International Journal of Innovative Computing, Information and Control
Volume **21**, Number **5**, October **2025**

pp. 1435–1450

NONLINEAR DIFFERENTIAL GEOMETRY CONTROLLER DESIGN FOR AIRCRAFT FLIGHT SYSTEM

Ruchun Wen, Qing Ouyang, Jianhong Wang and Bohua Zeng
School of Electronic Engineering and Automation
Jiangxi University of Science and Technology
No. 86, Hongqi Road, Ganzhou 341000, P. R. China
{ 9119940012; 6920230722 }@jxust.edu.cn; wangjianhong@nuaa.edu.cn; zeng341000@163.com

Received January 2025; revised May 2025

Abstract. *As all phenomena in nature are nonlinearity, and almost the existing linear control theory is suited only for linear system, to develop the advanced control theory for nonlinear system, one new strategy is needed for nonlinear controller design based on the idea of differential geometry during some practical engineers. More specifically, consider an aircraft flight system, corresponding to one nonlinear closed loop system with nonlinear aircraft system and nonlinear controller simultaneously, differential geometry is proposed to design that nonlinear controller directly after that nonlinear aircraft system is transformed into one special nonlinear piecewise form through some interesting nonlinear transformations. The merit of nonlinear control is to generate one nonlinear controller directly, without any linearizing process. Additionally, the sequent nonlinear stability is analyzed through our own mathematical derivations for our considered nonlinear closed loop system. Finally, a practical simulation example is given to show our theoretical results. Generally, this paper considers two important aspects from the nonlinear control theory, i.e., nonlinear control design and nonlinear stability analysis.*
Keywords: Nonlinear aircraft control, Nonlinear stability, Differential geometry

1. **Introduction.** With the continuous improvement and complexity of battlefield environment and mission requirements, the basic mode of future war is moving towards unmanned, intelligent and information-based. However, single aircraft is limited by its own conditions in the face of complex and diverse application environment. For civilian use, the scanning range of a single aircraft is limited by the on-board scanning sensors, making it impossible to complete tasks, such as forest patrol and surveying and mapping in a single voyage, resulting in inefficiency. For military use, when a single aircraft carries a limited amount of its own fuel, offensive weapons, etc., it will have a limited lethality, unable to form a sustained and beneficial strike force, and also unable to continue its mission due to a malfunction, a problem that ultimately leads to the loss of aircraft and affects the entire battle plan. As aircraft is popular in military or our normal life, the researches on aircraft are ongoing from different aspects, such as aircraft control, aircraft detection and aircraft attacking.

As aircraft control is to guarantee the considered aircraft fly according to the desired trajectory or path, while considering other flight missions, for example, minimum flight time or minimum short flight trajectory, this paper concerns on aircraft control for one general case, i.e., nonlinear control, after linearizing the original nonlinear aircraft system, and then based on this linearized state space system, the existing knowledge about linear quadratic Gaussian control or adaptive control can be applied directly. During these recent years, one innovative control method is studied very popularly, i.e., data driven

DOI: 10.24507/ijicic.21.05.1435

control strategy. As data-driven control only needs input-output data analysis process, it is also strongly supported in the control field. [1] designed the optimal output stable control strategy using the input-output data of the system. [2] identified the aircraft parameters using frequency-domain response combined with American commercial software, and designed a linear quadratic controller for attitude control, while the position controller is designed based on Lyapunov and backstepping theory. [3] designed a class of nonlinear discrete single input and single output systems using pseudo-gradients and input-output data, which have good robustness. [4] used the Kalman filter to estimate the parameters of the longitudinal model in detail, and then designed the linear quadratic controller based on the minimum energy principle. [5] used the excitation signal as input signal to identify the linear parameter varying model, and redesigned the position and attitude controller according to the cascade structure. [5] identified disturbances generated by changes in wind and load and predicted the output results. [6] combined data-driven and consistency principles, then a data-driven distributed optimal consistent controller is designed for unknown systems with input time-delay [7], and the optimal control of finite time periods is applied in time-delay systems [8]. [9] used an echo state network approach to predict the system's speed and angular speed based on identification of aircraft parameters. [10] used the system black box identification to obtain the mathematical model of aircraft attitude loop, and designed a robust controller, whose attitude control precision is superior to proportion integration differentiation control [11]. All above mentioned references consider aircraft controller design based on the mature linear system or linear control, but here the more extended nonlinear system or nonlinear control is studied for aircraft control.

During recent years, our team also obtains some new contributions about aircraft controller design. For example, our paper [12] proposed one nonlinear data driven control from theoretical analysis and practical engineering, i.e., aircraft formation flight system. [13] combined the direct data driven control and other safety property to form an innovative direct data driven safety control for aircraft flight system. In [14], a novel iterative learning data driven control strategy was established to efficiently design the flight controller for closed loop aircraft flight system, while avoiding the modeling process for that unknown aircraft system. Roughly speaking, above existing results about controller design for aircraft system are divided into two steps, i.e., the first linearization process and the second controller design process. More specifically, the considered aircraft system and the unknown controller are both linearized as their corresponding linear forms, then the existing results about linear controller design methods are applied directly, as there are lots of literature about linear control theory, i.e., the number of literature about linear control theory is vast. The formulation about nonlinear controller design for nonlinear aircraft system can be seen in [15], where a preliminary about the control process is given.

As all phenomena in nature are nonlinearity, meaning linear system does not exist, our considered aircraft is also one nonlinear system. Consider the problem of controller design for nonlinear aircraft system, it brings our studied nonlinear controller design without linearizing it as a linear controller. Based on our previous contributions about aircraft control, we design the nonlinear controller for nonlinear aircraft system, while not firstly approximating it with one linear controller. From above description, this paper proposes the nonlinear controller design for nonlinear aircraft flight system from theory and practice, respectively. More specifically, firstly after the detailed aircraft flight system is introduced into one classical closed loop system, including the nonlinear aircraft system and nonlinear controller simultaneously, differential geometry is applied to designing that nonlinear controller through feedback, i.e., the yielded controller is one nonlinear form, being deemed as the feedback part, while lie brace and lie derivative are used explicitly.

Moreover, this nonlinear controller is exemplified to achieve the goal of perfect tracking. Secondly, it is useless for unstable system, so to guarantee the nonlinear closed loop system is stable, nonlinear stability analysis is considered through our mathematical derivation, while here nonlinear stability means our defined finite gain stability. The sufficient conditions, guaranteeing the finite gain stability, are formulated into two theories. Thirdly, our proposed nonlinear controller is applied into one practical example.

Generally, the main contributions of this paper are formulated as follows. 1) For nonlinear aircraft system, existing in one nonlinear closed loop system, nonlinear controller is designed from the point of differential geometry, while the perfect tracking is achieved. 2) Consider above nonlinear closed loop flight system with nonlinear controller and nonlinear aircraft system, nonlinear stability condition is derived to satisfy the finite gain stability, which is equivalent to small gain theorem. 3) Apply our considered nonlinear control strategy into one practical aircraft control system.

