

DYNAMICS AND ENTROPY ANALYSIS OF A NOVEL FRACTIONAL CHAOTIC SYSTEM FOR COLOR MEDICAL IMAGE ENCRYPTION

UDAY JABBAR QUAEZ

Department of Mathematics
Education College
Al-Mustansiriyah University
Al Salihiya, Baghdad 10002, Iraq
ddr.uday@uomustansiriyah.edu.iq

Received December 2024; revised April 2025

ABSTRACT. *One of the most significant and widely recognized works in chaos theory pertains to chaos-based information encryption. This study proposed a new three-dimensional chaotic system with fractional orders and line equilibrium points, characterized by its flexible and rich dynamics. The system is investigated and analyzed in detail by the phase plots, bifurcation diagrams, Lyapunov exponents graph and entropy which show that the system has rich chaotic behavior. In the experimental aspects, we present a new dynamic algorithm based on the proposed system to protect medical data. The dynamics of this algorithm rely on mixing a logistic map with this system and make position changes highly sensitive to the chaotic sequence associated with pixel values of the image, where even minor alterations lead to entirely different outcomes. Additionally, the results of entropy and chi-square test indicate that our proposed algorithm for medical information encryption is suitable, robust and can withstand common attacks.*

Keywords: Entropy, Fractional chaotic system, Lyapunov exponents, Medical image encryption

1. Introduction. We would like to begin by referring to the fact that the application of fractional calculus to chaotic systems has emerged as one of the most rapidly developing directions due to its flexibility and accurate description. It has been utilized in various scientific fields, particular in applied mathematics, physics, engineering and medicine [1-5] which is evidenced through a collection of important articles, some of which will be reviewed. We start by the study of [6], which analyzed the chaotic behavior of fractional chaotic systems based on slandered system by using experimental bounds on place of fractional order. The fractional continuous chaotic system based on logistic map is designed by using a novel approach of delay time [7]. [8] used the Caputo fractional operator with Haar wavelet numerical scheme to investigate a three dimension solvability of chaotic system. [9] investigated control and synchronization of Caputo fractional difference of chaotic systems in three dimension space. [10] proposed fractional chaotic system which takes different shapes by adding nonlinear function and determine its parameters. [11] developed a three dimensional fractional chaotic system from integer order system and investigated the dynamic behavior of different fractional orders. A new fractional three dimensional chaotic system from a novel standard system was introduced [12]. The information and control of dynamical systems have been widely investigated [1,13,14]. It has been stated many times that chaotic trajectories are aperiodic and akin to random behaviors [15]. The application aspects of chaotic systems, particularly their use in image encryption, have

garnered significant attention from many authors across a broad spectrum. Here, we highlight some of these works. Let us start with the image encryption algorithm that relies on pixel position permutation and the randomness of key stream, which was introduced by [16]. The shuffling technique has been proposed by [17], introducing a good scheme of encryption that depends on a cat map. The authors in [18] mixed a chaotic map with DNA sequences to develop a new algorithm of image encryption. [19] introduced an algorithm based on a combination of a logistic and chaotic system in one dimensional space. [20] proposed a cryptographic algorithm based on a logistic map and ordinary chaotic system in two-dimension space. The loss of information is characterized by flow of information about the initial state by the evolution of a conditional Shannon entropy which generalizes the KS entropy for the case of observing the uncertainty more than one step ahead into the future [21]. In a theoretical perspective, our work is interested to examine the dynamics of the proposed fractional chaotic systems according to the perspective of information theory. The entropy can be considered as a measure to determine the chaotic range of dynamic. Consequently, we use this measure alongside other tools such as Lyapunov exponents, phase plots, bifurcation diagram and experimental bounds to discuss the chaotic behavior of fraction dynamical systems in continuous time. Moreover, we demonstrate that fractal orders have a significant impact and noticeable effect on chaotic behavior of study systems, also, we investigate and examine the impact of the fractional orders to gain a deeper understanding of how the system behaves differently from the integer-order fractional model. Therefore, it can be considered as a key player in that. In the context of the digitization of service systems, electronic health records are advanced way of collecting and managing medical information from multiple perspectives. However, they face a significant challenge in ensuring the security of these records. Despite the existence of laws and regulations, such as HIPAA in the United States, HIPAA, GDPR in Europe, and DPA in the United Kingdom, which provide privacy regulations, protecting these records from unauthorized access and attacks remains a major challenge. Therefore, encryption can be considered one of the recommended methods to address this issue. On the other hand, telemedicine and its related applications play an important and distinctive role in healthcare services by utilizing innovative techniques for diagnosis and providing an initial medical guideline in faster and less expensive way. However, there are several problems maintaining security of information transfer over the cloud such as access management, data breaches, lack of privacy, and weak encryption. Consequently, it becomes all the more important protecting that information from being accessed by non-authorizing individuals. Regarding color medical images, maintaining its integrity and credibility is extremely important, which ensures that the information is not accessed by an unauthorized individual. This prompts us to find rebuts and servable techniques through that information can be in transferred safely. Therefore, more importance should be given to maintain the security of medical information. In this context, several methods and techniques have been developed, particularly by authors in [22-25]. Encryption algorithm for color medical images was proposed by the authors in [26]. They used watermarking technique and their algorithm combined between reversible information caching methods and standard encryption methods for ensuring the security requirements, for transformed and stored data of images. The authors in [27] presented an encryption algorithm based on four-dimensional chaotic systems for protect color medical images against attacks. Their presented encryption algorithm combined between high speed permutation process and adaptive diffusion. A novel approach to medical images encryption was presented by [28] that combines the method of advanced encryption standard with technique of triple data encryption for this, and they used three chaotic maps logistic, Baker and Arnold cat. In fact, recently the medical image encryption has always received widespread attention

from scholars, since it not only ensures the safety of these images but also accelerates the efficiency of both process encryption and decryption. In [29], the authors proposed an encryption algorithm for specialized and efficient local medical images that it focuses on the interest regions within massive images. [30] presented an improved chaotic based cryptosystem to encrypt rapidly secret medical images. The chaotic sequence to provide an efficient encryption technique was proposed by [31] that enables the reconstruction of original medical picture from the encrypted picture with high quality and minimal distortion. Additionally, we would like to refer that one of the practical contributions of this work is to provide a new encryption algorithm of color medical image based on fractional chaotic systems. Our proposed algorithm contains new parameters, the fractional order and the shape parameter α . The algorithm is composed of two stages: a shuffling of pixel positions of original image using the logistic map and combined it with fractional chaotic sequence. The sensitivity of the proposed system to these parameters has been demonstrated, making the system more flexible and secure. Finally, for the comparison between the results obtained in this paper and studies presented in previous literature mentioned, it can be considered better in terms of entropy measure and chi-square test. The paper is organized as follows. Section 2 contains mathematical preliminaries about fractional calculus. In Section 3, the fractional chaotic system based on fractional order is proposed. Medical image encryption application is introduced in Section 4. Finally, conclusions are presented in Section 5.

2. Mathematical Preliminaries.

Definition 2.1. [32] *Let $x(t)$ be a function in $C^n(a, b)$ space. The Caputo fractional derivative of order γ can be formally defined as follows*

$$D_C^\gamma x(t) = \begin{cases} \frac{1}{\Gamma(n - \gamma)} \int_a^t \frac{x^{(n)}(s)}{(t - s)^{\gamma - n + 1}} ds, & 0 \leq n - 1 < \gamma < n \\ x^{(n)}(t), & \gamma = n \end{cases} \tag{1}$$

where $x^{(n)}(t)$ represents the n -th derivative of $x(t)$.

