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ROBUST FAULT TOLERANT CONTROL USING A DOUBLE FEEDBACK CONTROL SYSTEM FOR MINIMUM PHASE SYSTEMS

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Abstract. *To design the control system with low sensitivity characteristics and robust stability, a double feedback control system is proposed. The double feedback control system is a two-degree-of-freedom control system that is included in another two-degree-of-freedom control system. According to some studies, the double feedback control systems can simultaneously have low sensitivity characteristics and robust stability for a class of uncertainties. The double feedback control system also has the potential to maintain stability even if a component in the control system fails because the control system has multiple controllers. However, if a component in the double feedback control system fails, the control structure of its control system will be changed. This implies that for the failure of a component within the double feedback control system, the control system is not always stable, even if the stability condition of its double feedback control system without the failure is satisfied. Thus, it is important to obtain the stability condition of the double feedback control system in the case that a component within the double feedback control system fails. In the case that a component fails, several cases are considered. However, the stability condition of the double feedback control system in a case where a component fails is not always equal to that in another case. This paper considers a robust fault-tolerant control using a double feedback control system for Single-Output/Single-Input time-invariant minimum phase systems in the case that the output signal from a controller fails. In this paper, we clarify a robust stability condition of the double feedback control system in a case where a component failure is not always equal to that in another case. Based on the clarified robust stability condition, a design method for the double feedback control system that maintains stability in the case that the output signal from a controller fails.*

Keywords: Fault-tolerant control, Minimum phase system, Low sensitivity control, Robust stability, Systems with varying number of unstable poles

1. **Introduction.** A conventional feedback control design for a complex system may result in an unsatisfactory performance, or even instability, in the event of malfunctions in actuators, sensors, or other system components [1]. To maintain stability and performance of the control system for failure of a component, several researchers consider a control system design for fault tolerance [2, 3]. The main requirement for the control system

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with fault tolerance is that the performance of this controller at least approximates the performance of the nominal controller in times when the failures are absent [2, 4].

As one of the approaches to designing the control system with fault tolerance, the passive fault-tolerant control approach has been proposed [1, 3, 4, 5, 6]. The passive fault-tolerant control approach does not always require fault detection and diagnosis [6]. In the passive fault-tolerant control approach, the control system is designed under the criterion of a pre-defined performance and a pre-defined insensitivity to faults of the closed-loop system [2, 3]. In a passive fault-tolerant approach, the control law is not changed when the failure occurs [5]. The passive fault-tolerant control approach only deals with pre-specified scenarios in which the closed-loop system can be stable. However, the passive fault-tolerant control approach has an advantage that systems do not have computational performance. Several control system design methods based on a passive fault-tolerant control approach are proposed. Vidyasagar examines the simultaneous stabilization problem where one controller simultaneously stabilizes the nominal plant and plants after some sort of structural changes caused by the loss of a sensor or an actuator [7]. In [7], Vidyasagar also demonstrates that the simultaneous stabilization problem can be reduced to the strong stabilization problem, in which the control system is stabilized by a stable controller. Youla et al. show that the plant is strongly stabilizable, which means being stabilized by the stable controller, if and only if the plant satisfies the parity interlacing property condition, and examine a design procedure of a stable stabilizing controller [8]. Hoshikawa et al. clarify the class of strongly stabilizable plants that is stabilized by the stable controller [9]. Akuzawa et al. clarify the parameterization of all strongly stabilizing controllers [10]. However, the use of a stable controller means that the output from the control system cannot follow the reference input when uncertainty or a step disturbance. To overcome this weakness of a stable controller, Hoshikawa et al. propose the semi-strongly stabilizing controller that a controller having a pole at the origin and the rest of the poles in the open left half plane stabilizes a plant [11, 12]. In [13], the self-repairing control system with faulty sensors using the semi-strongly stabilizing controller is considered. Kimura et al. define the extended semi-strongly stabilizing controllers with a pole at the origin and two pairs of poles on the imaginary axis as a controller with a pole at the origin, two pairs of poles on the imaginary axis, and rest of the poles in the open left half plane [14]. The controller proposed in [14] can be used to stabilize a plant, and a control system that often uses a sinusoidal signal to detect failure with a function that eliminates the influence of the sinusoidal signal on the output. Using the result of [14], the extended semi-strongly stabilizing controller with failure detection is proposed [15].

