,\cdot,\cdot,\cdot):[t_0,+\infty)\times R^{n_x}\times R^{n_x}\times R^{n_\varpi}\to R^{n_x}$ and $g_i(\cdot,\cdot,\cdot):[t_0,+\infty)\times R^{n_x}\times R^{n_x}\to R^{n_x\times m}$ are two measurable functions, which denote the drift coefficient vector and the diffusion coefficient matrix in (1), respectively. Let $x(t,t_0,\xi,\chi(t_0))$ be the solution of systems (1) when $\varpi(t)=0$. For simplicity, $x(t)=x(t,t_0,\xi,\chi(t_0))$.

Hypothesis 2.1. *For any $k>0$, there exists a nonnegative function $l_i^k(t)$ with $l_i^k(\cdot)\in\mathcal{L}^1([t_0,+\infty);[0,+\infty))$ such that $|f_i(t,x,y,0)-f_i(t,\bar{x},\bar{y},0)|\vee|g_i(t,x,y)-g_i(t,\bar{x},\bar{y})|\le l_i^k(t)(|x-\bar{x}|+|y-\bar{y}|)$ for all $(t,i)\in[t_0,+\infty)\times\mathcal{N}$, $x,y,\bar{x},\bar{y}\in R^n$ with $|x|\vee|y|\vee|\bar{x}|\vee|\bar{y}|\le k$, and $f_i(t,0,0,0)=0$, $g_i(t,0,0)=0$.*

Remark 2.1. *Hypothesis 2.1 is also a local Lipschitz condition. Only under such condition, for any $i\in\mathcal{N}$, the solution of (1) may be explode at finite time* [3]. *In order to guarantee the existence and uniqueness for the global solution to systems (1), in this paper, the linear growth condition on $f_i(\cdot,\cdot,\cdot,0)$ and $g_i(\cdot,\cdot,\cdot)$ $(i\in\mathcal{N})$ is replaced by the Lyapunov monotonicity condition. The existence and uniqueness for the global solution to systems (1) can be shown when Hypothesis 2.1 and the Lyapunov monotonicity condition (see condition (b) in Theorem 3.1) are satisfied, see* [10].

Definition 2.1. [30]*: The global solution of systems (1) is said to be input-to-state stability (ISS) in the pth$(p\ge2)$-moment, if for any $\xi\in\mathcal{PC}([t_0-\tau,t_0];R^{n_x})$ and $\chi(t_0)\in\mathcal{N}$, there exist $\Theta\mathcal{L}$-function $\varrho(\cdot,\cdot)$ and Θ-function $\rho(\cdot)$ satisfying $\mathbb{E}\{|x(t)|^p\}\le\varrho\big(\|\xi\|_{\mathcal{C}}^p,t-t_0\big)+\rho\left(\|u\|_{[t_0,t)}\right)$, for any $t\ge t_0$.*

Definition 2.2. [30]*: The global solution of systems (1) is said to be stochastic input-to-state stable (SISS), if for any $\varepsilon\in(0,1)$, there exist $\Theta\mathcal{L}$-function $\varrho(\cdot,\cdot)$ and Θ-function $\rho(\cdot)$ such that for any $\xi\in\mathcal{PC}([t_0-\tau,t_0];R^{n_x})$ and $\chi(t_0)\in\mathcal{N}$, $\mathbb{P}\big\{|x(t)|^p<\varrho\big(\|\xi\|_{\mathcal{C}}^p,t-t_0\big)+\rho\left(\|u\|_{[t_0,t)}\right)\big\}\ge1-\varepsilon$.*

Definition 2.3. [18]*: Let the minimum dwell time of switching systems (1) be $\mathcal{T}_{\min}=\inf_{k=1,2,\ldots}\{t_k-t_{k-1}\}>0$. For any $t\ge s\ge t_0$, $N_{\chi(t)}(t,s)\le\frac{t-s}{\mathcal{T}_{\min}}+1$, where $N_{\chi(t)}(t,s)$ denotes the number of discontinuities of $\chi(t)$ over (s,t).*

3. Main Results.

Lemma 3.1. *Suppose that $z(\cdot)$ is a nonnegative function from $[t_0-\tau,+\infty)\to[0,+\infty)$, $\varpi(\cdot)$ is an n_ϖ-dimensional real value function defined on $[t_0,\infty)$, and $\varphi(\cdot):[0,+\infty)\to(0,+\infty)$ is a Θ_∞-class function. Let $\tau(\cdot)$ be a bounded function defined $[t_0,+\infty)$ with $0\le\tau(t)\le\tau$ $(\tau>0)$. Assume that there exist two integrable functions $\lambda_0(\cdot):[t_0,+\infty)\to R$ and $\lambda_1(\cdot):[t_0,+\infty)\to(0,+\infty)$, three constants $\hat{\lambda}_i>0$ $(i=0,1,2)$ such that the following inequality*

$$z(t)\le\begin{cases}\hat{\lambda}_0e^{\int_{t_0}^t\lambda_0(s)ds}+\displaystyle\int_{t_0}^te^{\int_s^t\lambda_0(u)du}\lambda_1(s)z_\tau(s)ds+\hat{\lambda}_1\int_{t_0}^te^{\int_s^t\lambda_0(u)du}\varphi(|\varpi(s)|)ds, & t\ge t_0,\\ \hat{\lambda}_2, & t\in[t_0-\tau,t_0],\end{cases}\tag{2}$$

is satisfied, where $z_\tau(t)=\sup_{\theta\in[-\tau,0]}z(t+\theta)$.

Furthermore, if there exist three constants $\gamma_1>0$, $\mu_1>0$, and $c_1\in R$ such that $\mu_1e^{c_1}\in(0,1)$, for any $t_0\le s<t$, $\int_s^t\left[\lambda_0(u)+\frac{\lambda_1(u)}{\mu_1}\right]du\le c_1-\gamma_1(t-s)$, then $z(t)\le$

$\bar{M}_0 e^{-\lambda^*(t-t_0)} + \bar{M}_1 \int_{t_0}^{t} e^{-\lambda^*(t-s)}\varphi(|\varpi(s)|)ds$ *holds for any* $t \in [t_0-\tau, +\infty)$, *where* $\lambda^* \in (0, \varsigma)$ *with* $\varsigma = \min\left\{\gamma_1, \frac{1}{\tau}\ln(1/(\mu_1 e^{c_1}))\right\} > 0$, $\bar{M}_0 = \max\left\{\hat{\lambda}_2, \frac{\hat{\lambda}_0 e^{c_1}}{\mu_1 e^{c_1+\lambda^*\tau}}\right\}$, *and* $M_1 \geq \frac{\hat{\lambda}_1 e^{c_1}}{1-\mu_1 e^{\lambda^*\tau+c_1}}$.