The remainder of this paper is organized as follows. In Section 2, we review aircraft flight system and point out the main flight control system structure, corresponding to one closed loop system with nonlinear aircraft system and nonlinear controller simultaneously. To design this unknown nonlinear controller directly without linearization process, differential geometry is applied to yielding one nonlinear controller in Section 3. Furthermore, the goal of perfect tracking is also talked about within the circumstance of nonlinearity. Section 4 considers the nonlinear stability analysis to guarantee nonlinear closed loop system satisfying the finite gain stability. Section 5 gives one practical example to show the efficiency of our considered nonlinear controller. Finally, Section 6 generalizes conclusion and points out our future work.

2. **Aircraft Flight System.**

The physical principle of aircraft flight control system is similar to that process of human pilot. Specifically, firstly the measurement device parts collect or measure the flight state of aircraft, then secondly the integrated computation device parts start to compare and compute according to the given instructions or orders. Thirdly the control signals are sent to the actuator parts in order to drive the steering surface, generating the aerodynamic forces and torques to control the flight state or mode of aircraft.

Classical aircraft flight control system includes three feedback loops [16], i.e., rudder loop, stabilization loop and control loop, plotted in the following Figure 1.

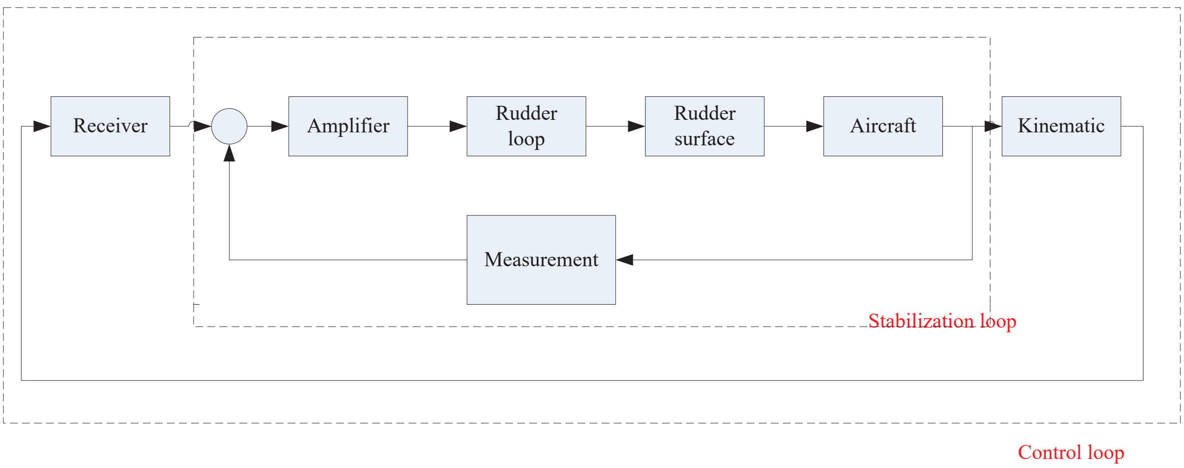


FIGURE 1. Aircraft flight control system

In above Figure 1, three feedback loops are explained in detail as follows. 1) Rudder loop is to improve the performance of rudder, so that the requirements of flight control system are satisfied. The output signals of rudder are returned back to the input part, and then one servo system with a negative feedback loop is constructed. Generally, rudder loop includes three units, i.e., rudder, feedback unit and amplifier. 2) Measurement device parts collect the accurate attitude information of aircraft. The measurement unit of attitude and rudder loop are combined together to be the autopilot. The autopilot and the aircraft control system constitute a stable loop, used to stabilize and control the aircraft attitude. 3) Control loop is composed of the stabilization loop, the measurement unit for gravity center position and the aircraft motion kinematics, describing the geometric relation or aircraft space. Its function is to stabilize and control the flight trajectory.

In order to better understand the detailed aircraft flight control system, we take an example of aircraft flight trajectory planning, i.e., aircraft trajectory control system, based on the attitude angle motion control system. The feedback loop of the trajectory control system can be closed in aircraft or through the ground equipment. The important stability and control of flight attitude play an important role in flight formation, cruise landing, terrain following and carrier based aircraft landing. In addition, the static attitude difference exists in the constant disturbance torque, so it brings attitude drift, meaning not applied into the altitude stabilization and control. Therefore, in the altitude stabilization and control system, it is necessary to measure the altitude direct by virtue of altitude difference sensor, controlling the aircraft altitude directly and changing the flight trajectory, and then the closed loop stability control is achieved for the flight height. For the sake of illustration, the aircraft flight control system is plotted in Figure 2.

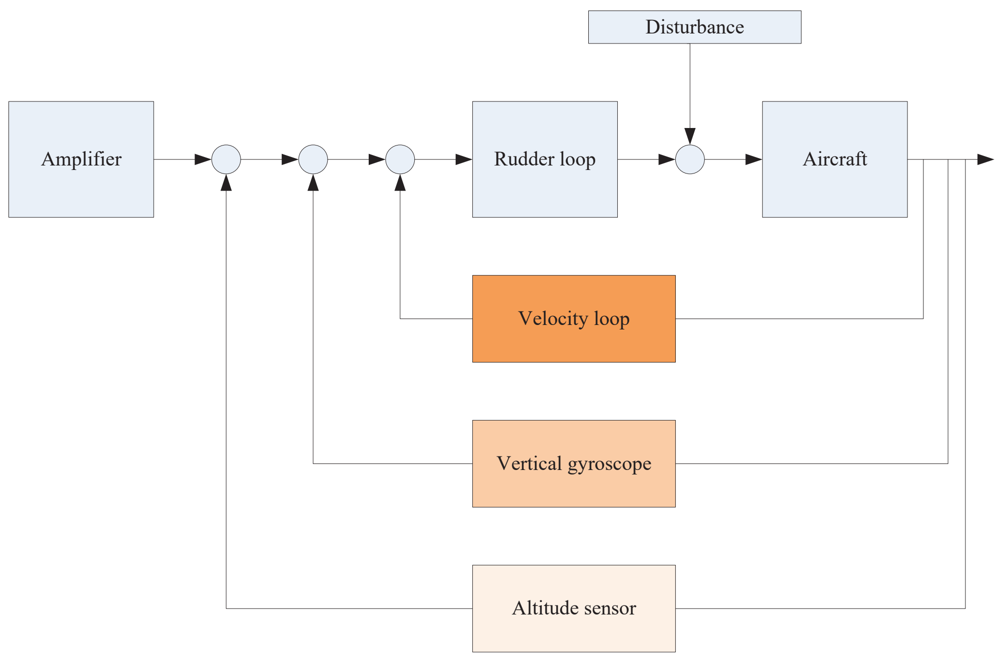


FIGURE 2. Aircraft altitude control system

3. **Nonlinear Controller Design.** From Figure 2, we see three feedback loops exist in that complete aircraft flight control system. As the inner loop includes one controller, needed to be designed, and other two loops are only unit feedback loops, the problem of controller design concerns on that inner loop.