Although [7, 13, 15] are suitable to the fault-tolerant control system design, these results do not always consider the uncertainty, which is the error between the plant and the nominal plant. The uncertainty often has a negative impact on the low-sensitivity characteristics and stability of control systems. To make a control system robustly stable for the plant with uncertainty, several researchers have considered the robust stabilization problem [16, 17, 18, 19, 20, 21, 22, 23]. The robust stabilization problem is to clarify a necessary and sufficient (robust stability) condition that the control system maintains stability for the plant with uncertainty, or consider a design method for a control system satisfying the robust stability condition. Yamada studies robust stability conditions for a class of uncertainty, which includes plants with varying number of unstable poles [23]. The robust stability condition clarified in [23] implies that low-sensitivity control guarantees robust stability. The result of [23] is suitable for the high-performance robust control system design. Expanding on the result of [23], the double feedback control system, which is a two-degree-of-freedom control system that is included in another two-degree-of-freedom one, is proposed [24, 25]. The structure of the double feedback control system implies that it is expected to apply the double feedback control system to a teleoperation control system, and so on. The double feedback control system can have low-sensitivity characteristics more than the two-degree-of-freedom control system, using the result of [23].

Studies in [24, 25] can be applied to designing the control system with fault tolerance for the failure defined as a part of the uncertainties. However, the applicability of the results presented in [24, 25] is limited in scenarios where feedback loops are broken down, as such failures induce structural changes in the double feedback control system. In addition, many control system designs based on the passive fault-tolerant control approach do not consider the failure of the controller, although there exists a failure of the controller in actual systems. For example, in the control for drones such as unmanned aerial vehicles, there exists a control system failure, mainly in the micro-computer board that carries out control calculation and peripheral devices [26]. The controller's failure makes the output signal from the stabilizing controller constant. For example, there exists a controller that has functions to output a constant value when a sensor fails in the actual system. These cases of failure are the same in the meaning of having the output from the controller constant value after failure.

In this paper, we examine a design method of the double feedback control system that maintains stability against the failure of making the output from the controller a constant value for the single-input and single-output minimum phase systems having uncertainty. This paper is organized as follows. In Section 2, we explain the structure of the double feedback control system. In addition, the class of uncertainty and failure considered in this paper is defined. The failure in this paper is defined as making the output signal from the stabilizing controller change to a constant value. In Section 3, the robust stability condition when the output signal of a stabilizing controller changes to a constant value is clarified. In Section 4, the robust stability conditions when the output signal from the other stabilizing controller changes to a constant value are clarified. In Section 5, we present a design procedure for the double feedback control system with robust stability for a failure of making the output signal from a feedback controller change to a constant value. In Section 6, we show a numerical example to illustrate the effectiveness of the proposed method. Section 7 gives concluding remarks.

2. Problem Formulation.

Consider the control system shown in Figure 1. Here, $G(s) \in R(s)$ is the SISO strictly proper plant, where $R(s)$ is the set of real rational functions with s. $G(s)$ is assumed to have no zero in the closed right half plane, that is, $G(s)$ is of minimum phase. The nominal plant of $G(s)$ is denoted by $F_0(s) \in R(s)$. $F_0(s)$ is also assumed to have no zero in the closed right half plane, that is, $F_0(s)$ is also of minimum phase. $F_1(s) \in RH_\infty$ and $F_2(s) \in RH_\infty$, in which RH_∞ means the set of stable proper real

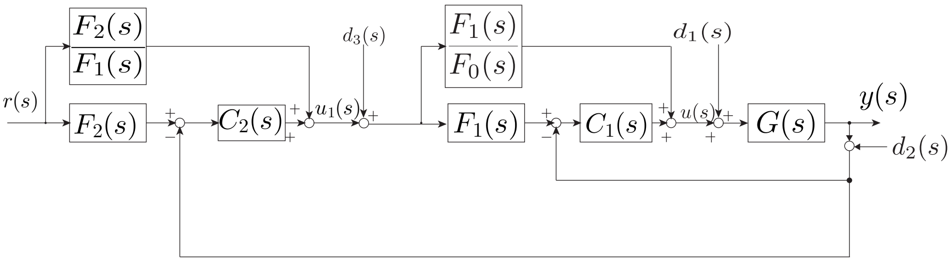

FIGURE 1. The double feedback control system

rational functions, are controllers satisfying $F_1(s)/F_0(s) \in RH_\infty$ and $F_2(s)/F_1(s) \in RH_\infty$, respectively. $C_1(s) \in R(s)$ is the stabilizing controller for $F_0(s)$, and $C_2(s) \in R(s)$ is the stabilizing controller for $F_1(s)$. $u(s) \in R(s)$ is the control input, $r(s) \in R(s)$ is the reference input, $y(s) \in R(s)$ is the output, $d_1(s) \in R(s)$, $d_2(s) \in R(s)$ and $d_3(s) \in R(s)$ are disturbances. The control system shown in Figure 1 is called the double feedback control system [24, 25].