Proof: Since $\mu_1 e^{c_1} \in (0,1)$ and $\frac{1}{\tau}\log\left(1/\left(\mu_1 e^{c_1}\right)\right) > 0$. Thus, $\varsigma > 0$. Define a function $H(\lambda) = \mu_1 e^{\lambda\tau + c_1} - 1$ with its being a strictly increasing function on $[0, +\infty)$, $H(0) < 0$ and $\limsup_{\lambda\to\infty} H(\lambda) = \infty$. Consequently, there exists $\lambda_0 \in (0, +\infty)$ such that $H(\lambda_0) = 0$ with $\lambda_0 = \frac{1}{\tau}\log\left(1/\left(\mu_1 e^{c_1}\right)\right)$. Thus, $\lambda^* \in (0, \varsigma)$, and $\mu_1 e^{\lambda^*\tau + c_1} \in (0,1)$. Besides, for any $\varepsilon > 0$, define $\bar{M}_{0,\varepsilon} = \max\left\{\hat{\lambda}_2 + \varepsilon, \frac{\hat{\lambda}_0 e^{c_1} + \varepsilon}{\mu_1 e^{\lambda^*\tau + c_1}}\right\} > 0$. Now, in order to prove the desired result, it is only required to show that for any $t \geq t_0 - \tau$,

$$z(t) \leq \bar{M}_{0,\varepsilon} e^{-\lambda^*(t-t_0)} + \bar{M}_1 \int_{t_0}^{t} e^{-\lambda^*(t-s)}\varphi(|\varpi(s)|)ds. \tag{3}$$

For any $t \in [t_0-\tau, t_0]$, $z(t) \leq \hat{\lambda}_2 < \bar{M}_{0,\varepsilon}$ is satisfied, which implies that Inequality (3) holds for any $t \in [t_0 - \tau, t_0]$. If Inequality (3) does not hold for any $t \geq t_0$, then there exists $t^* > t_0$ satisfying $t^* = \inf\left\{t > t_0 : z(t) > \bar{M}_{0,\varepsilon} e^{-\lambda^*(t-t_0)} + \bar{M}_1 \int_{t_0}^{t} e^{-\lambda^*(t-s)}\varphi(|\varpi(s)|)ds\right\}$. Consequently, for any $t \in [t_0 - \tau, t^*)$, $z(t) \leq \bar{M}_{0,\varepsilon} e^{-\lambda^*(t-t_0)} + \bar{M}_1 \int_{t_0}^{t} e^{-\lambda^*(t-s)}\varphi(|\varpi(s)|)ds$ and $z(t^*) = \bar{M}_{0,\varepsilon} e^{-\lambda^*(t^*-t_0)} + \bar{M}_1 \int_{t_0}^{t^*} e^{-\lambda^*(t^*-s)}\varphi(|\varpi(s)|)ds$.

However, from the first inequality in (2), we have

$$\begin{aligned} z(t^*) \leq\ & \hat{\lambda}_0 e^{c_1 - \gamma_1(t^*-t_0) - \frac{1}{\mu_1}\int_{t_0}^{t^*}\lambda_1(s)ds} + \bar{M}_{0,\varepsilon} e^{\lambda^*\tau + c_1} \int_{t_0}^{t^*} e^{-\gamma_1(t^*-s) - \frac{1}{\mu_1}\int_{s}^{t^*}\lambda_1(s)ds} \\ & \times \lambda_1(s) e^{-\lambda^*(s-t_0)} ds \\ & + \bar{M}_1 e^{\lambda^*\tau + c_1} \int_{t_0}^{t^*} e^{-\gamma_1(t^*-s) - \frac{1}{\mu_1}\int_{s}^{t^*}\lambda_1(s)ds} \lambda_1(s) \int_{t_0}^{s} e^{-\lambda^*(s-u)}\varphi(|\varpi(u)|)duds \\ & + \hat{\lambda}_1 e^{c_1} \int_{t_0}^{t^*} e^{-\gamma_1(t^*-s) - \frac{1}{\mu_1}\int_{s}^{t^*}\lambda_1(s)ds} \varphi(|\varpi(s)|)ds. \end{aligned} \tag{4}$$

The estimates of two terms in Inequality (4) can be obtained as

$$\begin{aligned} & \bar{M}_{0,\varepsilon} e^{\lambda^*\tau + c_1} \int_{t_0}^{t^*} e^{-\gamma_1(t^*-s) - \frac{1}{\mu_1}\int_{s}^{t^*}\lambda_1(s)ds} \times \lambda_1(s) e^{-\lambda^*(s-t_0)} ds \\ & \leq \bar{M}_{0,\varepsilon}\mu_1 e^{\lambda^*\tau + c_1} e^{-\lambda^*(t^*-t_0)} - \bar{M}_{0,\varepsilon}\mu_1 e^{\lambda^*\tau + c_1} e^{-\gamma_1(t_1-t_0) - \frac{1}{\mu_1}\int_{t_0}^{t^*}\lambda_1(s)ds}, \end{aligned}$$

and

$$\begin{aligned} & \bar{M}_1 e^{\lambda^*\tau + c_1} \times \int_{t_0}^{t^*} e^{-\gamma_1(t^*-s) - \frac{1}{\mu_1}\int_{s}^{t^*}\lambda_1(s)ds} \lambda_1(s) \int_{t_0}^{s} e^{-\lambda^*(s-u)}\varphi(|\varpi(u)|)duds \\ & < \bar{M}_1\mu_1 e^{\lambda^*\tau + c_1} \int_{t_0}^{t^*} e^{-\lambda^*(t^*-s)}\varphi(|\varpi(s)|)ds. \end{aligned}$$