Definition 2.2. *The γ -fractional integral of a function $x(t)$ is defined as follows*

$$I_C^\gamma x(t) = \frac{1}{\Gamma(\gamma)} \int_a^t \frac{x(s)}{(t - s)^{1 - \gamma}} ds, \quad 0 \leq \gamma < n \tag{2}$$

Proposition 2.1. [33] *Let $x(t), y(t)$ be two functions in $C^n(a, b)$ space. Then, for $\gamma_1, \gamma_2 > 0$, we have*

- i) $I_C^{\gamma_1} I_C^{\gamma_2} x(t) = I_C^{\gamma_1 + \gamma_2} x(t)$
- ii) $I_C^{\gamma_1} I_C^{\gamma_2} x(t) = I_C^{\gamma_2} I_C^{\gamma_1} x(t)$
- iii) $I_C^{\gamma_1} (x(t) + y(t)) = I_C^{\gamma_1} x(t) + I_C^{\gamma_1} y(t)$
- iv) $I_C^{\gamma_1} D_C^{\gamma_1} x(t) = x(t) - x_0, 0 < \alpha < 1$
- v) $D_C^{\gamma_1} I_C^{\gamma_1} x(t) = x(t)$

3. Fractional Chaotic System.

3.1. System description. We propose a novel three-dimensional chaotic system, clearly represented in Equation (3). This system has been meticulously designed to reflect the distinctive characteristics of chaos in three dimensions, leveraging fractional orders for their inherent flexibility in modifying the system’s core properties. This approach ensures that the mathematical formulation is both precise and clearly expressive, effectively capturing the complex nature of the system.

$$\begin{aligned}
D_C^{\gamma_1} x(t) &= \mu_1 y(t) + \mu_3 x(t) z(t) - z(t) \\
D_C^{\gamma_2} y(t) &= -\mu_5 x(t) + \mu_2 z(t) \\
D_C^{\gamma_3} z(t) &= x(t) - \mu_2 y^2(t) - \mu_3 x^2(t) + \mu_4 e^{-\alpha x(t)}
\end{aligned} \tag{3}$$

where x, y, z are the state variables, $D_C^{\gamma_i}$ denotes the Caputo fractional derivative of orders $\gamma_i, i = 1, 2, 3$ with respect to t and $\mu_j, \alpha > 0, j = 1, 2, 3, 4, 5$ are system constant parameters. Reformulate system (1), based on Caputo fractional integral (3) and using Proposition 2.1 iv), we get

$$\begin{aligned}
x(t) &= \frac{1}{\Gamma(\gamma_1)} \int_a^t \frac{\mu_1 y(s) + \mu_3 x(s) z(s) - z(s)}{(t-s)^{1-\gamma_1}} ds \\
y(t) &= \frac{1}{\Gamma(\gamma_2)} \int_a^t \frac{-\mu_5 x(s) + \mu_2 z(s)}{(t-s)^{1-\gamma_2}} ds \\
z(t) &= \frac{1}{\Gamma(\gamma_3)} \int_a^t \frac{x(s) - \mu_2 y^2(s) - \mu_3 x^2(s) + \mu_4 e^{-\alpha x(s)}}{(t-s)^{1-\gamma_3}} ds
\end{aligned} \tag{4}$$

Numerical methods are employed to find the approximate solutions to integrations in (4), such as Roberto Garrappa method [34], fractional Adams method [35], Grunwald-Letnikov method, and Galerkin method. For simplicity in this work, we will assume that the fractional orders are equal. The dynamical behaviors of the conservative chaotic system (3) are analyzed under both periodic and chaotic regimes. The parameter set for the periodic regime is defined as $\mu_1 = 2, \mu_2 = 4, \mu_3 = 2, \mu_4 = 1, \mu_5 = 2, \alpha = 0.01$ and fractional order $\gamma_i = 0.99, i = 1, 2, 3$ with the initial condition $(1, 1, 1)$. In the chaotic system, the parameter μ_4 is modified to 10, while the other parameters remain unchanged under the same initial condition. The phase portraits in $(x, y), (y, z)$ and (x, z) planes (Figures 1(a)-1(c)) display smooth topologies and characteristic of periodic motion. In contrast, Figures 1(d)-1(f) reveal irregular and complicate structures, signifying chaotic dynamics. The 0-1 test further helps to distinguish between the periodic (regular) (Figure 1(g)) and chaotic (irregular) (Figure 1(h)). As for Lyapunov exponents exhibit $(0, 0, 0)$ to periodic and $(0.212, -0.018, -0.243)$ for chaotic.

3.2. Chaotic dynamics.

3.2.1. Variation of phase trajectories with changes in parameters of the fractional orders.

This study highlights the dynamical behavior of system (3) by adjusting the fractional derivative parameters of the system. The Lyapunov exponents signify the average of convergence or divergence rate of trajectories in phase space. These exponents may be negative, zero or positive and their interpretation depends on their sign. Positive Lyapunov exponents are taken as an indication of chaos. The number of Lyapunov exponents corresponds to the number of dimensions in the underlying dynamical systems. As for a three-dimensional dynamical system, the Lyapunov exponents are calculated using the presented formula in Equation (5), providing the system's dynamic variations and a quantitative analysis of the level of chaos.

$$\left. \begin{aligned}
\lambda_1 &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \left(\frac{|\delta x(\tau)|}{|\delta x(0)|} \right) \\
\lambda_2 &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \left(\frac{|\delta y(\tau)|}{|\delta y(0)|} \right) \\
\lambda_3 &= \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln \left(\frac{|\delta z(\tau)|}{|\delta z(0)|} \right)
\end{aligned} \right\} \tag{5}$$

where δ represents the difference between two very close points in phase space and $\lambda_1 \geq \lambda_2 \geq \lambda_3$ refer to the Lyapunov exponents [36].

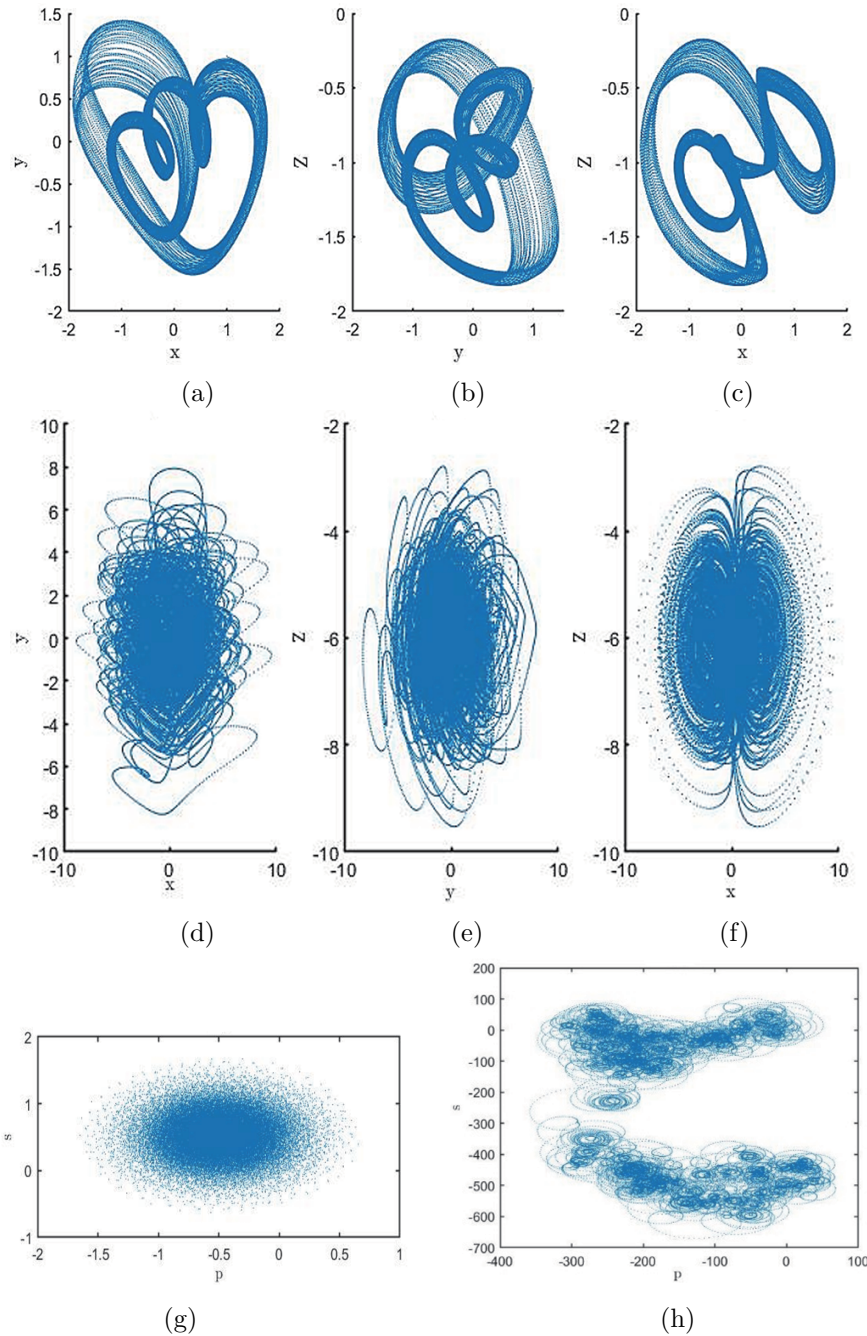


FIGURE 1. Phase space representation of system (3) across various planes: (a) xy_plane (periodic dynamics); (b) yz_plane (periodic dynamics); (c) xz_plane (periodic dynamics); (d) xy_plane (chaotic dynamics); (e) yz_plane (chaotic dynamics); (f) xz_plane (chaotic dynamics); (g) 0-1 test for periodic motion; (h) 0-1 test for chaotic motion