The nominal plant $F_0(s)$ is usually not equal to the plant $G(s)$. Let $G(s)$ be denoted as

$$G(s) = F_0(s)\left(1 + \Delta(s)\right), \tag{1}$$

where $\Delta(s) \in R(s)$ is the uncertainty. The influence of $\Delta(s)$ on the output $y(s)$ is a tendency to decrease if the sensitivity function denoted by $S(s) = S_1(s)S_2(s)$ is a small value, where $S_k(s)$ $(k = 1, 2)$ is a function denoted by

$$S_k(s) = \frac{1}{1 + C_k(s)F_{k-1}(s)} \quad (k = 1, 2). \tag{2}$$

According to [23], the low-sensitivity control guarantees robust stability for the following class of uncertainty $\Delta(s)$.

Definition 2.1. *The plant $G(s)$ is called the element of the class Ω if the following expressions hold* [23]*.*

1) The relative degree of $G(s)$ is equal to that of $F_0(s)$.

2)

$$\left|\frac{\Delta(j\omega)}{1 + \Delta(j\omega)}\right| \le |W(j\omega)| \quad (\forall \omega \in R), \tag{3}$$

where $W(s) \in R(s)$ is an upper bound of the uncertainty satisfying

$$\lim_{\omega\to\infty}\left|\frac{\Delta(j\omega)}{1 + \Delta(j\omega)}\right| \le \lim_{\omega\to\infty}|W(j\omega)| < 1. \tag{4}$$

The class of uncertainty Ω does not depend on the number of the poles of the plant $G(s)$ in the closed right half plane and that of the nominal plant $F_0(s)$. This means that the class Ω can treat the plant with varying number of unstable poles. [24, 25] clarify the robust stability condition of the double feedback control system in Figure 1 for $G(s) \in \Omega$ as follows.

Theorem 2.1. *The double feedback control system in Figure 1 is robustly stable for $G(s) \in \Omega$ if and only if*

$$\|S(s)W(s)\|_\infty = \|S_1(s)S_2(s)W(s)\|_\infty < 1 \tag{5}$$

holds true, where RH_∞ means the set of stable proper real rational functions [24, 25]*.*

When a failure such that the output $u_{ck}(t) = \mathcal{L}^{-1}\left[u_{ck}(s)\right]$ $(k = 1, 2)$ from one of the two feedback controller $C_k(s)$ $(k = 1, 2)$ changes to constant value $\hat{u}_{ck}$ $(k = 1, 2) \in R$ occurs, the structure of the double feedback control system shown in Figure 1 will be changed. This means even if the double feedback control system in Figure 1 satisfies Theorem 2.1, the control system is not always stable. To consider the failure that has the output from one of the two feedback controllers becomes a constant value, we define the failure of making the output signal from the stabilizing controller $C_k(s)$ $(k = 1, 2)$ a constant value as follows.

Definition 2.2 (The failure of making the output signal from the stabilizing controller $C_k(s)$ $(k = 1, 2)$ a constant value)**.** *It is called the failure of making the output signal from* *the feedback controller* $C_k(s)$ $(k = 1, 2)$ *a constant value if the output* $u_{ck}(t)$ $(k = 1, 2)$ *of* $C_k(s)$ $(k = 1, 2)$ *changes to* $u_{ck}(t) = \hat{u}_{ck}$ $(k = 1, 2)$*, where* $\hat{u}_{ck}$ *is any real number.*

Definition 2.2 can deal with not having $u_{ck}(t)$ change to 0 but also $u_{ck}(t)$ change to a constant value. The problem considered in this paper is to provide a design method for the double feedback control system with robust stability and low sensitivity even if a failure of making the output signal from the stabilizing controller $C_k(s)$ a constant value occurs.