Furthermore, it yields from Inequality (4) that

$$\begin{aligned} z(t^*) <\ & \left[\bar{\lambda}_0 - \bar{M}_{0,\varepsilon}\mu_1 e^{\lambda^*\tau}\right] e^{c_1 - \gamma_1(t^*-s) - \frac{1}{\mu_1}\int_{s}^{t^*}\lambda_1(s)ds} + \bar{M}_{0,\varepsilon}\mu_1 \times e^{\lambda^*\tau + c_1} e^{-\lambda^*(t^*-t_0)} \\ & + \left[\bar{M}_1\mu_1 e^{\lambda^*\tau + c_1} + \hat{\lambda}_1 e^{c_1}\right] \int_{t_0}^{t^*} e^{-\lambda^*(t^*-s)}\varphi(|\varpi(s)|)ds. \end{aligned}$$

From the definitions of $\bar{M}_{0,\varepsilon}$ and $\bar{M}_1$, we have $\bar{\lambda}_0 e^{c_1} - \bar{M}_{0,\varepsilon}\mu e^{\lambda^*\tau + c_1} < 0$ and $\bar{M}_1\mu_1 e^{\lambda^*\tau + c_1} + \hat{\lambda}_1 e^{c_1} \leq \bar{M}_1$. Then, due to the fact that $\mu_1 e^{\lambda^*\tau + c_1} \in (0,1)$, $z(t^*) < \bar{M}_{0,\varepsilon} e^{-\lambda^*(t^*-t_0)} +$

$\bar{M}_1 \int_{t_0}^{t^*} e^{-\lambda^*(t^*-s)}\varphi(|\varpi(s)|)ds$ holds, which contradicts with inequality $z(t^*) = \bar{M}_{0,\varepsilon}e^{-\lambda^*(t^*-t_0)} + \bar{M}_1 \int_{t_0}^{t^*} e^{-\lambda^*(t^*-s)}\varphi(|\varpi(s)|)ds$. Therefore, Inequality (3) holds for any $t \geq t_0 - \tau$. As $\varepsilon \to 0^+$ in Inequality (3), the desired result is obtained. □

Theorem 3.1. *Let Hypothesis 2.1 be satisfied. Suppose that there exist multiple Lyapunov functions* $V_{\iota_k}(\cdot,\cdot) \in C^{1,2}([t_0,+\infty) \times R^{n_x}; (0,+\infty))$, $\rho_1(\cdot) \in \mathcal{V}\Theta_\infty$, $\rho_2(\cdot) \in \mathcal{C}\Theta_\infty$, *some functions* $\lambda_{0,\iota_k}(\cdot) : [t_0-\tau,+\infty) \to R$, $\lambda_{1,\iota_k}(\cdot) : [t_0-\tau,+\infty) \to [0,+\infty)$ *and* $\phi_{\iota_k}(\cdot) \in \Theta_\infty$, *some constants* $\beta_{\iota_k} > 0$, $\varsigma > 0$, $c_{\iota_k} \in R$, $\gamma_{\iota_k} > 0$, *and* $\mu_{\iota_k} \geq 1$ $(\iota_k \in \mathcal{N})$ *such that*

(a) for any $x \in R^{n_x}$, *and* $i \in \mathcal{N}$, $\rho_1(|x|^p) \leq V_{\iota_k}(t,x) \leq \rho_2(|x|^p)$ $(p \geq 2)$;

(b) for any $t \in [t_k, t_{k+1})$ $(k = 0,1,2,\ldots)$, $x, y \in R^{n_x}$ *and* $\varpi \in R^{\varpi}$, $\mathcal{L}V_{\iota_k}(t,x,y,\varpi) \leq \lambda_{0,\iota_k}(t)V_{\chi(t)}(t,x) + \lambda_{1,\iota_k}(t)V_{\iota_k}(t-\tau(t), y) + \phi_{\iota_k}(|\varpi|)$, *where* $\mathcal{L}V.(\cdot,\cdot,\cdot,\cdot)$ *is given in* [29];

(c) for any $k \in \mathbb{N}$, $V_{\chi(t_k)}(t_k, x(t_k)) \leq \mu_{\iota_k} V_{\chi(t_k^-)}\left(t_k^-, x\left(t_k^-\right)\right)$;

(d) for any $t_k \leq s < t < t_{k+1}$ $(k = 0,1,2,\ldots)$, $\int_s^t [\lambda_{0,\iota_k}(u) + \lambda_{1,\iota_k}(u)/\beta_{\iota_k}]\, du \leq c_{\iota_k} - \gamma_{\iota_k}(t-s)$, *where* $\beta_{\iota_k}e^{c_{\iota_k}} \in (0,1)$;

(e) $\mathcal{T}_{\min} \geq \max\{\tau, \ln(\varrho_{\iota_k})/\varsigma\}$ *and* $\varsigma \in \left(0, \min_{\iota_k\in\mathcal{N}}\{\lambda^*_{\iota_k}\}\right)$, *where* $\varrho_{\iota_k} = \max\left\{e^{\lambda^*_{\iota_{k-1}}\tau}, \frac{\mu_{\iota_k}}{\beta_{\iota_k}e^{c_{\iota_k}+\lambda^*_{\iota_k}\tau}}\right\}$, *and* $\lambda^*_{\iota_k} \in \left(0, \min\left\{\gamma_{\iota_k}, \frac{1}{\tau}\ln(1/(\beta_{\iota_k}e^{c_{\iota_k}}))\right\}\right)$ $(k = 1,2,\ldots)$.

Then, for any $\xi \in \mathcal{PC}([t_0-\tau, t_0]; R^{n_x})$ *and* $\sigma(t_0) \in \mathcal{N}$, *the global solution of systems (1) is ISS and SISS.*

Proof: The proof of this theorem is divided into three steps, as follows.