Figures 2 presents in (a) the Lyapunov exponents and (b) bifurcation diagram showing the local maxima of x plotted against fractional order parameter γ . It is observed that both diagrams are consistent in characterizing the dynamical behavior of the system. To analyze the system's sensitivity with respect to γ , we note that even small changes in parameter γ induce move from periodic behavior to chaotic behavior, thus showing

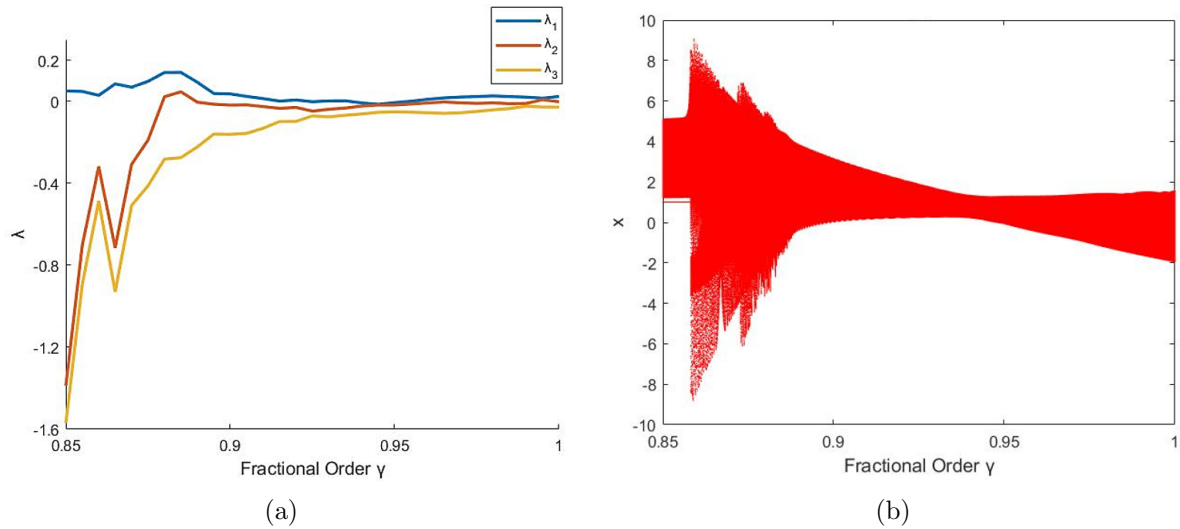


FIGURE 2. Lyapunov exponents and bifurcation diagram with respect to $\gamma = \gamma_1 = \gamma_2 = \gamma_3$: (a) Lyapunov exponents; (b) bifurcation diagram

its role at the beginning of chaotic behavior, also the effectiveness of γ is changed with attractor's size and complicity which in turn controls the amount and intensity of chaos.

3.2.2. *Large-scale chaos influenced by μ_3 and μ_4 .* Through numerical simulation, we note that the system (3) has a wide region of chaos with respect to the change of parameters μ_3 and μ_4 . Figure 3(a) and Figure 4(a) show the spectrum of Lyapunov exponents for μ_3 and μ_4 . Obviously, the values of maximal Lyapunov exponent exhibit positive within the range $\mu_3 > 3$ and $\mu_4 > 2$ which indicate that the system has chaotic behavior. It is important to refer that in Figure 3 and Figure 4. We display only limit range ($1 \leq \mu_3 \leq 10$) and ($1 \leq \mu_4 \leq 10$) but when μ_3 exceeds 10 and μ_4 exceeds 10 consistently exhibits chaotic behavior. Additionally, through Figure 3(b) and Figure 4(b), it has been observed that the results obtained by Lyapunov exponents analysis are compatible with irregular points in bifurcation diagrams.

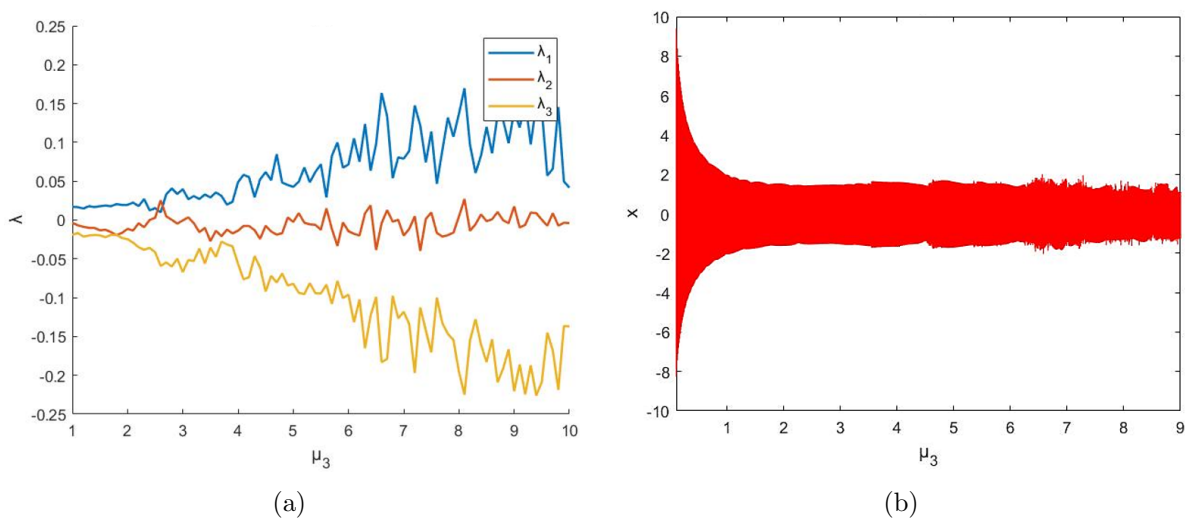


FIGURE 3. Lyapunov exponents and bifurcation diagram with respect to μ_3 : (a) Lyapunov exponents; (b) bifurcation diagram

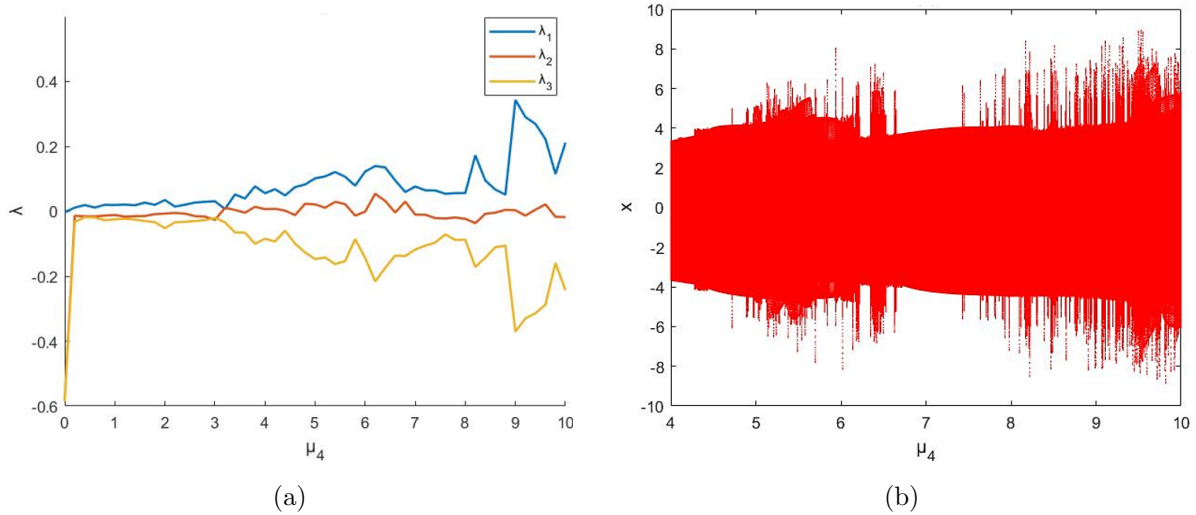


FIGURE 4. Lyapunov exponents and bifurcation diagram with respect to μ_4 : (a) Lyapunov exponents; (b) bifurcation diagram

3.2.3. *Investigating the influence of parameters μ_2 , μ_5 and α on chaos.* Fix parameters $\mu_1 = 2$, $\mu_3 = 2$, $\mu_4 = 1$, $\mu_5 = 2$, $\alpha = 0.01$ and let μ_2 vary. The spectrum of Lyapunov exponents of the proposed system (3) versus μ_2 are computed, and the results are shown in Figure 5(a). It is observed that the effectiveness of μ_2 on chaotic behavior of this system is less significant compared to parameters μ_3 and μ_4 . These results are consistent with bifurcation diagram in Figure 5(b). The remaining parameters μ_5 and α exhibit a very minimal effect on the chaotic behavior as clearly illustrated by spectrum of Lyapunov exponents in Figures 6(a) and 7(a) and bifurcation diagram in Figures 6(b) and 7(b).