3. **Failure of Making the Output Signal from the Stabilizing Controller $C_1(s)$ a Constant Value.** In this section, we examine the failure of making the output signal from the stabilizing controller $C_1(s)$ a constant value in the double feedback control system in Figure 1.

When the output $u_{c1}(t)$ changes to $\hat{u}_{c1}$, the double feedback control system in Figure 1 is changed such as Figure 2. The robust stability condition of the double feedback control system in Figure 2 for $G(s) \in \Omega$ in (1) is summarized as follows.

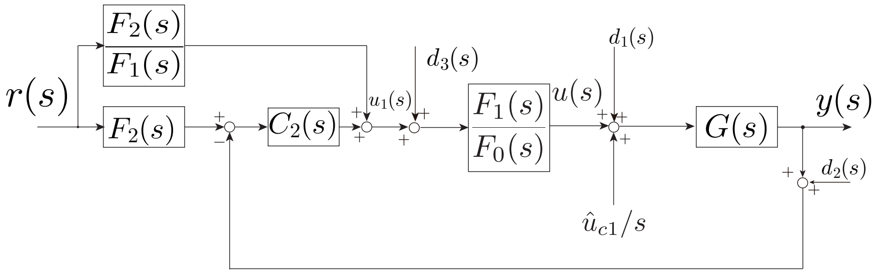

FIGURE 2. The double feedback control system when the output $u_{c1}(t)$ changes to $\hat{u}_{c1}$ $C_1(s)$

Theorem 3.1. *Assume that* $C_2(s)$ *stabilizes* $F_1(s)$*. The double feedback control system shown in Figure 2 is internally stable for* $G(s) \in \Omega$ *if and only if*

$$F_0(s) \in RH_\infty, \tag{6}$$

and

$$\|S_2(s)W(s)\|_\infty < 1. \tag{7}$$

In order to prove Theorem 3.1, the following lemmas are needed.

Lemma 3.1. *Assume that* $W(s)$ *satisfies (4). In addition, assume that* $F_0(s) \in R(s)$ *has no zero in the closed right half plane and* p_m*-th number of poles in the closed right half plane,* $G(s) \in R(s)$ *has no zero in the closed right half plane and* p*-th number of poles in the closed right half plane, and the relative degree of* $G(s)$ *is equal to the relative degree of* $F_0(s)$*. Then the Nyquist plot of* $1 + \Delta(j\omega)$ *for* $-\omega \leq \omega \leq \infty$ *encircles the origin* $(0, 0)$ $p - p_m$ *times in the counter-clockwise direction* [23].

Lemma 3.2. *Assume that* $\Delta(s)$ *in (1) is equal to 0, and the failure of* $C_1(s)$ *occurs, that is* $u_{c1}(s) = \hat{u}_{c1}/s$ *and* $C_2(s)$ *stabilizes* $F_1(s)$*. Then the double feedback control system shown in Figure 1 is internally stable for* $F_0(s) \in R(s)$ *if and only if*

$$F_0(s) \in RH_\infty \tag{8}$$

holds true.

Proof: The necessity is shown. That is, we show that if the double feedback control system shown in Figure 2 is stable, then $F_0(s) \in RH_\infty$. Since the double feedback control system in Figure 2 is internally stable, all functions $V_j(s)$ $(j = 1, 2, \ldots, 12)$ denoted as

$$\begin{bmatrix} y(s) \\ u(s) \\ u_{c2}(s) \end{bmatrix} = \begin{bmatrix} V_1(s) & V_2(s) & V_3(s) & V_4(s) \\ V_5(s) & V_6(s) & V_7(s) & V_8(s) \\ V_9(s) & V_{10}(s) & V_{11}(s) & V_{12}(s) \end{bmatrix} \begin{bmatrix} r(s) \\ d_1(s) \\ d_2(s) \\ d_3(s) \end{bmatrix} \tag{9}$$

belong to RH_∞, where $V_1(s) = F_2(s)$, $V_2(s) = F_0(s)/(1+F_1(s)C_2(s))$, $V_3(s) = F_1(s)/(1+F_1(s)C_2(s))$, $V_4(s) = V_6(s) = V_{11}(s) = -F_1(s)C_2(s)/(1+F_1(s)C_2(s))$, $V_5(s)= F_2(s)/F_0(s)$, $V_7(s) = (F_1(s)/F_0(s))/(1 + F_1(s)C_2(s))$, $V_8(s) = (F_1(s)/F_0(s))C_2(s)/(1 + F_1(s)C_2(s))$, $V_9(s) = F_2(s)/F_1(s)$, $V_{10}(s) = F_0(s)C_2(s)/(1 + F_1(s)C_2(s))$, and $V_{12}(s) = C_2(s)/(1 + F_2(s)C_2(s))$. From the assumption that $F_1(s)$ is in RH_∞ and $C_2(s)$ stabilizes $F_1(s)$, $V_2(s) \in RH_\infty$ and $V_{10}(s) \in RH_\infty$ implies that (6) holds true. The necessity was shown.