Step 1: For any $\iota_k \in \mathcal{N}$ $(k = 0,1,2,\ldots)$, we can define a function $H_{\iota_k}(\lambda) = \beta_{\iota_k}e^{c_{\iota_k}+\lambda\tau} - 1$. Since $\beta_{\iota_k}e^{c_{\iota_k}} \in (0,1)$, $H_{\iota_k}(0) = \beta_{\iota_k}e^{c_{\iota_k}} - 1 < 0$, $\limsup_{\lambda\to\infty} H_{\iota_k}(\lambda) = \infty$, and $H_{\iota_k}(\cdot)$ is a strictly increasing function on $[0,\infty)$. Hence, for any $\iota_k \in \mathcal{N}$ $(k = 0,1,2,\ldots)$, there exists one corresponding value λ_{ι_k} such that $\beta_{\iota_k}e^{c_{\iota_k}+\lambda_{\iota_k}\tau} \in (0,1)$, i.e., $\lambda_{\iota_k} = \frac{1}{\tau}\log(1/\beta_{\iota_k}e^{c_{\iota_k}}) > 0$. Then, for any $\iota_k \in \mathcal{N}$, $\lambda^*_{\iota_k} \in (0, \min\{\lambda_{\iota_k}, \gamma_{\iota_k}\})$. Consequently, for any $\iota_k \in \mathcal{N}$, $\lambda^*_{\iota_k} < \gamma_{\iota_k}$ and $\beta_{\iota_k}e^{c_{\iota_k}+\lambda^*_{\iota_k}\tau} \in (0,1)$.

Step 2: For any $t \in [t_k, t_{k+1})$, $\chi(t) = \iota_k \in \mathcal{N}$ $(k = 0,1,2,\ldots)$, let $M_{1,\iota_k} \geq \frac{e^{c_{\iota_k}}}{1-\beta_{\iota_k}e^{c_{\iota_k}+\lambda^*_{\iota_k}\tau}}$ and $M_{0,\iota_0} = \max\left\{\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}\{V_{\iota_0}(\theta,\xi(\theta))\}, \frac{\mathbb{E}\{V_{\iota_0}(t_0,x(t_0))\}}{\beta_{\iota_k}e^{c_{\iota_0}+\lambda^*_{\iota_0}\tau}}\right\}$. Now, it is necessarily shown that for any $t \in [t_k, t_{k+1})$, when $\chi(t) = \iota_k$,

$$\begin{aligned}
&\mathbb{E}\{V_{\iota_k}(t,x(t))\} \\
&\leq M_{0,\iota_0}\prod_{l=1}^{k}\varrho_{\iota_l}e^{-\sum_{i=0}^{k-1}\lambda^*_{\iota_i}(t_{i+1}-t_i)-\lambda^*_{\iota_k}(t-t_k)} + \sum_{l=0}^{k-1}M_{1,\iota_l}\prod_{m=l+1}^{k}\varrho_{\iota_m}e^{-\sum_{i=l+1}^{k-1}\lambda^*_{\iota_i}(t_{i+1}-t_i)} \\
&\quad\times\int_{t_l}^{t_{l+1}}e^{-\lambda^*_{\iota_l}(t_{l+1}-s)}\varphi_{\iota_l}(|\varpi(s)|)ds\,e^{-\lambda^*_{\iota_k}(t-t_k)} + M_{1,\iota_k}\int_{t_k}^{t}e^{-\lambda^*_{\iota_k}(t-s)}\varphi_{\iota_k}(|\varpi(s)|)ds. \quad (5)
\end{aligned}$$

For systems (1), define one Lyapunov-Krasovskii function $e^{-\int_{t_0}^t \lambda_{0,\chi(s)}(s)ds}V_{\iota_0}(t,x(t))$ $(t \in [t_0,t_1))$. By using the Itô formula and taking the mathematical expectation in turn, we have

$$\begin{aligned}
&e^{-\int_{t_0}^t \lambda_{0,\chi(s)}(s)ds}\mathbb{E}\{V_{\iota_0}(t,x(t))\} \\
&\leq \mathbb{E}\{V_{\iota_0}(t_0,x(t_0))\} + \int_{t_0}^{t}e^{-\int_{t_0}^{s}\lambda_{0,\chi(s)}(u)du}\lambda_{1,\chi(s)}(s)\sup_{\theta\in[-\tau,0]}\mathbb{E}\{V_{\iota_0}(s+\theta,x(s+\theta))\}ds \\
&\quad+\int_{t_0}^{t}e^{-\int_{t_0}^{s}\lambda_{0,\chi(s)}(u)du}\times\phi_{\iota_0}(|\varpi(s)|)ds,
\end{aligned}$$

where condition (b) is used, which yields that for any $t \in [t_0, t_1)$,

$$\begin{aligned}&\mathbb{E}\{V_{\iota_0}(t, x(t))\}\\ &\le \mathbb{E}\{V_{\iota_0}(t_0, x(t_0))\}e^{\int_{t_0}^t \lambda_{0,\chi(s)}(s)ds} + \int_{t_0}^t e^{\int_s^t \lambda_{0,\chi(u)}(u)du}\lambda_{1,\chi(s)}(s)\\ &\times \sup_{\theta\in[-\tau,0]} \mathbb{E}\{V_{\iota_0}(s+\theta, x(s+\theta))\}ds + \int_{t_0}^t e^{\int_s^t \lambda_{0,\chi(s)}(u)du}\phi_{\iota_0}(|\varpi(s)|)ds. \qquad (6)\end{aligned}$$

In addition, for any $t \in [t_0 - \tau, t_0]$, $\mathbb{E}\{V_{\iota_0}(t, x(t))\} \le \sup_{\theta\in[t_0-\tau,t_0]} \mathbb{E}\{V_{\iota_0}(\theta, \xi(\theta))\}$. Furthermore, from Lemma 3.1, for any $t \in [t_0 - \tau, t_1)$, it yields from condition (d) that

$$\mathbb{E}\{V_{\iota_0}(t, x(t))\} \le M_{0,\iota_0}e^{-\lambda^*_{\iota_0}(t-t_0)} + M_{1,\iota_0}\int_{t_0}^t e^{-\lambda^*_{\iota_0}(t-s)}\varphi_{\iota_0}(|\varpi(s)|)ds, \qquad (7)$$

where $\lambda^*_{\iota_0} \in (0, \min\{\gamma_{\iota_0}, \lambda_{\iota_0}\})$ with $\lambda_{\iota_0} = \frac{1}{\tau}\log(1/\beta_{\iota_0}e^{c_{\iota_0}}) > 0$.