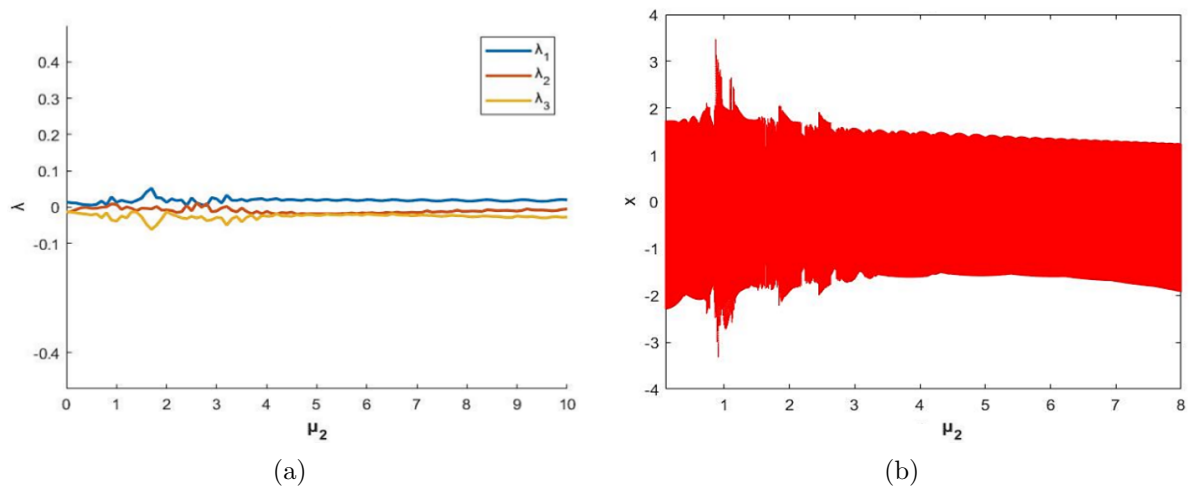


FIGURE 5. Lyapunov exponents and bifurcation diagram with respect to μ_2 : (a) Lyapunov exponents; (b) bifurcation diagram

3.3. **Entropy analysis.** Entropy measures are extensively applied to quantifying the complexity of dynamical systems in various fields. Shannon [37] introduced a measure of uncertain quantify as follows

$$H(x; \theta) = -E(\ln(f(x; \theta))) \tag{6}$$

where x and $f(x; \theta)$ represent random variable and its probability function, respectively. Kolmogorov-Sinai (KS) entropy measures the rate of information production during the evolution of a chaotic system, reflecting the amount of information either gained or lost over time. Pincus [38] proposed approximate entropy as a method for quantifying the complexity of time series data. [39] presented the formula elucidating the relationship between KS entropy and Lyapunov exponents $\lambda_i, i = 1, 2, \dots, n$ as follows:

$$KSH = \sum_{\lambda_i > 0} \lambda_i \tag{7}$$

In fact, as entropy evolves over time, it plays a crucial role in dynamical systems, serving as a key indicator to determine whether a trajectory is regular [40]. Shannon entropy serves as a fundamental basis in the computation of Kolmogorov-Sinai (KS) entropy. While Shannon entropy quantifies the uncertainty of random variables without considering temporal evolution, KS entropy extends this concept to dynamical systems by measuring the rate of information production over time. Consequently, it acts as an indicator of

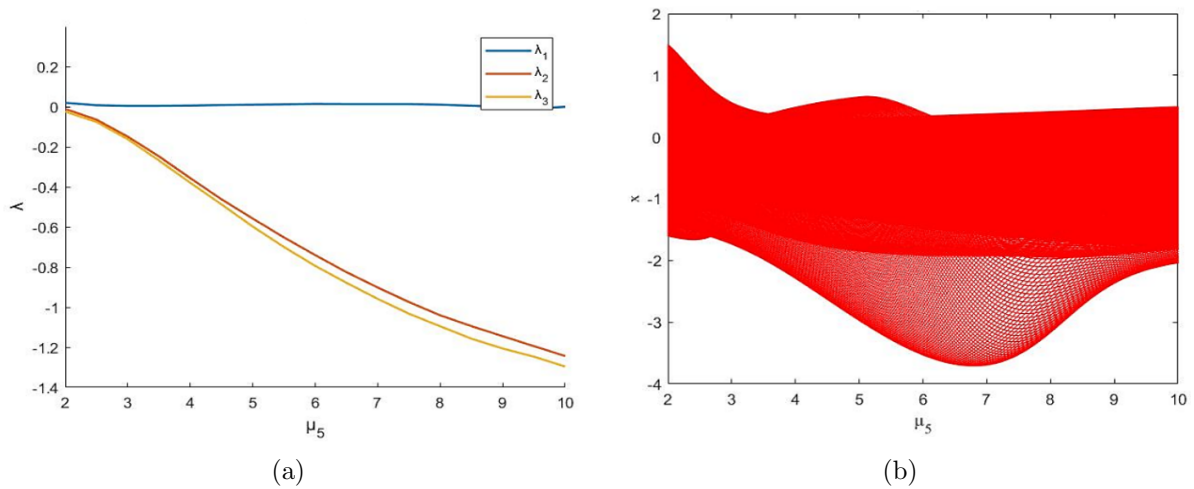


FIGURE 6. Lyapunov exponents and bifurcation diagram with respect to μ_5 : (a) Lyapunov exponents; (b) bifurcation diagram

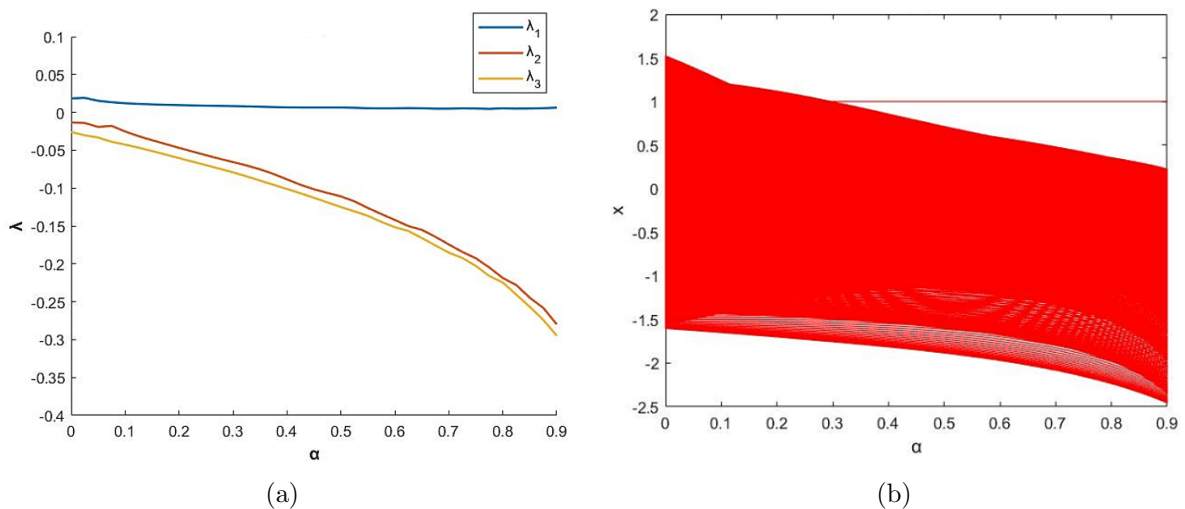


FIGURE 7. Lyapunov exponents and bifurcation diagram with respect to α : (a) Lyapunov exponents; (b) bifurcation diagram

chaos within the system. Mathematically, KS entropy can be considered a supremum of Shannon’s entropy [41]. The entropy of system (3) for parameters $\mu_2, \mu_3, \alpha, \mu_4, \mu_5$ and γ are shown in Figure 8 when the initial state $(x_0, y_0, z_0) = (1, 1, 1)$. It is important to refer, if the values of entropy converge to zero, then it indicates that the system is either non-chaotic or that the chaotic behavior of the system has nearly vanished. On the contrary, when it moves away from 0, then it indicates that the system is chaotic. Obviously, through Figures 1-7, the results obtained by chaotic dynamics analysis are compatible with entropy results in Figure 8.