The sufficiency is shown. When $\Delta(s) = 0$, We show that the double feedback control system shown in Figure 2 is internally stable if $F_0(s) \in RH_\infty$. The control system shown in Figure 2 is internally stable if all function $V_j(s) \in RH_\infty$ $(j = 1, 2, \dots, 12)$. From the assumption that $F_0(s) \in RH_\infty$, and $C_2(s)$ stabilizes $F_1(s)$, it is clear that all function $V_j(s)$ $(j = 1, 2, \dots, 12)$ is in RH_∞ if $F_0(s) \in RH_\infty$. The sufficiency was shown.

We have thus proved Lemma 3.2. □

We will prove Theorem 3.1 using above lemmas.

Proof: Even if $\Delta(s) = 0$, the double feedback control system in Figure 2 needs to be stable. From Lemma 3.2, the double feedback control system in Figure 2 is internally stable for $F_0(s)$, which implies $\Delta(s) = 0$, if and only if $F_0(s)$ is in RH_∞.

The double feedback control system shown in Figure 2 is internally stable for $G(s) \in \Omega$ if and only if all transfer function $V_i(s)$ $(i = 1, 2, \dots, 12)$ in (9) are in RH_∞. Since the characteristic polynomial of the double feedback control system is given by

$$1 + G(s)\frac{F_1(s)}{F_0(s)}C_2(s) = 1 + F_1(s)C_2(s)\left(1 + \Delta(s)\right), \tag{10}$$

the condition that all functions $V_j(s)$ $(j = 1, 2, \dots, 12)$ in (9) belong to RH_∞ is the same that the Nyquist plot of $1 + F_1(j\omega)C_2(j\omega)(1 + \Delta(j\omega))$ in (10) for $-\infty < \omega < \infty$ encircles the origin $(0, 0)$ $p + p_{c2}$ times counter-clockwise direction, where p_{c2} is the number of the poles of $C_2(s)$ in the closed right half plane. Thus, the double feedback control system shown in Figure 2 is robustly stable for $G(s) \in \Omega$ if and only if $F_0(s) \in RH_\infty$ and the Nyquist plot of (10) encircles the origin $(0, 0)$ $p + p_{c2}$ times in the counter-clockwise direction. The characteristics polynomial in (10) is rewritten by $1 + G(s)(F_1(s)/F_0(s))C_2(s) = (1 + F_1(s)C_2)(1 + \Delta(s))(1 - S_2(s)(\Delta(s)/(1 + \Delta(s))))$. From the assumption that $C_2(s)$ stabilizes $F_1(s) \in RH_\infty$, the Nyquist plot of $1+C_2(s)F_1(s)$ encircles the origin $(0, 0)$ p_{c2} times in the counter-clockwise direction. From Lemma 3.1, the Nyquist plot of $1+\Delta(s)$ encircles the origin $(0, 0)$ $p-p_m$ times in the counter-clockwise direction. Thus, the double feedback control system shown in Figure 2 is internally stable if the Nyquist plot of

$$1 - S_2(s)\frac{\Delta(s)}{1 + \Delta(s)} \tag{11}$$

does not encircle the origin $(0, 0)$ any time. The remaining problem is to prove that the Nyquist plot of (11) does not encircle the origin $(0, 0)$ any times equal to (7).

The necessity is shown. The proof is to show if $\|S_2(s)W(s)\|_\infty \geq 1$, then there exists $\Delta(s) \in \Omega$ to let the Nyquist plot of (11) encircle the origin. Since $F_0(s)$ is strictly proper, some $\omega \in \Omega$ satisfying $|S_2(j\omega)W(j\omega)| = 1+\epsilon_1$ $(\epsilon_1 > 0)$ exists. If we set $|\Delta(j\omega)/(1+\Delta(j\omega))|$ as $|\Delta(j\omega)/(1 + \Delta(j\omega))| = |W(j\omega)|/(1+\epsilon_1) \leq |W(j\omega)|$, then we have $1-S_2(j\omega)\Delta(j\omega)/(1+$

$\Delta(j\omega)) = 0$. This means that the Nyquist plot of $1 - S_2(j\omega)\Delta(j\omega)/(1+\Delta(j\omega))$ passes on the origin. Therefore, the control system in Figure 2 is unstable.