When $t = t_1^-$, from (7), it follows that

$$\mathbb{E}\{V_{\iota_0}(t_1^-, x(t_1^-))\} \le M_{0,\iota_0}e^{-\lambda^*_{\iota_0}(t_1-t_0)} + M_{1,\iota_0}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)} \times \varphi_{\iota_0}(|\varpi(s)|)ds.$$

From condition (c), it yields that $\mathbb{E}\{V_{\iota_1}(t_1, x(t_1))\} \le M_{0,\iota_0}\mu_{\iota_1}e^{-\lambda^*_{\iota_0}(t_1-t_0)} + M_{1,\iota_0}\mu_{\iota_1} \times \int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}\varphi_{\iota_0}(|\varpi(s)|)ds$. Besides, for any $t \in [t_1 - \tau, t_1)$, we have $\mathbb{E}\{V_{\iota_1}(t, x(t))\} \le M_{0,\iota_0}e^{\lambda^*_{\iota_0}\tau}e^{-\lambda^*_{\iota_0}(t_1-t_0)} + M_{1,\iota_0}e^{\lambda^*_{\iota_0}\tau}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}\varphi_{\iota_0}(|\varpi(s)|)ds$. Thus, when $t \in [t_1, t_2)$ and $\chi(t) = \iota_1$, similar to the reasoning process used in (6) and (7), it gives that

$$\begin{aligned}&\mathbb{E}\{V_{\iota_1}(t, x(t))\}\\ &\le \mathbb{E}\{V_{\iota_1}(t_1, x(t_1))\}e^{\int_{t_1}^t \lambda_{0,\chi(s)}(s)ds} + \int_{t_1}^t e^{\int_s^t \lambda_{0,\chi(u)}(u)du}\lambda_{1,\chi(s)}(s)\\ &\times \sup_{\theta\in[-\tau,0]} \mathbb{E}\{V_{\iota_1}(s+\theta, x(s+\theta))\}ds + \int_{t_1}^t e^{\int_s^t \lambda_{0,\chi(s)}(u)du}\phi_{\iota_1}(|\varpi(s)|)ds, \qquad (8)\end{aligned}$$

and for any $t \in [t_1 - \tau, t_1]$,

$$\begin{aligned}\mathbb{E}\{V_{\iota_1}(t, x(t))\} \le\ & M_{0,\iota_0}\max\left\{\mu_{\iota_1}, e^{\lambda^*_{\iota_0}\tau}\right\}e^{-\lambda^*_{\iota_0}(t_1-t_0)} + M_{1,\iota_0}\max\left\{\mu_{\iota_1}, e^{\lambda^*_{\iota_0}\tau}\right\}\\ &\times \int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}\varphi_{\iota_0}(|\varpi(s)|)ds.\end{aligned}$$

In view of Lemma 3.1 again, for any $t \in [t_1 - \tau, t_2)$, from (8), we have

$$\begin{aligned}&\mathbb{E}\{V_{\iota_1}(t, x(t))\}\\ &\le M_{0,\iota_0}\varrho_{\iota_1}e^{-\lambda^*_{\iota_0}(t_1-t_0)-\lambda^*_{\iota_1}(t-t_1)} + M_{1,\iota_0}\varrho_{\iota_1}e^{-\lambda^*_{\iota_1}(t-t_1)}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}\varphi_{\iota_0}(|\varpi(s)|)ds\\ &+ M_{1,\iota_1}\int_{t_1}^t e^{-\lambda^*_{\iota_1}(t-s)}\varphi_{\iota_1}(|\varpi(s)|)ds, \qquad (9)\end{aligned}$$

where $\lambda^*_{\iota_1} \in (0, \min\{\gamma_{\iota_1}, \lambda_{\iota_1}\})$.

When $t = t_2^-$, from (9), it follows that $\mathbb{E}\{V_{\iota_1}(t_2^-, x(t_2^-))\} \le M_{0,\iota_0}\varrho_{\iota_1}e^{-\lambda^*_{\iota_0}(t_1-t_0)-\lambda^*_{\iota_1}(t_2-t_1)} + M_{1,\iota_0}\varrho_{\iota_1} \times e^{-\lambda^*_{\iota_1}(t_2-t_1)}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}\varphi_{\iota_0}(|\varpi(s)|)ds + M_{1,\iota_1}\int_{t_1}^{t_2} e^{-\lambda^*_{\iota_1}(t_2-s)}\varphi_{\iota_1}(|\varpi(s)|)ds$. From condition (c), it yields that

$$\mathbb{E}\{V_{\iota_2}(t_2, x(t_2))\} \le M_{0,\iota_0}\mu_{\iota_2}\varrho_{\iota_1}e^{-\lambda^*_{\iota_0}(t_1-t_0)-\lambda^*_{\iota_1}(t_2-t_1)} + M_{1,\iota_0}\mu_{\iota_2}\varrho_{\iota_1}e^{-\lambda^*_{\iota_1}(t_2-t_1)}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}$$

$$\times \varphi_{\iota_0}(|\varpi(s)|)ds + M_{1,\iota_1}\mu_{\iota_2}\int_{t_1}^{t_2} e^{-\lambda^*_{\iota_1}(t_2-s)}\varphi_{\iota_1}(|\varpi(s)|)ds.$$

From (9) again, it gives that for any $t \in [t_2-\tau, t_2)$,

$$\mathbb{E}\{V_{\iota_2}(t,x(t))\} \le M_{0,\iota_0}\varrho_{\iota_1}e^{\lambda^*_{\iota_1}\tau}e^{-\lambda^*_{\iota_0}(t_1-t_0)-\lambda^*_{\iota_1}(t_2-t_1)} + M_{1,\iota_0}\varrho_{\iota_1}e^{\lambda^*_{\iota_1}\tau}e^{-\lambda^*_{\iota_1}(t_2-t_1)}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}$$
$$\times \varphi_{\iota_0}(|\varpi(s)|)ds + M_{1,\iota_1}e^{\lambda^*_{\iota_1}\tau}\int_{t_1}^{t_2} e^{-\lambda^*_{\iota_1}(t_2-s)}\varphi_{\iota_1}(|\varpi(s)|)ds.$$