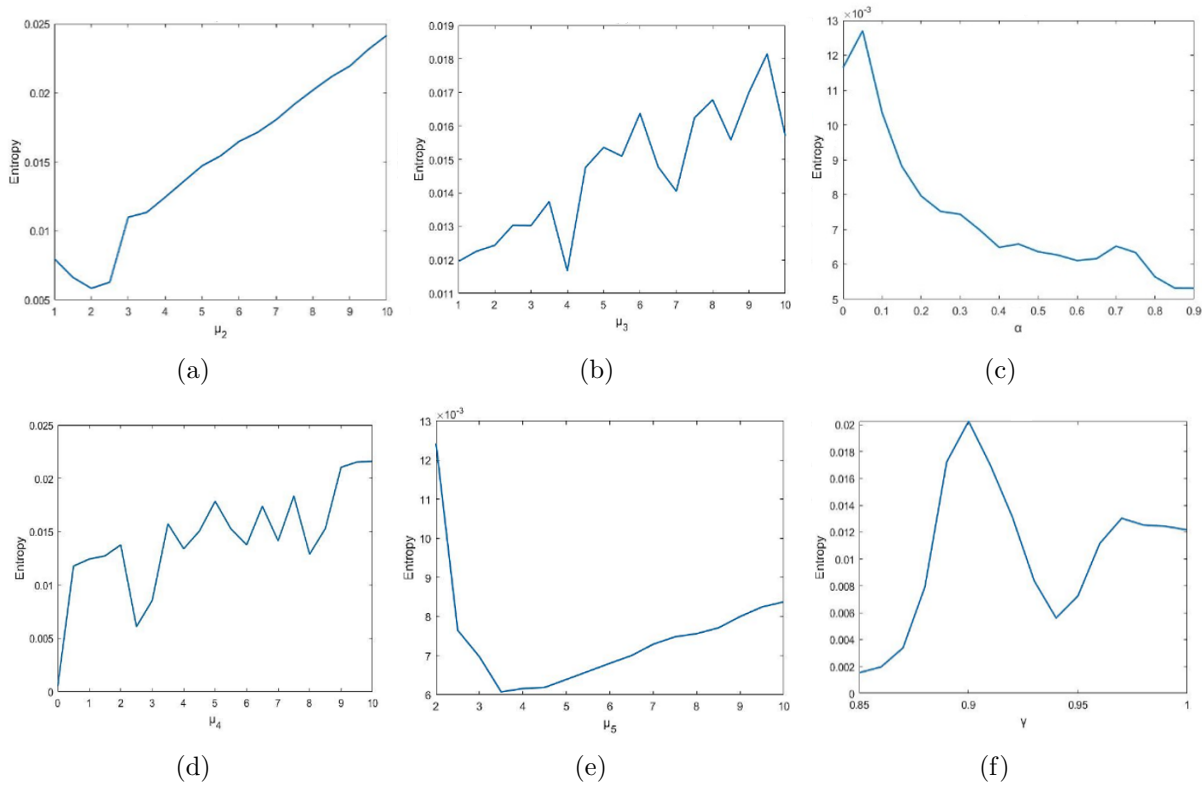


FIGURE 8. Evolutions of entropy varying as (a) $\mu_2 \in [1, 10]$; (b) $\mu_3 \in [1, 10]$; (c) $\alpha \in (0, 0.9)$; (d) $\mu_4 \in [0, 10]$; (e) $\mu_5 \in [2, 10]$; (f) $\gamma \in [0.85, 1]$ when the initial state $(x_0, y_0, z_0) = (1, 1, 1)$

3.4. Key sensitivity. The time series of x generated from two initial states $(1, 1, 1)$ and $(1.001, 1, 1)$ are plotted in Figure 9 that represents the test of key sensitivity results for system (3). It is observed that at the beginning, the two paths are almost the same, but their differences are increasing after a number of iterations, this means that the system (3) has high sensitivity that depends on initial states and its future behavior cannot be unpredictable in the long term rendering it suitable in practical aspects of medical image encryption.

4. Medical Image Encryption Application.

4.1. Design and implementation of the proposed algorithm. Fractional chaotic systems possess unique and distinctive features properties such as high sensitivity to initial states, deterministic but unpredictability, complexity in their structure, non-repetition of its states, and homogeneous distribution of states, making it difficult to predict their

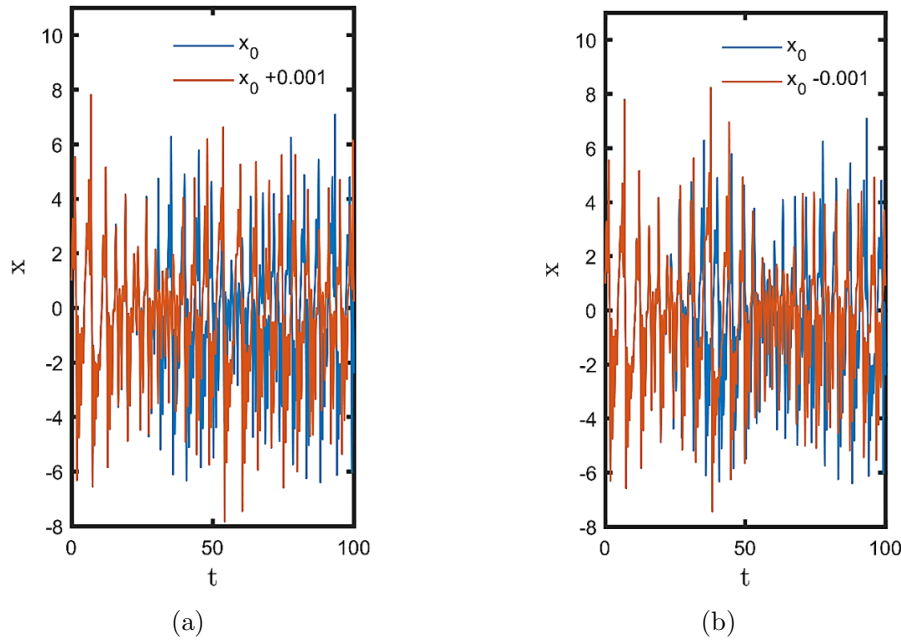


FIGURE 9. Test of sensitivity results of system (3): (a) $(x_0, y_0, z_0) = (1, 1, 1)$; (b) $(x_0, y_0, z_0) = (1.001, 1, 1)$

outputs. Therefore, using them in medical image encryption achieves high levels of security. It is well known that the relationship between pixels' position in image array is strong and important and this relation can lead to the loss of essential information in the image. One of the techniques used to break this correlation is shuffling, which is applied to the array of the original image. Let A be a matrix with size $m \times n$. The shuffled version of A denoted by Sh_A^{rc} generates as follows. 1) (Rows shuffling) The logistic map is used to generate the values $i_1, i_2, \dots, i_m, i_k \neq i_l, k \neq l$. We change the components position of a matrix

A according to these values in the following form $Sh_A^r = \begin{bmatrix} A_{i_1,1} & A_{i_1,2} & \dots & A_{i_1,n} \\ A_{i_2,1} & A_{i_2,2} & \dots & A_{i_2,n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{i_m,1} & A_{i_m,2} & \dots & A_{i_m,n} \end{bmatrix}$. 2)

(Columns shuffling) The similar method in 1) is used to generate the values $j_1, j_2, \dots, j_m, j_k \neq j_l, k \neq l$. Accordingly, we obtain final form of A shuffled version of as follows:

$$Sh_A^{rc} = \begin{bmatrix} A_{i_1,j_1} & A_{i_1,j_2} & \dots & A_{i_1,j_n} \\ A_{i_2,j_1} & A_{i_2,j_2} & \dots & A_{i_2,j_n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{i_m,j_1} & A_{i_m,j_2} & \dots & A_{i_m,j_n} \end{bmatrix}.$$

Now, steps of our proposed algorithm of medical image encryption are described as follows.

Step 1: The original image is read into an array \mathcal{M} having dimensions $m \times n \times 3$.

Step 2: Extract the color image RGB into the matrices $\mathcal{M}_R, \mathcal{M}_G, \mathcal{M}_B$ with dimensions $m \times n$.

Step 3: Generate shuffled versions of arrays $Sh_{\mathcal{M}_R}^{rc}, Sh_{\mathcal{M}_G}^{rc}$ and $Sh_{\mathcal{M}_B}^{rc}$.

Step 4: Convert the elements of arrays $Sh_{\mathcal{M}_R}^{rc}, Sh_{\mathcal{M}_G}^{rc}$ and $Sh_{\mathcal{M}_B}^{rc}$ from decimal to binary $SR = \{SR_1, SR_2, \dots, SR_{mn}\}, SG = \{SG_1, SG_2, \dots, SG_{mn}\}$ and $SB = \{SB_1, SB_2, \dots, SB_{mn}\}$, respectively.

Step 5: Inject the values x_0, y_0, z_0 of initial state of fractional chaotic system () to generate the sequences $x = \{x_1, x_2, \dots, x_{mn}\}$, $y = \{y_1, y_2, \dots, y_{mn}\}$ and $z = \{z_1, z_2, \dots, z_{mn}\}$.