The sufficiency that the Nyquist plot of (11) does not encircle the origin $(0, 0)$ any times as the same as (7) shown. It is clear that the Nyquist plot of (11) can encircle the origin $(0, 0)$ no times even if we select any $\Delta(s) \in \Omega$.

From the above discussion, the proof of Theorem 3.1 was shown. □

Note that the double feedback control system in Figure 2 satisfying Theorem 3.1 is robustly stable even if $G(s) \in R(s)$ is unstable and $F_0(s) \in RH_\infty$. This is because the class of uncertainty Ω does not depend on the number of poles of $G(s)$ and that of $F_0(s)$ in the closed right half plane.

4. **Stability against the Failure of Making the Output Signal from the Stabilizing Controller $C_2(s)$ a Constant Value.** In this section, we examine the failure of making the output signal $u_{c2}(t)$ from the stabilizing controller $C_2(s)$ a constant value in the double feedback control system in Figure 1.

When the output signal $u_{c2}(t)$ from $C_2(s)$ changes to $\hat{u}_{c2}$, the double feedback control system in Figure 1 is rewritten as Figure 3. The robust stability condition that the double feedback control system shown in Figure 3 is stable for $G(s) \in \Omega$ in (1) is summarized as follows.

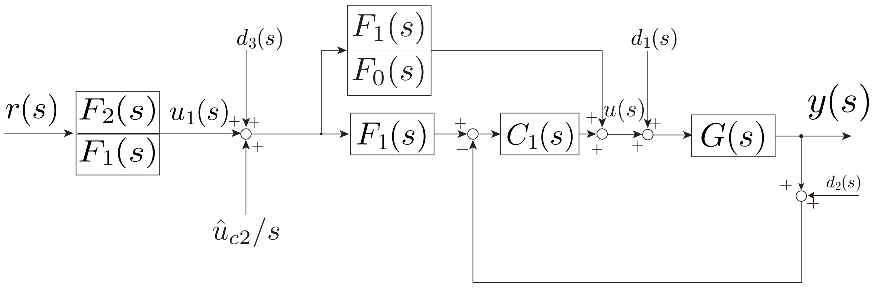

FIGURE 3. The double feedback control system when the output signal $u_{c2}(t)$ from $C_2(s)$ changes to $\hat{u}_{c2}$

Theorem 4.1. *Assume that $C_1(s)$ stabilizes $F_0(s)$. The double feedback control system shown in Figure 3 is stable for $G(s) \in \Omega$ if and only if*

$$\|S_1(s)W(s)\|_\infty < 1. \tag{12}$$

Theorem 4.1 is proved by using Lemma 3.1.

Proof: The characteristic polynomial of the double feedback control system in Figure 3 is given by

$$1 + G(s)C_1(s). \tag{13}$$

From the Nyquist theorem, the control system in Figure 3 is stable if and only if the Nyquist plot of $1+G(j\omega)C_1(j\omega)$ in (13) for $-\infty < \omega < \infty$ encircles the origin $(0, 0)$ $p+p_{c1}$ times in the counter-clockwise direction, where ω is any real number and p_{c1} means the number of poles of $C_1(s)$ in the closed right half plane. The characteristic polynomial in (13) can be rewritten as $1 + G(s)C_1(s) = (1 + F_0(s)C_1(s))(1 + \Delta(s))(1 - S_1(s)(\Delta(s)/(1+\Delta(s))))$. From the assumption that $C_1(s)$ stabilizes $F_0(s) \in R(s)$, the Nyquist plot of $1 + F_0(s)C_1(s)$ for $-\infty \leq \omega \leq \infty$ encircles the origin $(0, 0)$ $p + p_{c1}$ times in the counter-clockwise direction. From Lemma 3.1, the Nyquist plot of $1 + \Delta(s)$ encircles the origin $p - p_m$ times. Therefore, the necessary and sufficient condition that the control system in Figure 3 is stable for $G(s) \in \Omega$ is that the Nyquist plot of

$$1 - S_1(s)\frac{\Delta(s)}{1+\Delta(s)} \tag{14}$$

does not encircle the origin any time.