Consequently, for $t \in [t_2-\tau, t_2]$,

$$\begin{aligned}\mathbb{E}\{V_{\iota_2}(t,x(t))\} \le{}& M_{0,\iota_0}\varrho_{\iota_1}\max\left\{\mu_{\iota_2}, e^{\lambda^*_{\iota_1}\tau}\right\}e^{-\lambda^*_{\iota_0}(t_1-t_0)-\lambda^*_{\iota_1}(t_2-t_1)}\\ &+ M_{1,\iota_0}\varrho_{\iota_1}\max\left\{\mu_{\iota_2}, e^{\lambda^*_{\iota_1}\tau}\right\}e^{-\lambda^*_{\iota_1}(t_2-t_1)}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}\\ &\times \varphi_{\iota_0}(|\varpi(s)|)ds + M_{1,\iota_1}\max\left\{\mu_{\iota_2}, e^{\lambda^*_{\iota_1}\tau}\right\}\int_{t_1}^{t_2} e^{-\lambda^*_{\iota_1}(t_2-s)}\varphi_{\iota_1}(|\varpi(s)|)ds.\end{aligned}$$

From Lemma 3.1 again, for any $t \in [t_2-\tau, t_3)$, we have

$$\begin{aligned}\mathbb{E}\{V_{\iota_2}(t,x(t))\} \le{}& M_{0,\iota_0}\varrho_{\iota_1}\varrho_{\iota_2}e^{-\lambda^*_{\iota_0}(t_1-t_0)-\lambda^*_{\iota_1}(t_2-t_1)}\\ &\times e^{-\lambda^*_{\iota_2}(t-t_2)} + M_{1,\iota_0}\varrho_{\iota_1}\varrho_{\iota_2}\int_{t_0}^{t_1} e^{-\lambda^*_{\iota_0}(t_1-s)}\varphi_{\iota_0}(|\varpi(s)|)de^{-\lambda^*_{\iota_1}(t_2-t_1)-\lambda^*_{\iota_2}(t-t_2)}\\ &+ M_{1,\iota_1}\varrho_{\iota_2}\int_{t_1}^{t_2} e^{-\lambda^*_{\iota_1}(t_2-s)}\varphi_{\iota_1}(|\varpi(s)|)ds\\ &\times e^{-\lambda^*_{\iota_2}(t-t_2)} + M_{1,\iota_2}\int_{t_2}^{t} e^{-\lambda^*_{\iota_2}(t-s)}\varphi_{\iota_2}(|\varpi(s)|)ds,\end{aligned}$$

where $\lambda^*_{\iota_2} \in (0, \min\{\gamma_{\iota_2}, \lambda_{\iota_2}\})$.

Suppose that Inequality (5) is satisfied for any $t \in [t_{k-1}, t_k)$ and $\chi(t) = \iota_{k-1}$. When $t \in [t_k, t_{k+1})$ and $\chi(t) = \iota_k$ $(k = 0, 1, 2, \ldots)$, similarly, from condition (b), we have

$$\begin{aligned}\mathbb{E}\{V_{\iota_k}(t,x(t))\} \le{}& \mathbb{E}\{V_{\iota_k}(t_k,x(t_k))\} \times e^{\int_{t_k}^{t}\lambda_{0,\chi(s)}(s)ds}\\ &+ \int_{t_k}^{t} e^{\int_s^t \lambda_{0,\chi(u)}(u)du}\lambda_{1,\chi(s)}(s)\sup_{\theta\in[-\tau,0]}\mathbb{E}\{V_{\iota_k}(s+\theta, x(s+\theta))\}ds\\ &+ \int_{t_k}^{t} e^{\int_s^t \lambda_{0,\chi(s)}(u)du}\phi_{\iota_k}(|\varpi(s)|)ds.\end{aligned}$$

In addition, for any $t \in [t_{k-1}-\tau, t_k]$,

$$\begin{aligned}&\mathbb{E}\{V_{\iota_k}(t,x(t))\}\\ &\le M_{0,\iota_0}\prod_{l=1}^{k-1}\varrho_{\iota_l}\max\left\{\mu_{\iota_k}, e^{\lambda^*_{\iota_{k-1}}\tau}\right\}e^{-\sum_{i=0}^{k-2}\lambda^*_{\iota_i}(t_{i+1}-t_i)} \times e^{-\lambda^*_{\iota_{k-1}}(t-t_{k-1})}\\ &+ \sum_{l=0}^{k-1}M_{1,\iota_l}\prod_{m=l+1}^{k-1}\varrho_{\iota_m}\max\left\{\mu_{\iota_k}, e^{\lambda^*_{\iota_{k-1}}\tau}\right\}e^{-\sum_{l=l}^{k-2}\lambda^*_{\iota_l}(t_{l+1}-t_l)}\int_{t_l}^{t_{l+1}} e^{-\lambda^*_{\iota_l}(t_{l+1}-s)}\varphi_{\iota_l}(|\varpi(s)|)ds\\ &+ M_{1,\iota_{k-1}}\max\left\{\mu_{\iota_k}, e^{\lambda^*_{\iota_{k-1}}\tau}\right\}\int_{t_{k-1}}^{t} e^{-\lambda^*_{\iota_{k-1}}(t-s)}\varphi_{\iota_{k-1}}(|\varpi(s)|)ds.\end{aligned}$$

Furthermore, from Lemma 3.1, it yields that for any $t \in [t_k - \tau, t_{k+1})$ and $\chi(t) = \iota_k$,

$$\mathbb{E}\{V_{\iota_k}(t, x(t))\} \leq \max\left\{\sup_{\theta\in[t_k-\tau,t_k]} \mathbb{E}\{V_{\iota_k}(\theta, x(\theta))\}, \frac{\mathbb{E}\{V_{\iota_k}(t_k, x(t_k))\}}{\beta_{\iota_k} e^{c_{\iota_k}+\lambda^*_{\iota_k}\tau}}\right\} e^{-\lambda^*_{\iota_k}(t-t_k)} + M_{1,\iota_k}\int_{t_k}^{t} e^{-\lambda^*_{\iota_k}(t-s)}\varphi_{\iota_k}(|\varpi(s)|)ds \quad (10)$$

which implies that Inequality (2) is satisfied.