For clarity of presentation, that inner loop in Figure 2 is extracted to show many physical variables, seeing the following Figure 3, where $P(z)$ is the aircraft kinematic, and $C(z)$ is the unknown controller. $r(t)$ is the given input signal, and $u(t)$ is the input for aircraft $P(z)$. $y_0(t)$ is the aircraft output without disturbance or noise $\tilde{y}(t)$, i.e., noiseless output. Noise $\tilde{y}(t)$ corresponds to the external effect, such as wind or atmosphere. $y(t)$ is the corrupted output, and $e(t)$ is the error signal, i.e., $e(t) = r(t) - y(t)$. z is the shift operator.

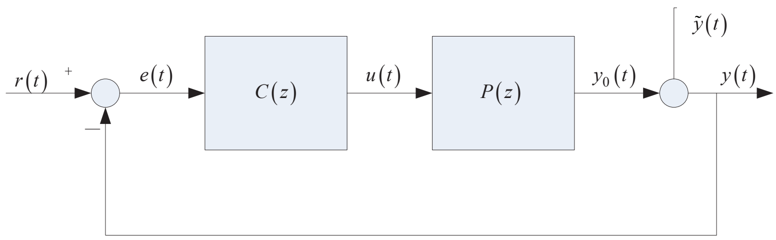


FIGURE 3. Inner loop system

Here as aircraft system $P(z)$ and unknown controller $C(z)$ are all nonlinearity, we cannot figure out their linear transfer function forms, meaning their appropriate forms be $P(z, u(t))$ and $C(z, u(t))$. Observing that nonlinear aircraft system $P(z)$, from the nonlinear control or nonlinear system theory, it can be changed into one special nonlinear form through some nonlinear transformation from the point of differential geometry. This special nonlinear form is piecewise with respect to its input variable $u(t)$, i.e., shown as

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t); \quad y(t) = h(x(t)) \tag{1}$$

where $x(t)$ means the corresponding state vector, used to embody the internal variables for aircraft system, for example, position, velocity and acceleration.

Based on above nonlinear internal relation (1) for aircraft system $P(z)$, Figure 3 is replotted in Figure 4. In our previous paper [14], these two forms in Figure 4 are proven

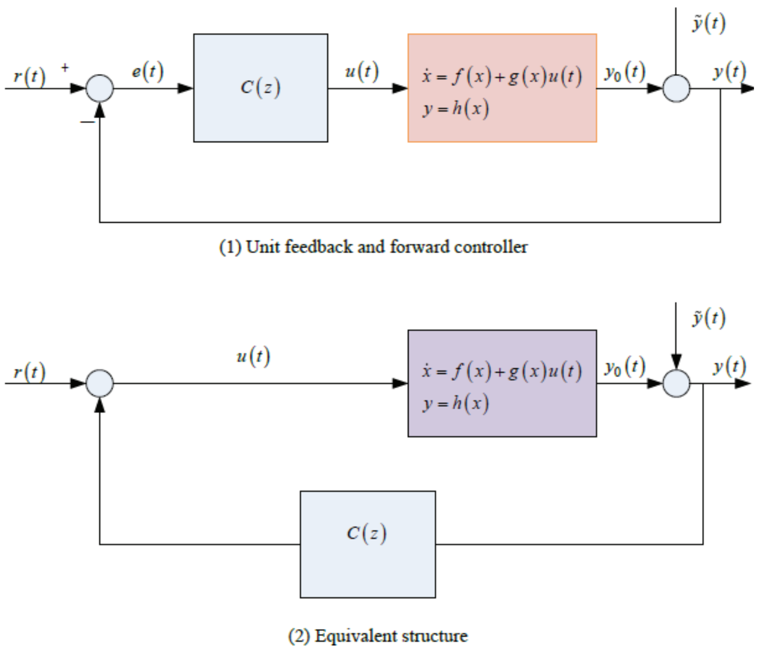


FIGURE 4. Considered inner loop system

to be equivalent to each other. Consequently, our main concern is to design that nonlinear controller $C(z)$ in case of that special nonlinear aircraft system (1).

For notational simplicity, time variable t is neglected in latter mathematical derivations, i.e., Equation (1) is rewritten as

$$\dot{x} = f(x) + g(x)u; \quad y = h(x) \tag{2}$$

where $f(x)$, $g(x)$ are smooth vector fields and $h(x)$ a smooth nonlinear function, i.e., $f(x)$, $g(x)$, $h(x)$ are all infinity differentiable functions.

Taking the partial derivative with respect to time variable t, it holds that

$$\dot{y} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = L_f h(x) + L_g h(x)u \tag{3}$$

where $L_f h(x)$ and $L_g h(x)$ are Lie derivatives of $h(x)$ with respect to $f(x)$ and $g(x)$, respectively, so $L_f h(x)$ is a function, giving the rate of change of $h(x)$ along the flow of the vector field $f(x)$. If $L_g h(x)$ is bounded away from zero, the state feedback law is chosen by

$$u = \frac{1}{L_g h(x)}(-L_f h(x) + v) \tag{4}$$

From the differential geometry theory, nonlinear controller $C(z)$ is designed in three steps.
Step 1: differential output y until input u appears.
Step 2: choose u to cancel the nonlinearities and guarantee the expected goal, for example, perfect tracking.
Step 3: study the nonlinear stability of the nonlinear closed loop system.

Observing Equation (4), if $L_g h(x) = 0$ for all x, then we differentiate $\dot{y}$ to obtain

$$y^{(2)} = L_f^2 h(x) + L_g L_f h(x)u \tag{5}$$

Iterative above differentiate operation until for some integer r, satisfying that $L_g L_f^{r-1} h(x) \neq 0$, i.e.,

$$y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x)u \tag{6}$$

then the nonlinear control law is chosen as

$$u = \frac{1}{L_g L_f^{r-1} h}\left(-L_f^r h(x) + v\right) \tag{7}$$

Substituting Equation (7) into Equation (6), we see

$$y^{(r)} = v \tag{8}$$

Remark 3.1. *Variable v must be chosen approximately as follows. Let*

$$v = -k_{r-1}y^{(r-1)} - \cdots - k_1\dot{y} - k_0 y \tag{9}$$

where coefficients $\{k_i\}_{i=1}^{r-1}$ are chosen such that the following polynomial

$$k(z) = z^r + k_{r-1}z^{r-1} + \cdots + k_1 z + k_0 \tag{10}$$

has all its roots strictly in the left haaf plane.