Step 6: Rewrite the sequences derived from Step 5 in the following form $x = \text{floor}(\text{mod}(x \times 10^{14}), 256)$, $y = \text{floor}(\text{mod}(y \times 10^{14}), 256)$, $z = \text{floor}(\text{mod}(z \times 10^{14}), 256)$.

Step 7: Transform the values derived of x, y and z in Step 6 from decimal to binary as follows $\mathcal{B}^x, \mathcal{B}^y$ and \mathcal{B}^z , respectively.

Step 8: (a) Use a bitwise XOR operation by using $\mathcal{B}^x, \mathcal{B}^y$ and \mathcal{B}^z as follows $\mathcal{B}_1 = \text{bitxor}(\mathcal{B}^x, \mathcal{B}^y)$, $\mathcal{B}_2 = \text{bitxor}(\mathcal{B}^y, \mathcal{B}^z)$, $\mathcal{B}_3 = \text{bitxor}(\mathcal{B}^x, \mathcal{B}^z)$.

(b) Use a bitwise XOR operation between SR, SG and SB with $\mathcal{B}_1, \mathcal{B}_2$ and \mathcal{B}_3 , respectively, as follows $e_1 = \text{bitxor}(SR, \mathcal{B}_1)$, $e_2 = \text{bitxor}(SG, \mathcal{B}_2)$, $e_3 = \text{bitxor}(SB, \mathcal{B}_3)$.

Step 9: Transform the values of e_1, e_2 and e_3 from binary to decimal, D_R, D_G and D_B , respectively.

Step 10: Resize the vectors in Step 9 to be arrays C_R, C_G, C_B having dimensions $m \times n$ as the cipher images of $\mathcal{M}_R, \mathcal{M}_G, \mathcal{M}_B$, respectively.

Step 11: Merge the arrays C_R, C_G, C_B into a single color array C , representing the encrypted image of the original image.

Additionally, the steps of our decryption algorithm are outlined in detail as follows.

Step 1: Extract the encrypted image C into the matrices C_R, C_G, C_B with dimensions $m \times n$.

Step 2: Resize the matrices C_R, C_G, C_B to the vectors D_R, D_G and D_B , respectively.

Step 3: Transform the values of D_R, D_G and D_B from decimal to binary as follows e_1, e_2 and e_3 , respectively.

Step 4: Repeat Steps 5-7 in the encryption algorithm to obtain $\mathcal{B}^x, \mathcal{B}^y$ and \mathcal{B}^z .

Step 5: Perform the bitwise XOR between the following values: $\mathcal{B}_1 = \text{bitxor}(\mathcal{B}^x, \mathcal{B}^y)$, $\mathcal{B}_2 = \text{bitxor}(\mathcal{B}^y, \mathcal{B}^z)$, $\mathcal{B}_3 = \text{bitxor}(\mathcal{B}^x, \mathcal{B}^z)$, $F_R = \text{bitxor}(\mathcal{B}_1, e_1)$, $F_G = \text{bitxor}(\mathcal{B}_2, e_2)$, $F_B = \text{bitxor}(\mathcal{B}_3, e_3)$.

Step 6: Convert the values of F_R, F_G and F_B from binary to decimal, say Z_R, Z_G and Z_B , respectively.

Step 7: Resize Z_R, Z_G and Z_B to be arrays $Sh_{Z_R}^{rc}, Sh_{Z_G}^{rc}$ and $Sh_{Z_B}^{rc}$, respectively.

Step 8: Use the shuffling method and arrays merging to obtain the decryption image, say DE.

4.2. Experimental analysis. In this subsection, we evaluate the performance of presented algorithm through a series of experiments conducted on various medical images. To assess its efficacy, we applied the algorithm to four standard test images which they are saved in the following sources: “Montgomery country chest X-ray dataset” [42], “Ultrasonic Brachial Plexus dataset” [43] and “BraTS18 dataset” [44]. It is also pointed out that these images have also been utilized by authors in [2]. Additionally, these images were selected for their widespread use and they represent different anatomical the regions. Also, all chosen images have the length of (256×256) pixels). The analysis aims to demonstrate the algorithm’s effectiveness and robustness in practice.

Through medical images in Figure 10, the experimental results obtained indicate that the effectiveness of the image encryption process under investigation is good and acceptable when compared with previous studies. The proposed three-dimensional chaotic system generates a chaotic sequence that is very close to randomness, which shows high sensitivity to the initial conditions and encryption key. Therefore, any change in its parameters results in a significantly different encrypted image, which is sufficient to resist brute force attacks and provides enhanced security.

Image histograms visually illustrate the distribution of a continuous numeric variable by measuring the distribution of pixels at which certain values appear in the image. It

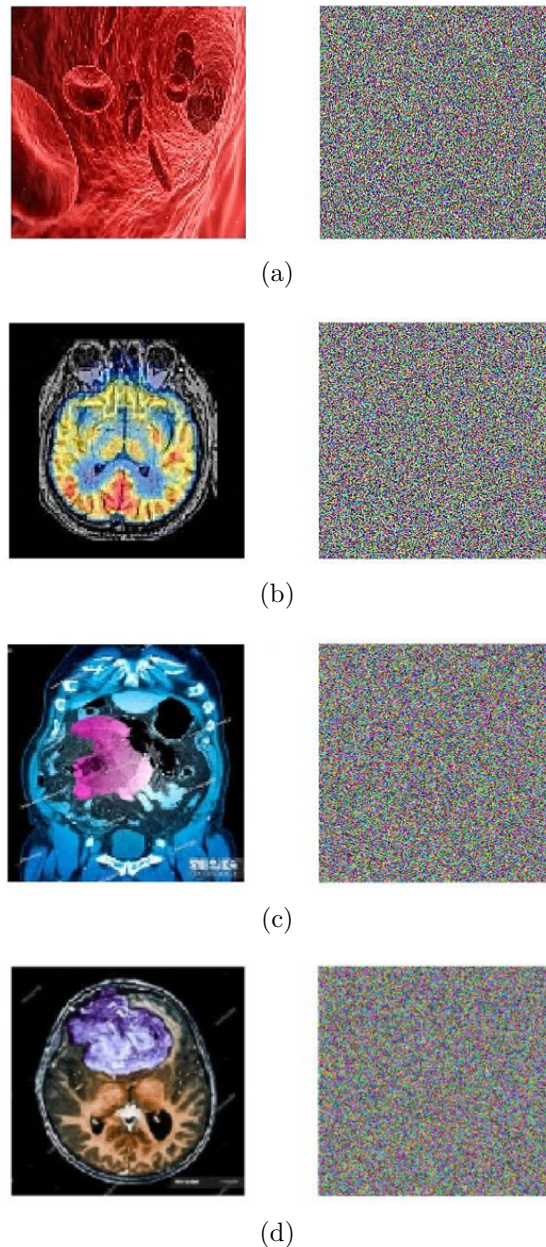


FIGURE 10. (a), (b), (c) and (d) refer to the original images and their corresponding cipher images of Image 1, Image 2, Image 3 and Image 4, respectively.

plots the number of pixels for each tonal. A substantial amount of statistical information about the image can be derived from its histogram. The histogram of the encrypted image should exhibit a uniform distribution and should differ from the histogram of the original image. The analysis of the proposed algorithm using a histogram is introduced in Figure 11. The histograms of four test medical images on the left side, and their encrypted images on the right side are illustrated. It can be observed that the histograms of original images are shown irregular in pixel level frequency distributions. However, the cipher images produced using proposed algorithm have quite uniform and flat pixel-level frequency distribution. Therefore, it is highly effective at resisting histogram attacks.