Step 3: From condition (e), it gives that $\mathcal{T}_{\min} \geq \frac{\log(\varrho_{\iota_k})}{\varsigma}$. Then, $\varrho_{\iota_k} \leq e^{\varsigma(t_k - t_{k-1})}$ ($k = 1, 2, \ldots$). Furthermore, it concludes from (10) that for any $t \geq t_0$,

$$\mathbb{E}\{V_{\chi(t)}(t, x(t))\} \leq M_{0,\iota_0} e^{-\lambda^*(t-t_0)} + M_1\int_{t_0}^{t} e^{-\lambda^*(t-s)}\bar{\varphi}(|\varpi(s)|)ds, \quad (11)$$

where $\lambda^* = \min_{\iota_k\in\mathcal{N}}\{\lambda^*_{\iota_k} - \varsigma\} > 0$, $M_1 = \max_{\iota_k\in\mathcal{N}}\{M_{0,\iota_k}\}$ and $\bar{\varphi}(|\varpi|) = \max_{\iota_k\in\mathcal{N}}\{\varphi_{\iota_k}(|\varpi|)\}$.

Then, by using Jensen's inequality, from condition (a) and Inequality (11), it follows that for any $t \geq t_0$,

$$\mathbb{E}\{|x(t)|^p\} \leq \rho_1^{-1}\left(\frac{2\sup_{\theta\in[t_0-\tau,t_0]}\mathbb{E}\{\rho_2(|\xi(\theta)|^p)\}}{\beta_{\iota_0} e^{c_{\iota_0}+\lambda^*_{\iota_0}}} e^{-\lambda^*(t-t_0)}\right) + \rho_1^{-1}\left(\frac{2M_1}{\lambda^*}\|\bar{\varphi}\|_{[0,t]}\right), \quad (12)$$

which concludes that the global solution of systems (1) is ISS in p-moment. For any $\epsilon > 0$, let $\varrho(x, t - t_0) = \frac{1}{\epsilon}\rho_1^{-1}\left(\frac{2\rho_2(x)}{\beta_{\iota_0} e^{c_{\iota_0}+\lambda^*_{\iota_0}}} e^{-\lambda^*(t-t_0)}\right)$ and $\rho(y) = \rho_1^{-1}\left(\frac{2M_1}{\lambda^*}y\right)$, where $\varrho(\cdot,\cdot) \in \Theta\mathcal{L}$ and $\rho(\cdot) \in \Theta$. By using Chebyshev's inequality and Inequality (12), it yields $\mathbb{P}\{|x(t)|^p \geq \varrho\left(\sup_{\theta\in[-\tau,0]}\mathbb{E}\{|\xi(\theta)|^p\}, t - t_0\right) + \rho(\|\varpi\|_{[0,t]})\} \leq \frac{\mathbb{E}\{|x(t)|^p\}}{\varrho\left(\sup_{\theta\in[-\tau,0]}\mathbb{E}\{|\xi(\theta)|^p\}, t-t_0\right) + \rho(\|\varpi\|_{[0,t]})} \leq \epsilon$, which means that $\mathbb{P}\left\{|x(t)|^p < \varrho\left(\sup_{\theta\in[-\tau,0]}\mathbb{E}\{|\xi(\theta)|^p\}, t - t_0\right) + \rho\left(\|\varpi\|_{[0,t]}\right)\right\} \geq 1 - \epsilon$. Thus, the global solution of system (1) is SISS. □

4. Example.

Example 4.1. *We consider the following one-dimensional highly nonlinear stochastic time-varying switched delay systems:*

$$dx(t) = f_{\chi(t)}(t, x(t), x(t - \tau(t)), \varpi(t))dt + g_{\chi(t)}(t, x(t), x(t - \tau(t)))d\omega(t), \quad (13)$$

on $t \geq t_0 = 0$, *with the initial value* $\{x(t) : t_0 - \tau \leq t \leq t_0\} = \xi \in \mathcal{PC}([t_0-\tau, t_0]; R)$ ($\tau > 0$), *where the switching signal* $\chi(t)$ *takes its value on* $\mathcal{N} = \{1, 2\}$, $x(t)$, $x(t - \tau(t)) \in R$, $f_1(t, x, y, \varpi) = (-t - 0.3)x - t|\cos(t)|x^3 + 0.5ty + \sin^2(t)\varpi$, $g_1(t, x, y) = \sqrt{2|\cos(t)|}x^2$, $f_2(t, x, y, \varpi) = (-t - t^2 - 0.4)x - t|\sin(t)|xy^2 + 0.6ty + \cos^2(t)\varpi$, *and* $g_2(t, x, y) = 0.9\sqrt{t}x + \sqrt{t|\sin(t)|}xy$. *In (13),* $\varpi(t)$ *denotes the external bounded disturbance.*

For systems (13), two Lyapunov functions are taken as $V_1(t, x) = |x|^2$ and $V_2(t, x) = 0.9|x|^2$, respectively, with $\mu_{\iota_k} = 1.12$ ($\iota_k \in \mathcal{N}$). By computing the Itô differentiable operator, we have $\mathcal{L}V_1(t, x, y, \varpi) \leq \lambda_{0,1}(t)|x|^2 + \lambda_{1,1}(t)|y|^2 + \sin^2(t)|\varpi|^2$, and $\mathcal{L}V_2(t, x, y, \varpi) \leq \lambda_{0,2}(t)|x|^2 + \lambda_{1,2}(t)|y|^2 + 0.9\cos^2(t)|\varpi|^2$, where $\lambda_{0,1}(t) = -1.5t - 0.1 - 0.5\cos(2t)$, $\lambda_{1,1}(t) = 0.5t$, $\lambda_{0,2}(t) = -1.46t - 0.27 + 0.45\cos(2t)$, and $\lambda_{1,2}(t) = 0.54t$. Note that since functions $\lambda_{0,1}(t)$ and $\lambda_{0,2}(t)$ have variable signs and are unbounded, the operators $\mathcal{L}V_1(t, x, y, \varpi)$ and $\mathcal{L}V_2(t, x, y, \varpi)$ are indefinite. When $\varpi(t) = 0$, it can readily be verified that $f_i(t, x, y, 0)$ and $g_i(t, x, y)$ ($i \in \mathcal{N}$) both satisfy Hypothesis 2.1 and condition (b) in Theorem 3.1. Let $\beta_1 = 0.5$ and $\beta_2 = 0.6$, for any $t_k \leq s < t \leq t_{k+1}$ ($k = 0, 1, 2, \ldots$), we have