Substituting Equation (9) into Equation (7), the actual nonlinear control law is

$$u = \frac{1}{L_g L_f^{r-1} h}\left(-L_f^r h(x) - k_{r-1}y^{(r-1)} - \cdots - k_1\dot{y} - k_0 y\right) \tag{11}$$

Observing Equation (8), it holds that similarly

$$y^{(r)} = -k_{r-1}y^{(r-1)} - \cdots - k_1\dot{y} - k_0 y \tag{12}$$

which means that nonlinear control law (11) gives a locally asymptotically stable closed loop system.

Comment: By construction, consider the problem of perfect tracking a given expected trajectory $y_d(t)$, and let

$$\mu_d = \left[y_d \ \dot{y}_d \ \cdots \ y_d^{(r-1)} \right] \tag{13}$$

Define the tracking error vector by

$$\tilde{\mu}(t) = \mu(t) - \mu_d = \left[y \ \dot{y} \ \cdots \ y^{(r-1)} \right] - \left[y_d \ \dot{y}_d \ \cdots \ y_d^{(r-1)} \right] \tag{14}$$

Choose that nonlinear control law as

$$u = \frac{1}{L_g L_f^{r-1}\mu_1}\left(-L_f^r \mu_1 + y_d^{(r)} - k_{r-1}\tilde{\mu}_r - \cdots - k_0\tilde{\mu}_1 \right) \tag{15}$$

where

$$\begin{aligned} \left[y \ \dot{y} \ \cdots \ y^{(r-1)} \right] &= \left[\mu_1 \ \mu_2 \ \cdots \ \mu_r \right] \\ \left[\tilde{\mu}_1 \ \tilde{\mu}_2 \ \cdots \ \tilde{\mu}_r \right] &= \left[y - y_d \ \dot{y} - \dot{y}_d \ \cdots \ y^{(r-1)} - y_d^{(r-1)} \right] \end{aligned} \tag{16}$$

The detailed explanation about comment is seen the knowledge of differential geometry. Equation (15) gives one nonlinear control law to achieve the goal of perfect tracking, i.e., $\left\{ y^{(i)} \to y_d^{(i)} \right\}_{i=1}^{r}$, depending on the tracking error vector explicitly.

Combining those two nonlinear control laws (7) and (15), nonlinear closed loop system structure is replotted in the following Figure 5, where that derived nonlinear controller is in the feedforward path with unit feedback or in the feedback path. Whatever feedforward path or feedback path, their equivalence can be proved only through the simple computation.

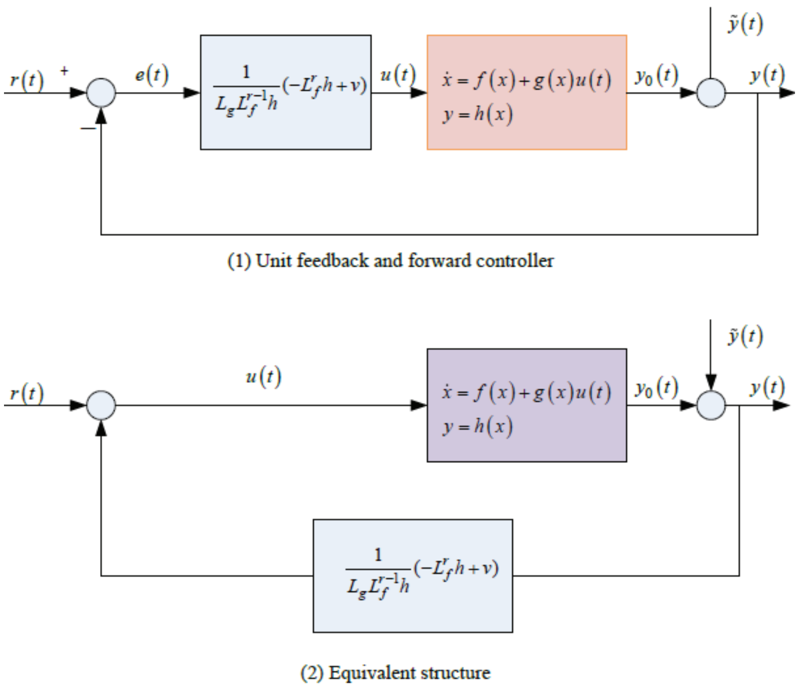


FIGURE 5. Nonlinear control law

4. **Nonlinear Stability Analysis.** Stability is an index for the considered system, as the unstable system is useless. Stability means the output trajectory will converge to one equilibrium point or move around one region with the time increases. To analyze the stability of closed loop system, plotted in Figure 4, firstly the concept of finite gain stability is defined.

Definition 4.1. [*Finite gain stability*]: *A causal nonlinear operator f is said to be finite gain stable if for given input x, the output $f(x)$, there exist $k, \beta > 0$ such that for given x, the output trajectory $f(x)$ satisfies*

$$|f(x)| \le k|x| + \beta \tag{17}$$

where notation $|x|$ denotes one norm.

4.1. **Preliminary.** Based on above definition about finite gain stability, two useful theorems are important in our later nonlinear stability analysis.

Theorem 4.1. *Consider one interconnected feedback system with input r_1, r_2 and output y_1, y_2 in Figure 6. Assume that the systems P, C are both causal and finite gain stable, that is there exist k_1, k_2, β_1, β_2 such that*

$$|P(e_1)| \le k_1|e_1| + \beta_1; \quad |C(e_2)| \le k_2|e_2| + \beta_2 \tag{18}$$

Provide that if $k_1k_2 < 1$, then the closed loop system is also finite gain stable from r_1, r_2 to y_1, y_2.

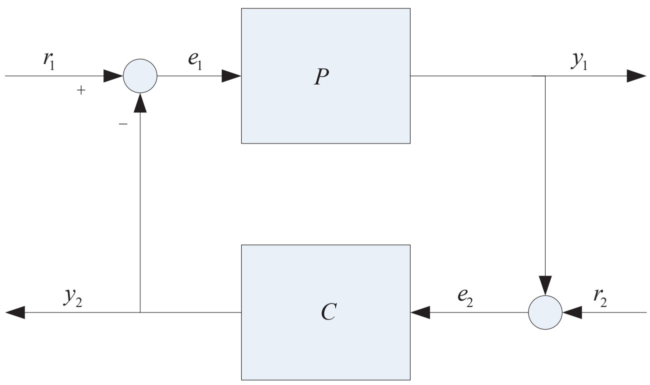


Figure 6. Interconnected feedback system

Proof: From Figure 6, we have easily

$$e_1 = r_1 - y_2 = r_1 - C(e_2); \quad e_2 = y_1 + r_2 = r_2 + P(e_1) \tag{19}$$

Using the priori known condition (18), it holds that

$$|e_1| \le |r_1| + k_2|e_2| + \beta_2; \quad |e_2| \le |r_2| + k_1|e_1| + \beta_1 \tag{20}$$