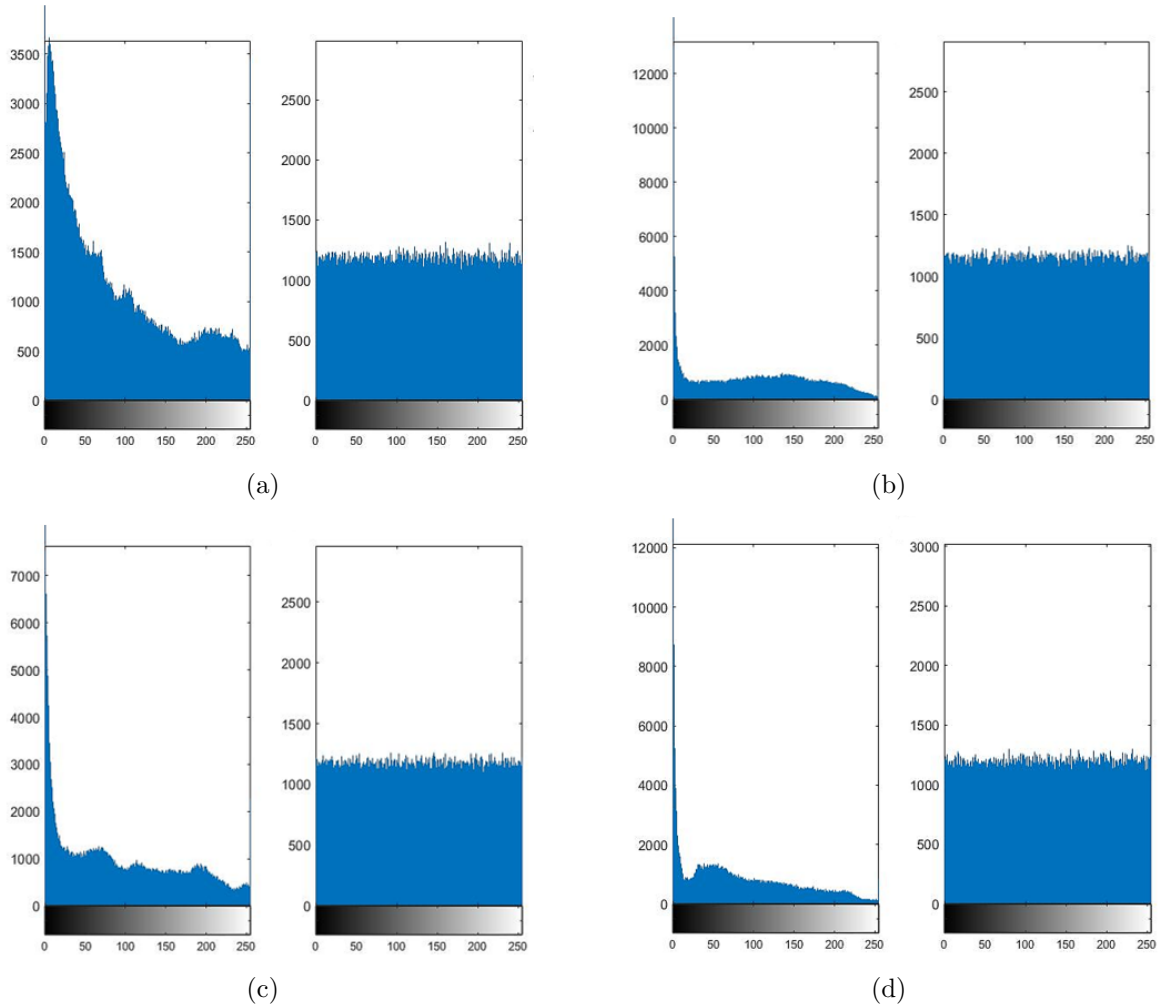


FIGURE 11. (a), (b), (c) and (d) refer to original image and cipher image histograms of Image 1, Image 2, Image 3 and Image 4, respectively.

4.3. Statistical and entropy analysis. It is known that statistical tests, particularly the Chi-square indicate that the histogram of encrypted image does not differ significantly from a uniform distribution. This is achieved when the Chi-square test value is less than the tabulated value, indicating the acceptance of the null hypothesis, which suggests that the distributions approximate a uniform distribution.

This test [45] can be used to determine the uniformity and homogeneity of data by the following formula:

$$x^2 = \sum_{r=1}^{\tau} \frac{(M_r - m_r)^2}{m_r} \tag{8}$$

where $\tau = 256$ and M_r and m_r represent the observed and the expected occurrence frequencies of each gray level (0-255). Here, the value 0.05 is used as a level of significance. This test allows the assessment of discrepancies between observed and expected distributions, helping to determine whether the data conform to or deviate significantly from an expected pattern. The results are listed in Table 1, and it is observed that the computed values of x^2 of cipher images are less than the critical value $x^2(255, 0.05)$. It is evident that $x^2(\tau - 1, 0.05) = x^2(255, 0.05) = 293$. Consequently, the test image passes the test successfully at this significance level, indicating that the histograms of the cipher images exhibit a uniform distribution.

TABLE 1. Calculating x^2 of original and cipher images at $\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 2$, $\mu_4 = 10$, $\mu_5 = 2$, $\alpha = 0.01$ and initial vector $(x_0, y_0, z_0) = (1, 1, 1)$ with fractional orders: $\gamma_1 = \gamma_2 = \gamma_3 = 0.99$

	x^2	
Image	Original image	Cipher image
Image 1	145880	302
Image 2	5826182	224
Image 3	1711330	219
Image 4	4686188	245

TABLE 2. Calculating entropy of original and cipher images at $\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 2$, $\mu_4 = 10$, $\mu_5 = 2$, $\alpha = 0.01$ and initial vector $(x_0, y_0, z_0) = (1, 1, 1)$ with fractional orders: $\gamma_1 = \gamma_2 = \gamma_3 = 0.99$

	<i>Entropy</i>	
Image	Original image	Cipher image
Image 1	7.7317	7.9995
Image 2	6.1872	7.9997
Image 3	7.1563	7.9997
Image 4	6.3120	7.9996

Entropy is a measure of the uncertainty of an information source. Clearly, the entropy value of ideal information of a random message consisting of 8 bit equals 8. Here, we use the following formula in (9) to compute the entropy of the information source $y = (y_0, y_1, \dots, y_{N-1})$

$$H(x) = - \sum_{r=1}^{N-1} p(y_i) \log_2 p(y_i) \quad (9)$$

The entropy test of four medical images are listed in Table 2. The results show that the values of entropy are acceptable when compared to previous works and are very close to the theoretical value 8.

5. Conclusions. At first glance, the chaotic behavior of first order dynamical system in [36] may appear quite similar to that of a fractional system in Equation (3). However, the dynamics associated with the fractional-order is considerably more complicated due to their enhanced sensitivity to fractional orders. In practical aspects such as encryption of medical information, the fractional orders introduce a new parameter that can play a pivotal and significant role, due to its flexibility as it expands the system range. Our results demonstrate that our proposed system has a wide range of chaotic behavior as demonstrated through Lyapounov exponents and entropy diagrams. Furthermore, our findings reveal that the designed system exhibits the property of coexisting attractors. Furthermore, our work introduces a meticulously designed algorithm for medical image encryption. This algorithm was designed in two stages. The first stage is the shuffling of the elements of pixel matrix for the image intended encryption and the second stage is the subsequent encryption performed utilizing the proposed fractional chaotic systems, as conceptualized in this study. Experimental analysis conducted on a diverse set of medical pictures reveals that the proposed algorithm not only achieves a high level of effectiveness but also demonstrates exceptional robustness, rendering it highly resistant to cryptanalytic attacks and thereby significantly enhancing overall security. As a future

goal, we recommend applying the algorithm used into another medical information other than medical images.

Acknowledgment. The authors gratefully acknowledge the helpful comments and suggestions of the reviewers, which have improved the presentation.