$$\int_s^t [\lambda_{0,i}(u) + \lambda_{1,i}(u)/\beta_i]du \leq \begin{cases} c_1 - \gamma_1(t - s), & \text{if } i = 1, \\ c_2 - \gamma_2(t - s), & \text{if } i = 2, \end{cases}$$

with $c_1 = 0.5$, $\gamma_1 = 0.1$, $c_2 = 0.45$, $\gamma_2 = 0.27$, $\beta_1 e^{c_1} = 0.8243 \in (0,1)$, and $\beta_2 e^{c_2} = 0.9409 \in (0,1)$. Furthermore, $\lambda_1^* \in (0, 0.1)$ and $\lambda_2^* \in (0, 0.06)$. Consequently, $\lambda_1^* = 0.099$, $\lambda_2^* = 0.059$ and $\varsigma \in (0, 0.059)$. Thus, $\varrho_1 = 1.2428$ and $\varrho_2 = 1.1287$. When $\varsigma = 0.058$, we have $\mathcal{T}_{\min} \geq 3.6841$. Then, the ISS and the SISS of systems (13) are guaranteed. It is found that the theoretical results given in [29] are not suitable, since the Lyapunov monotonicity condition has a sign-changed and unbounded time-varying coefficient in this example. When the bounded external disturbance $\varpi(t) = \sin(t)$, $\tau(t) = 0.6|\sin(t)| + 0.3$, the initial value $x(t) = -1.0$ ($t \in [-0.9, 0]$) and $\chi(0) = 1 \in \mathcal{N}$, Figure 1 shows the switching signal $\chi(t)$ used in systems (13), and Figure 2 and Figure 3 show the dynamical responses on the ISS in mean square and SISS of systems (13), respectively.

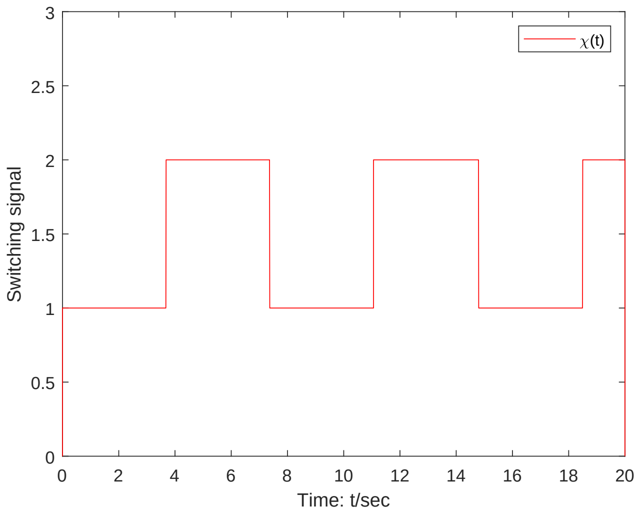

FIGURE 1. Switching signal $\chi(t)$ in systems (13)

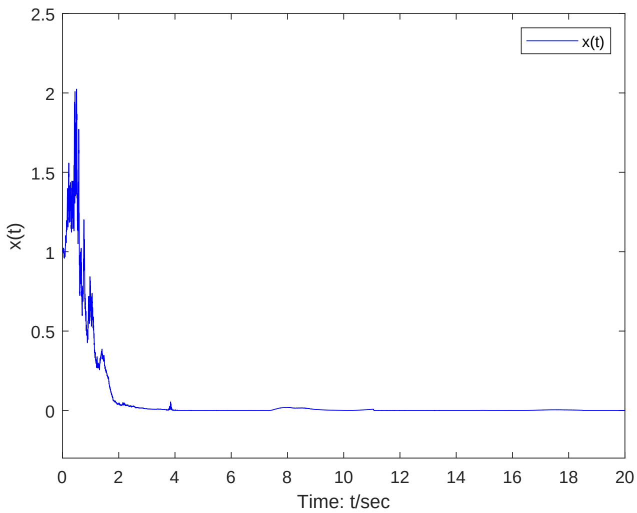

FIGURE 2. Dynamical response on the ISS of systems (13)

5. **Conclusion.** This paper has studied the ISS/SISS for the solution of time-varying nonlinear switched SDS. We have established the result on the ISS/SISS of such systems by using the Lyapunov-Khasminskii technique. The general monotonicity condition can have a sign-changed and unbounded time-varying coefficient. We have provided a theoretical

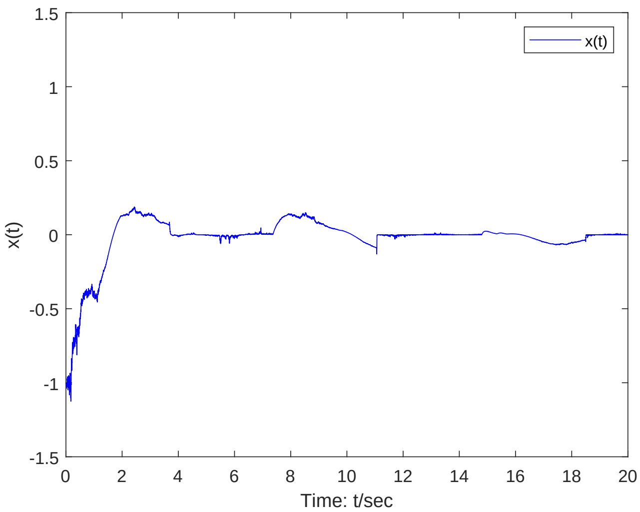

FIGURE 3. Dynamical response on the SISS of systems (13)

framework to analyze the ISS/SISS of time-varying switched SDS, which has been well shown by one example. In the future, we will devote ourselves to discussing the event-triggered control and sampled-data control for time-varying switched SDS.

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