Through the common substitution operation obtain

$$\begin{aligned} |e_1| &\le |r_1| + k_2|r_2| + k_1k_2|e_1| + k_2\beta_1 + \beta_2; \\ |e_2| &\le |r_2| + k_1k_2|e_2| + k_1|r_1| + k_1\beta_2 + \beta_1 \end{aligned} \tag{21}$$

i.e.,

$$|e_1| \le \frac{|r_1| + k_2|r_2| + k_2\beta_1 + \beta_2}{1 - k_1k_2} \tag{22}$$

Then the output y_1 satisfies that

$$|y_1| = |P(e_1)| \leq k_1|e_1| + \beta_1 \leq \frac{k_1|r_1| + k_1k_2|r_2| + k_1\beta_1 + \beta_2}{1 - k_1k_2} \tag{23}$$

According to Definition 4.1, the closed loop system is finite gain stable from r_1, r_2 to y_1. Similarly, the closed loop system r_1, r_2 to y_2 is also finite gain stable.

Theorem 4.2. *Consider that feedback loop in Figure 6 again, set the input $r_2 = 0$, suppose that there exist constants k_{21}, k_1, β_{21}, β_1 with $k_{21}, k_1 > 0$ such that*

$$|CP(e_1)| \leq k_{21}|e_1| + \beta_{21}; \quad |P(e_1)| \leq k_1|e_1| + \beta_1 \tag{24}$$

Provide that if $k_{21} < 1$, then the system has finite gain stable from $r_1 \neq 0$ to e_1, y_1.

Proof: (1) In case of $r_2 = 0$.
It is easy to see that

$$e_1 = r_1 - C(y_1) = r_1 - CP(e_1); \quad r_1 = e_1 + CP(e_1) \tag{25}$$

and

$$\begin{aligned} &|y_1| = |P(e_1)| \leq k_1|e_1| + \beta_1 = k_1|r_1 - C(y_1)| + \beta_1; \\ &|e_1| = |r_1 - CP(e_1)| \leq |r_1| + |CP(e_1)| \leq |r_1| + k_{21}|e_1| + \beta_{21} \end{aligned} \tag{26}$$

Then

$$|e_1| \leq \frac{|r_1| + \beta_{21}}{1 - k_{21}} \tag{27}$$

Substituting Equation (27) into Equation (26), we have

$$|y_1| \leq |P(e_1)| \leq k_1|e_1| + \beta_1 \leq k_1\frac{|r_1| + \beta_{21}}{1 - k_{21}} + k_1\beta_1 \leq \frac{k_1|r_1| + k_1\beta_{21} + k_1\beta_1 - k_1k_{21}\beta_1}{1 - k_{21}} \tag{28}$$

From Definition 4.1, the closed loop system from r_1 to y_1 is finite gain stable.
(2) In case of $r_2 \neq 0$.
Similarly, we have

$$e_1 = r_1 - C(e_2) = r_1 - C(r_2 + P(e_1)); \quad e_2 = r_2 + P(e_1) \tag{29}$$

Then

$$\begin{aligned} &|e_1| = |r_1 - C(r_2 + P(e_1))| \leq |r_1| + |C(r_2 + P(e_1))| \leq |r_1| + |C(r_2)| + |CP(e_1)|; \\ &|e_1| \leq |r_1| + |C(r_2)| + k_{21}|e_1| + \beta_1 \end{aligned} \tag{30}$$

Similarly, the latter proof is the same as above case of $r_2 = 0$.

4.2. **Our stability analysis.** Section 3 proposes the knowledge of differential geometry into designing one nonlinear controller after the considered nonlinear aircraft system is changed to one special form through some nonlinear transformations. When to do nonlinear stability analysis for that nonlinear closed loop system, and for the sake of completeness, our preliminaries about finite gain stable are formulated into two theorems. Now we start to analyze that nonlinear stability with our own derivations.

To better understand the latter analysis, we plot Figure 4 again and do some adaptation in Figure 7.

In Figure 7, $\{\tilde{r}(t), \tilde{y}(t)\}$ are two kinds of disturbances or noises, existing in input and output part, respectively.

If $\tilde{r}(t) = 0$, then Figure 7 is reduced to Figure 4. $r_0(t)$ and $y_0(t)$ are noiseless input and noiseless output. Our main result about the nonlinear stability analysis for nonlinear closed loop system in Figure 7 is yielded into the following Theorem 4.3.

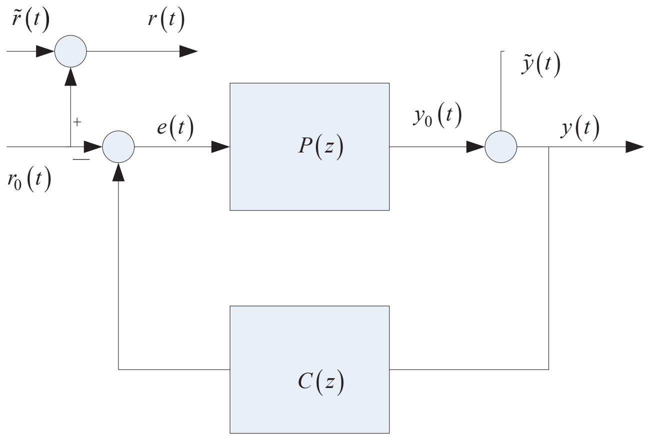


FIGURE 7. Adaptive nonlinear closed loop system

Theorem 4.3. *Consider that nonlinear closed loop system in Figure 7 with two kinds of noises, suppose that there exist constants* k_1, k_2, β_1, β_2 *such that*

$$|C(y(t))| \leq k_2|y(t)| + \beta_2; \quad |P(e(t))| \leq k_1|e(t)| + \beta_1 \tag{31}$$

Provide that if $k_1k_2 < 1$, *then that nonlinear system has finite gain stable from* $r(t)$ *to* $y(t)$.

Proof: Observing Figure 7, we have

$$\begin{aligned} e(t) &= r_0(t) - C(z, y(t)) = r(t) - \tilde{r}(t) - C(z, y(t)) \\ y(t) &= y_0(t) + \tilde{y}(t) = P(z, e(t)) + \tilde{y}(t) \end{aligned} \tag{32}$$

Using Equation (31) yield that

$$|e(t)| \leq |r(t)| + |\tilde{r}(t)| + k_2|y(t)| + \beta_2; \quad |y(t)| \leq k_1|e(t)| + |\tilde{y}(t)| + \beta_1 \tag{33}$$

Furthermore, we have

$$|e(t)| \leq |r(t)| + |\tilde{r}(t)| + k_2k_1|e(t)| + k_2|\tilde{y}(t)| + k_2\beta_1 + \beta_2 \tag{34}$$

i.e.,

$$|e(t)| \leq \frac{|r(t)| + |\tilde{r}(t)| + k_2|\tilde{y}(t)| + k_2\beta_1 + \beta_2}{1 - k_1k_2} \tag{35}$$

so the output trajectory $y(t)$ satisfies that

$$|y(t)| \leq \frac{k_1|r(t)| + k_1|\tilde{r}(t)| + |\tilde{y}(t)| + k_2\beta_1 + \beta_2 - k_1k_2\beta_1}{1 - k_1k_2} \tag{36}$$

In practice, two noises $\{\tilde{r}(t), \tilde{y}(t)\}$ are bounded, so from our previous Definition 4.1, Theorem 4.1 and Theorem 4.2, the output trajectory $y(t)$ from input $r(t)$ is finite gain stable.