REFERENCES

- [1] S. H. Abid and U. J. Quaez, Capacity of control for stochastic dynamical systems perturbed by mixed fractional Brownian motion with delay in control, *International Journal of Innovative Computing, Information and Control*, vol.15, no.5, pp.1913-1934, DOI: 10.24507/ijicic.15.05.1913, 2019.
- [2] S. Inam, S. Kanwal, R. Firdous and F. Hajjej, Blockchain based medical image encryption using Arnold's cat map in a cloud environment, *Scientific Reports*, vol.14, no.1, 5678, 2024.
- [3] S. H. Abid and U. J. Quaez, Rényi entropy for mixture model of multivariate skew Laplace distributions, *Journal of Physics: Conference Series*, 12037, 2020.
- [4] Z.-A. S. A. Rahman, B. H. Jasim, Y. I. A. Al-Yasir, Y.-F. Hu, R. A. Abd-Alhameed and B. N. Alhasnawi, A new fractional-order chaotic system with its analysis, synchronization, and circuit realization for secure communication applications, *Mathematics*, vol.9, no.20, 2593, 2021.
- [5] A. Daoui et al., Biomedical Multimedia encryption by fractional-order Meixner polynomials map and quaternion fractional-order Meixner moments, *IEEE Access*, vol.10, pp.102599-102617, 2022.
- [6] A.-A. Khennaoui, A. Ouannas, S. Bendoukha, X. Wang and V.-T. Pham, On chaos in the fractional-order discrete-time unified system and its control synchronization, *Entropy*, vol.20, no.7, 530, 2018.
- [7] A. K. Bagheedh, S. H. Abid and S. A. Mehdi, Design a fractional chaotic logistic dynamical system, *Journal of Physics: Conference Series*, 12055, 2021.
- [8] E. F. D. Goufo, Solvability of chaotic fractional systems with 3D four-scroll attractors, *Chaos, Solitons & Fractals*, vol.104, pp.443-451, 2017.
- [9] A. Ouannas, A.-A. Khennaoui, Z. Odiat, V.-T. Pham and G. Grassi, On the dynamics, control and synchronization of fractional-order Ikeda map, *Chaos, Solitons & Fractals*, vol.123, pp.108-115, 2019.
- [10] G. A. Beyene, F. Rahma, K. Rajagopal, A.-B. A. Al-Hussein and S. Boulaaras, Dynamical analysis of a 3D fractional-order chaotic system for high-security communication and its electronic circuit implementation, *Journal of Nonlinear Mathematical Physics*, vol.30, no.4, pp.1375-1391, 2023.
- [11] F. Yu, S. Xu, Y. Lin, T. He, C. Wu and H. Lin, Design and analysis of a novel fractional-order system with hidden dynamics, hyperchaotic behavior and multi-scroll attractors, *Mathematics*, vol.12, no.14, 2227, 2024.
- [12] A. Sambas et al., A novel 3D chaotic system with line equilibrium: Multistability, integral sliding mode control, electronic circuit, FPGA implementation and its image encryption, *IEEE Access*, vol.10, pp.68057-68074, 2022.
- [13] J. Dambre, D. Verstraeten, B. Schrauwen and S. Massar, Information processing capacity of dynamical systems, *Scientific Reports*, vol.2, no.1, 514, 2012.
- [14] S. Tiomkin, D. Polani and N. Tishby, Control capacity of partially observable dynamic systems in continuous time, *arXiv Preprint*, arXiv: 1701.04984, 2017.
- [15] P. Gaspard and X.-J. Wang, Sporadicity: Between periodic and chaotic dynamical behaviors, *Proceedings of the National Academy of Sciences*, vol.85, no.13, pp.4591-4595, 1988.
- [16] T. Sivakumar and R. Venkatesan, Image encryption based on pixel shuffling and random key stream, *Int. J. Comput. Inf. Technol*, vol.3, no.6, pp.1468-1476, 2014.
- [17] W. Zhu and Y. Shen, Encryption algorithms using chaos and CAT methodology, *2010 International Conference on Anti-Counterfeiting, Security and Identification*, pp.20-23, 2010.
- [18] J. Zhang, D. Fang and H. Ren, Image encryption algorithm based on DNA encoding and chaotic maps, *Mathematical Problems in Engineering*, vol.2014, no.1, 917147, 2014.
- [19] L. Xu, Z. Li, J. Li and W. Hua, A novel bit-level image encryption algorithm based on chaotic maps, *Optics and Lasers in Engineering*, vol.78, pp.17-25, 2016.
- [20] S. S. Askar, A. A. Karawia, A. Al-Khedhairi and F. S. Al-Ammar, An algorithm of image encryption using logistic and two-dimensional chaotic economic maps, *Entropy*, vol.21, no.1, 44, 2019.
- [21] G. Deco and B. Schurmann, Information flow and chaotic dynamics, *Proc. of International Workshop on Neural Networks for Identification, Control, Robotics and Signal/Image Processing*, pp.321-329, 1996.

- [22] M. Y. M. Parvees, J. A. Samath and B. P. Bose, Medical images are safe – An enhanced chaotic scrambling approach, *Journal of Medical Systems*, vol.41, pp.1-16, 2017.
- [23] M. Benssalah, Y. Rhaskali and M. S. Azzaz, Medical images encryption based on elliptic curve cryptography and chaos theory, *2018 International Conference on Smart Communications in Network Technologies (SaCoNeT)*, pp.222-226, 2018.
- [24] J. Rinkel, D. Magalhes, F. Wagner, E. Frojdh and R. B. Sune, Equalization method for Medipix3RX, *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, vol.801, pp.1-6, 2015.
- [25] E. N. Gimenez, V. Astromskas, I. Horswell, D. Omar, J. Spiers and N. Tartoni, Development of a Schottky CdTe Medipix3RX hybrid photon counting detector with spatial and energy resolving capabilities, *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, pp.101-103, DOI: 101016/j.nima201510092, 2016.
- [26] H. Abdel-Nabi and A. Al-Haj, Medical imaging security using partial encryption and histogram shifting watermarking, *2017 8th International Conference on Information Technology (ICIT)*, pp.802-807, 2017.
- [27] S. S. Moafimadani, Y. Chen and C. Tang, A new algorithm for medical color images encryption using chaotic systems, *Entropy*, vol.21, no.6, 577, 2019.
- [28] A. Z. Hussain, Securing medical images using chaotic map encryption and LSB steganography, *Revue d'Intelligence Artificielle*, vol.38, no.1, 313, 2024.
- [29] L. Zhang, X. Song, A. A. A. El-Latif, Y. Zhao and B. Abd-El-Atty, Reversibly selective encryption for medical images based on coupled chaotic maps and steganography, *Complex & Intelligent Systems*, vol.10, no.2, pp.2187-2213, 2024.
- [30] M. Gafsi, N. Abbassi, M. A. Hajjaji, J. Malek and A. Mtibaa, Improved chaos-based cryptosystem for medical image encryption and decryption, *Scientific Programming*, vol.2020, no.1, 6612390, 2020.
- [31] M. T. Gatta and S. T. Abd Al-Latif, Medical image security using modified chaos-based cryptography approach, *Journal of Physics: Conference Series*, 12036, 2018.
- [32] S. H. Abid, S. Q. Hasan and U. J. Quaez, Approximate controllability of fractional Sobolev type stochastic differential equations driven by mixed fractional Brownian motion, *Journal of Mathematical Sciences and Applications*, vol.3, no.1, pp.3-11, 2015.
- [33] A. Anguraj and M. L. Maheswari, Existence of solutions for fractional impulsive neutral functional infinite delay integrodifferential equations with nonlocal conditions, *J. Nonlinear Sci. Appl.*, vol.5, pp.271-280, 2012.
- [34] R. Garrappa, Numerical solution of fractional differential equations: A survey and a software tutorial, *Mathematics*, vol.6, no.2, 16, 2018.
- [35] C. Li and C. Tao, On the fractional Adams method, *Computers & Mathematics with Applications*, vol.58, no.8, pp.1573-1588, 2009.
- [36] Y. Wang, X. Leng, C. Zhang and B. Du, Adaptive fast image encryption algorithm based on three-dimensional chaotic system, *Entropy*, vol.25, no.10, 1399, 2023.
- [37] C. E. Shannon, A mathematical theory of communication, *The Bell System Technical Journal*, vol.27, no.3, pp.379-423, 1948.
- [38] S. M. Pincus, Approximate entropy as a measure of system complexity, *Proceedings of the National Academy of Sciences*, vol.88, no.6, pp.2297-2301, 1991.
- [39] J. B. Pesin, Ljapunov characteristic exponents and ergodic properties of smooth dynamical systems with an invariant measure, *Hamiltonian Dynamical Systems*, pp.512-515, 2020.
- [40] M. Mondal, F. Rahaman and K. Singh, Lyapunov exponent, ISCO and Kolmogorov-Senai entropy for Kerr-Kiselev black hole, *The European Physical Journal C*, vol.81, no.1, pp.1-16, 2021.
- [41] V. H. Gaidhane, Y. V Hote and V. Singh, A new approach for estimation of eigenvalues of images, *International Journal of Computer Applications*, vol.26, no.9, pp.1-6, 2011.
- [42] S. Jaeger, S. Candemir, S. Antani, Y.-X. J. Wang, P.-X. Lu and G. Thoma, Two public chest X-ray datasets for computer-aided screening of pulmonary diseases, *Quantitative Imaging in Medicine and Surgery*, vol.4, no.6, 475, 2014.
- [43] M. Baby and A. S. Jereesh, Automatic nerve segmentation of ultrasound images, *2017 International Conference of Electronics, Communication and Aerospace Technology (ICECA)*, pp.107-112, 2017.
- [44] B. H. Menze et al., The multimodal brain tumor image segmentation benchmark (BRATS), *IEEE Transactions on Medical Imaging*, vol.34, no.10, pp.1993-2024, 2014.
- [45] H. S. Kwok and W. K. S. Tang, A fast image encryption system based on chaotic maps with finite precision representation, *Chaos, Solitons & Fractals*, vol.32, no.4, pp.1518-1529, 2007.

Author Biography



Uday Jabbar Quaez obtained his Ph.D. degree in Mathematics from Al-Mustansiriyah University, Iraq in 2020; he worked at the Iraqi Ministry of Education as the postdoctoral from 2020 to 2023.

Dr. Quaez is currently a full-time professor at Department of Mathematics, Education College, Al-Mustansiriyah University, Baghdad, Iraq. His main research interests include the entropy, information theory in dynamical systems, chaotic theory, time series and multivariate analysis. He has published numerous research papers on well-known journals.