5. **Simulation Example.** To prove our above theoretical results about cooperative distributed model predictive control, we use aircraft control framework to achieve it. One practical real aircraft is seen in Figure 8, which is a real product for air force in our lab. Before to apply nonlinear control, one mathematical model of aircraft is needed through physical principle or system identification method. For convenience, Figure 9 gives a detailed force effect diagram. Through some forces analysis, we use the classical Lagrangian tool to construct the force balance equation. More specifically, in Figure 9, F_L is lift, D is drag, Y is side force, and G is gravity. Furthermore, V_{aero} denotes air velocity, V_k is

FIGURE 8. Real aircraft

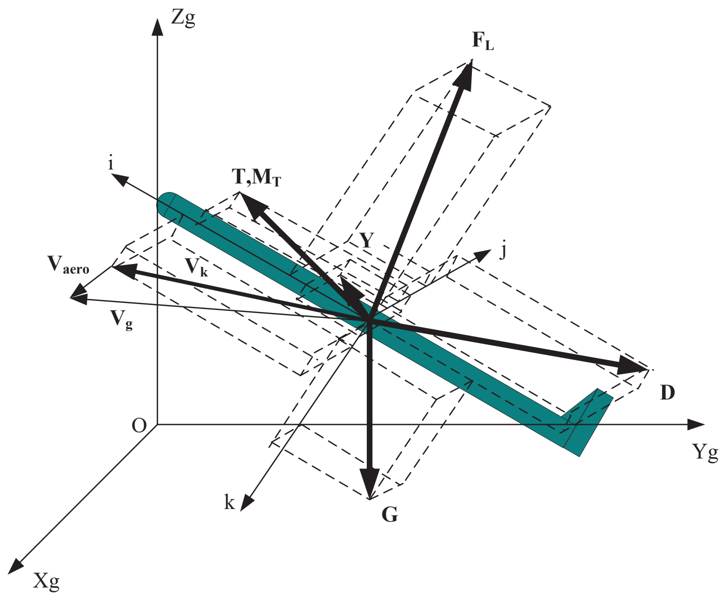


FIGURE 9. Force effect diagram

relative speed, and V_g represents speed of aircraft relative to the earth. If the thrust is moved to the center of mass, then the thrust is denoted by T, and the equivalent torque M_T is caused by the thrust movement.

The motion of aircraft in air can be divided into two parts, i.e., the translational motion of aircraft's center of mass and the fixed point rotation of aircraft around the center of mass. Under certain assumptions, Newton's second law leads to the vector form of aircraft dynamics equation.

$$F = m\frac{dN}{dt} \tag{37}$$

Then vector forms along three axis are

$$\begin{cases} F = F_x i + F_y j + F_z k; \quad V = ui + vj + wk \\ w = pi + qj + rk; \quad M = Li + Mj + Nk \end{cases} \tag{38}$$

Due to a symmetrical plane on aircraft, its dynamical equations are listed as follows.

$$\begin{cases} F_x = m(\dot{u} - vr + wq); \quad F_y = m(\dot{v} - wp + ur); \quad F_z = m(\dot{w} - up + vp) \\ L = I_x\dot{p} - I_{xz}\dot{r} + (I_x - I_y)qr - I_{xz}pq; \quad M = I_y\dot{q} + (I_x - I_z)pr + I_{xz}\left(p^2 - r^2\right) \\ L = I_z\dot{r} - I_{xz}\dot{r} + (I_y - I_x)pq + I_{xz}qr \end{cases} \tag{39}$$

Furthermore, the thrust component and deflection angle in airframe coordinate system satisfy the following geometric relations.

$$T^2 = T_x^2 + T_y^2 + T_z^2; \quad \tan\beta_T = \frac{T_y}{T_x}; \quad \tan\alpha_T = \frac{T_z}{T_x} \tag{40}$$

Then the three axis components of the thrust are expressed as

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} T\left(1 + \tan^2\alpha_T + \tan^2\beta_T\right) \\ T\left(1 + \tan^2\alpha_T + \tan^2\beta_T\right)\tan\beta_T \\ T\left(1 + \tan^2\alpha_T + \tan^2\beta_T\right)\tan\alpha_T \end{bmatrix} \tag{41}$$

In practice, flight velocity V is chosen as state variable, and then the relation between flight velocity V and velocity component in body coordinate is as follows.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = S_{ba}\begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

After differentiating both sides of Equation (42), it holds that

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = S_{ba}\begin{bmatrix} \dot{V} \\ V\dot{\beta} \\ V\cos\beta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} \dot{V}\cos\alpha\cos\beta - \dot{\beta}V\cos\alpha\sin\beta \\ \dot{V}\sin\beta + \dot{\beta}V\cos\beta \\ V\sin\beta\cos\beta + \dot{\alpha}V\cos\alpha\cos\beta \end{bmatrix} \tag{43}$$

Choose the state variable as

$$X = (V, \beta, \alpha, u, v, w) \tag{44}$$

Then the nonlinear model for aircraft is

$$f(\dot{x}, x, u) = 0 \tag{45}$$

Equation (45) corresponds to one dynamical model for only one aircraft. Now five aircrafts are used to form a formation mode, where these five aircrafts have the same physical devices. That formation mode is given in Figure 10, where the front aircraft is deemed as the leader, and the other four aircrafts are followers.

During the whole simulation, the maximal aircraft velocity is 80 m/s, maximal acceleration is 10 m/s^2, and communication radius is 300 m. The expected velocity is 60 m/s, and

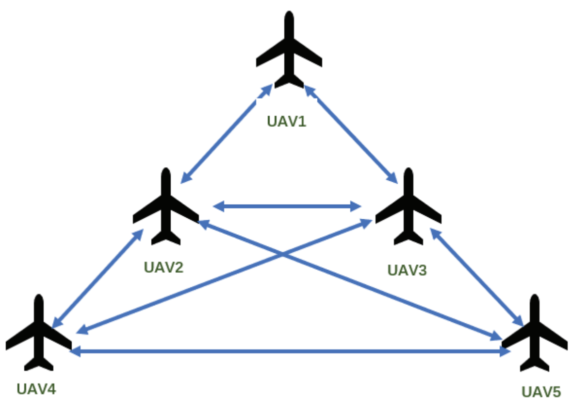
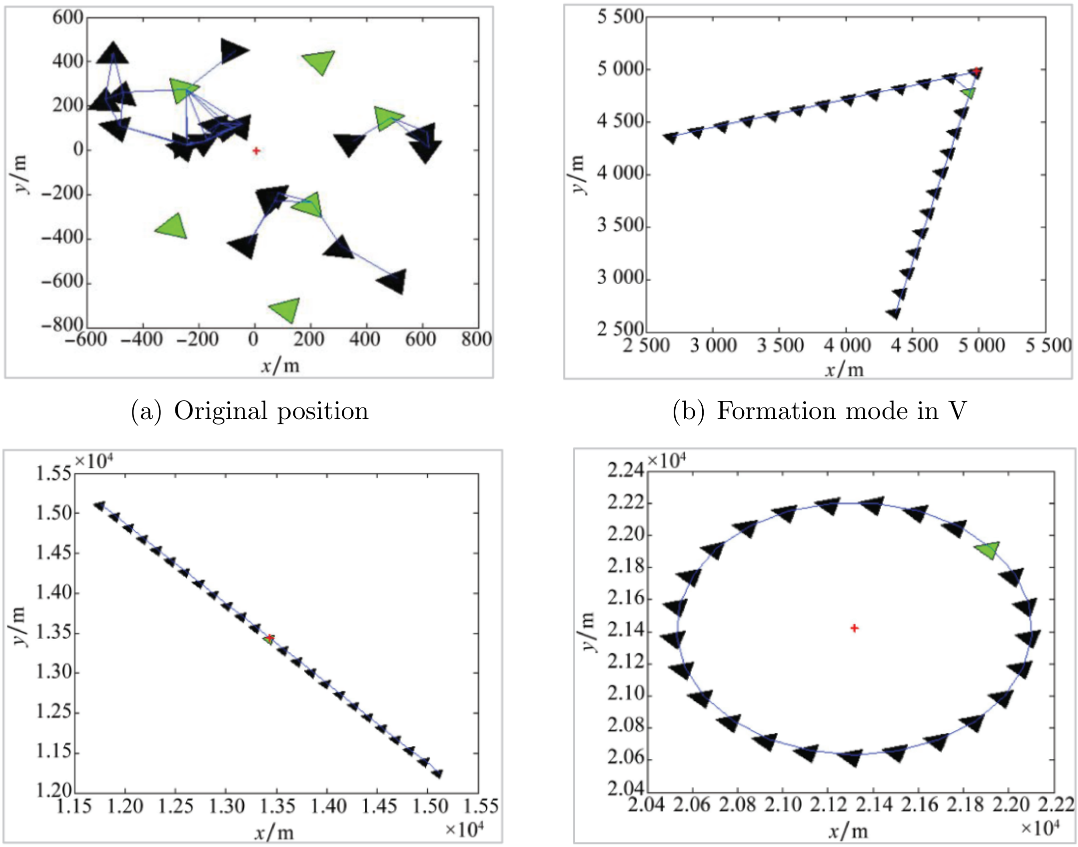


FIGURE 10. Formation mode

the expected yaw angle is 45°. Set the adjoint distance between two aircrafts as 200 m. The original position and original velocity of each aircraft are yielded randomly based on Gaussian white noise, i.e., $[-100, 100]$ m $\times$ $[-100, 100]$ m and $[0, 30]$ m/s $\times$ $[0, 30]$ m/s.

Suppose the desired formation in the time interval $[0\text{ s}, 200\text{ s}]$ is v-shaped. The formation task changes to horizontal formation in $[200\text{ s}, 400\text{ s}]$, and the formation task changes again in $[400\text{ s}, 600\text{ s}]$. The desired formation is circular formation. The randomly given initial position distribution of the cluster is shown in Figure 11, where the red cross marked as the virtual navigator at position $[0, 0]$ on the x-y axis, and the green triangles represent randomly selected navigation information in different neighborhoods. The black triangle is the followers, and the triangles indicate that there is information interaction between aircrafts, and that the followers in different alliances do not have information interaction, as the alliances are divided by neighborhood.



(a) Original position

(b) Formation mode in V

(c) Formation mode in line

(d) Formation mode in circle

FIGURE 11. (color online) Different formation modes

Figure 11 tells us that those aircrafts can switch their different formation modes freely by virtue of cooperative distributed model predictive control strategy, meaning each aircraft is controlled by its own distributed controller. This merit of switching formation mode corresponds to our first guaranteed safety property, i.e., stability.

Next, we need to testify the second guaranteed safety property, i.e., positively invariance. Let aircrafts fly within a tube or limited range, so aircrafts cannot cross the upper and lower boundary of the required tube in Figure 12. Our goal is to design each controller for each aircraft in order to fly during that tube as one by one, not crash together and cross the boundary.

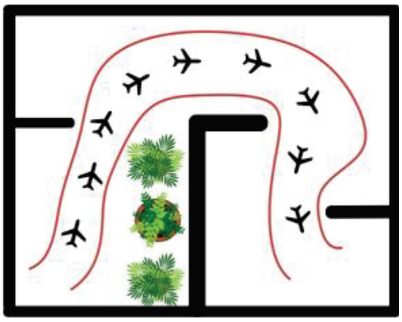


FIGURE 12. Requited tube or limited range

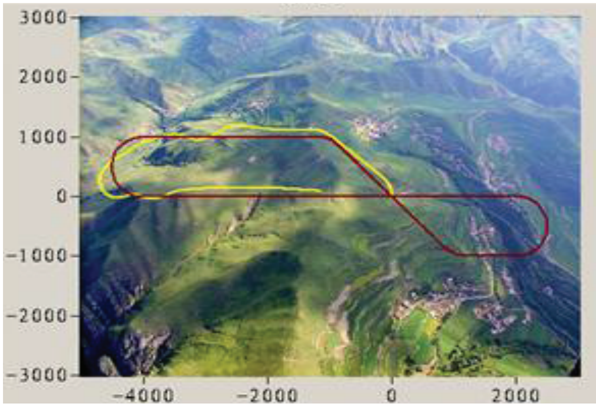


FIGURE 13. Aircraft trajectory in screen

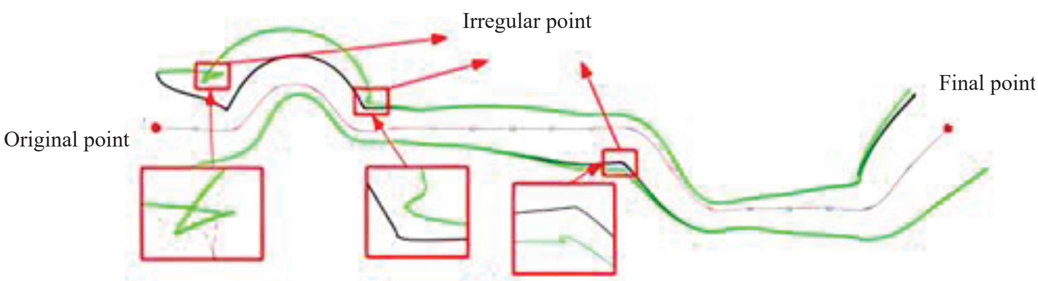


FIGURE 14. Aircraft trajectory in limited range

From our goal above, we design each distributed controller for each aircraft to guarantee that safety property. Each aircraft trajectory is observed in one computer screen, where aircraft trajectory corresponds to one irregular curve in Figure 13. Additionally, the detailed analysis of each aircraft trajectory is seen in Figure 14, where aircraft starts from one original point and stops at one final point. Combining Figures 13 and 14, we apply our proposed nonlinear differential geometry controller in devising each distributed nonlinear controller for each aircraft. Based on this nonlinear controller, after giving the expected flight route information and mission requirement, aircraft will take off in several seconds and then fly perfectly within the desired flight trajectory.

6. **Conclusion.** This paper considers two important aspects from the control theory, i.e., nonlinear control design and nonlinear stability analysis. First, after the detailed aircraft flight system is introduced into one classical closed loop system, including the nonlinear aircraft system and nonlinear controller simultaneously, differential geometry is

applied to designing that nonlinear controller through feedback. Moreover, this nonlinear controller is exemplified to achieve the goal of perfect tracking. Second, to guarantee the nonlinear closed loop system is stable, nonlinear stability analysis is considered through our mathematical derivation. As this paper is our first work about nonlinear differential geometry control, latter in future we will study the nonlinear direct data driven control strategy from the theory and practice respectively, paving a new road for future research about nonlinear control system.

REFERENCES

[1] J. Berberich, J. Kohler and M. Muller, Data driven model predictive control with stability and robustness guarantees, *IEEE Transactions on Automatic Control*, vol.66, no.4, pp.1702-1717, 2021.

[2] A. Alanwar, J. J. Rath et al., Distributed set based observers using diffusion strategies, *Journal of the Franklin Institute*, vol.360, no.11, pp.6976-6993, 2023.

[3] U. Validya and D. T. Castro, Data driven stochastic optimal control with safety constraints using linear transfer operators, *IEEE Transactions on Automatic Control*, vol.69, no.4, pp.2100-2115, 2024.

[4] J. Kohler, M. A. Muller and F. Allgower, A computationallly efficient robust model predictive control framework for uncertain nonlinear systems, *IEEE Transactions on Automatic Control*, vol.66, no.2, pp.794-801, 2021.

[5] A. S. Amorez, P. Chanfreut and J. M. Maestre, Robust coalitional model predictive control with negotiation of mutual interactions, *Journal of Process Control*, vol.123, no.4, pp.64-75, 2023.

[6] R. A. Gonzalez, C. R. Rojas, S. Pan and J. S. Welsh, Refined instrumental variable methods for unstable continuous time systems in closed loop, *International Journal of Control*, vol.96, no.10, pp.2527-2541, 2023.

[7] M. H. Zadeh, B. Sinopoli et al., Robust-to-early termination model predictive control, *IEEE Transactions on Automatic Control*, vol.69, no.4, pp.2507-2513, 2024.

[8] A. Perodou, A. Korniienko, M. Zarudniev and G. Scorletti, Frequency synthesis of interconnected homogeneous LTI systems, *IEEE Transactions on Automatic Control*, vol.69, no.3, pp.1480-1491, 2024.

[9] A. Perrusquia and W. Guo, Trajectory inference of unknown linear systems based on partial states measurements, *IEEE Transactions on Systems, Man and Cybernetics: System*, vol.54, no.4, pp.2276-2286, 2024.

[10] Z. Ding, Distributed time-varying optimization – An output regulation approach, *IEEE Transactions on Cybernetics*, vol.54, no.4, pp.2155-2165, 2024.

[11] M. Bisiacco and G. Pillonetto, Sliding mode theory under feedback constraints and the problem of epidemic control, *SIAM Journal on Applied Mathematics*, vol.83, no.6, pp.2189-2211, 2023.

[12] J. Wang, R. A. Ramirez-Mendoza and R. Morales-Menendez, *Data Driven Strategy: Theory and Application*, CRC Press, Boca Raton, FL, USA, 2023.

[13] J. Wang, Y. Wang, R. A. Ramirez-Mendoza and R. Morales-Menendez, Adaptive iterative correlation tuning for closed loop system with two parameterized controllers, *International Journal of Systems Science*, vol.52, no.9, pp.1835-1849, 2020.

[14] J. Wang and R. A. Ramirez-Mendoza, Finite sample properties of virtual reference feedback tuning with two degrees of freedom controllers, *ISA Transactions*, vol.99, no.6, pp.37-49, 2020.

[15] J. Wang, Y. Zhang and R. A. Ramirez-Mendoza, Direct data driven scheme for UAV flight control, *IEEE Access*, vol.10, no.1, pp.108241-108250, 2022.

[16] Z. Jin, F. Luo and G. Wu, Research on improved dual-channel fuzzy logic controller for aircraft, *International Journal of Innovative Computing, Information and Control*, vol.19, no.3, pp.893-906, 2023.

Author Biography

Ruchun Wen received the bachelor's degree from Jiangxi University of Science and Technology, China, in 1994. In 1998, she received master's degree in the same university. She is currently an associate professor with Jiangxi University of Science and Technology, China. Her current research interests include machine leaning, direct data driven control, convex optimization and nonlinear control theory.

Qing Ouyang received the bachelor's degree in Control Engineering from Dalian Jiaotong University, China, in 2017. She is currently pursuing the master's degree with the Jiangxi University of Science and Technology, China. Her research interest includes advanced control strategy.

Jianhong Wang received the diploma in Engineering Cybernetics from Yunnan University, China, in 2007. In 2011, he received the Dr.Sc. degree in College of Automation Engineering from Nanjing University of Aeronautics and Astronautics, China. From 2013 to 2015, he was a postdoctoral fellow in Informazione Politecnico di Milano, Italy. From 2016 to 2018, he was a professor in University of Seville, Spain. From 2019 to 2020, he was a professor in Technology of Monterrey, Mexico. From 2019 until now, he is a professor in Jiangxi University of Science and Technology, China. From 2025, he will join in Universidad del Caribe (UNICARIBE, Santo Domingo, Dominican Republic), to be a research professor. His current research interests include real-time distributed control, nonlinear adaptive control, differential geometry control and topology game control.

Bohua Zeng received the master's degree in Control Engineering from the Jiangxi University of Science and Technology, China, in 2021. He is currently teaching with Jiangxi University of Science and Technology. His research interest includes advanced control strategy.