The remaining problem is to show that the necessary and sufficient condition that the Nyquist plot of (14) does not encircle the origin any time is expressed as (7). The sufficiency is proven as follows. Assume that $\|S_1(s)W(s)\|_\infty < 1$. It is clear that the Nyquist plot of (14) can encircle the origin no times even if we select any $\Delta(s) \in \Omega$. The necessity is shown. We show that if $\|S_1(s)W(s)\|_\infty \geq 1$, then there exists $\Delta(s) \in \Omega$ to let the Nyquist plot of (14) encircle the origin. Since $F_0(s)$ is strictly proper, some ω satisfying $|S_1(j\omega)W(j\omega)| = 1+\epsilon$ $(\epsilon > 0)$ exists, and some $\omega \in \Omega$ satisfying $|S_1(j\omega)W(j\omega)| = 1+\epsilon_2$ $(\epsilon_1 > 0)$ exists. If we set $|\Delta(j\omega)/(1+\Delta(j\omega))|$ as $|\Delta(j\omega)/(1+\Delta(j\omega))| = |W(j\omega)|/(1+\epsilon_2) \leq |W(j\omega)|$, then we have $1 - S_1(j\omega)\Delta(j\omega)/(1+\Delta(j\omega)) = 0$. This means the Nyquist plot of $1 - S_1(j\omega)\Delta(j\omega)/(1+\Delta(j\omega))$ passes on the origin. Therefore, the control system in Figure 3 is unstable.

From the above discussion, the proof of this theorem was shown. □

5. Design Procedure.

In this section, we provide a design procedure for the double feedback control system that maintains stability for a failure of making $u_{ck}(t)$ change to a constant value. Using the provided design procedure, we have controllers $C_1(s)$, $C_2(s)$, $F_1(s)$ and $F_2(s)$ satisfying Theorem 2.1, Theorem 3.1 and Theorem 4.1, simultaneously.

The design procedure is summarized as follows.

Design Procedure

Step 1. $W(s)$ is settled satisfying Definition 2.1.

Step 2. Design the feedback controller $C_1(s)$ satisfying (12).

Step 3. $F_1(s)$ is settled as

$$F_1(s) = Q_{f1}(s)F_0(s), \tag{15}$$

where $Q_{f1}(s) \in RH_\infty$ is any function.

Step 4. $F_2(s)$ is settled as

$$F_2(s) = Q_{f2}(s)F_1(s), \tag{16}$$

where $Q_{f2}(s) \in RH_\infty$ is any function.

Step 5. $C_2(s)$ satisfying both (5) and (7) is obtained by solving

$$\left\|S_2(s)\hat{W}(s)\right\|_\infty < 1. \tag{17}$$

$\hat{W}(s)$ is settled to satisfy

$$|W(j\omega)| \leq \left|\hat{W}(j\omega)\right| \quad (\forall\omega \in R), \tag{18}$$

and

$$|S_1(j\omega)W(j\omega)| \leq \left|\hat{W}(j\omega)\right| \quad (\forall\omega \in R). \tag{19}$$

From (18) and (19), it is obvious that if (17) is satisfied, then $C_2(s)$ satisfies both (5) and (7).

6. Numerical Example.

In this section, a numerical example is illustrated to show the effectiveness of the proposed method.

Consider the problem of designing a double feedback control system in Figure 1 for the plant $G(s) \in \Omega$ in (1) using the procedure described in the previous section, where $F_0(s)$ and $W(s)$ are given as

$$F_0(s) = \frac{0.5(s+10)(s+1)}{(s+3)(s+4)(s+2)}, \tag{20}$$

and

$$W(s) = \frac{0.7(s+73.72)(s+16.28)}{(s+6.449)(s+1.551)}. \tag{21}$$

We have the controller $C_1(s)$ satisfying (12) by using Linear Matrix Inequality (LMI). The controller $C_1(s)$ is designed as

$$C_1(s) = \frac{430109549.5969(s+14.47)(s+4)(s+3)(s+2)}{(s+2609000)(s+9.999)(s+6.496)(s+1.169)(s+0.9126)}. \tag{22}$$

$F_1(s)$ and $F_2(s)$ are designed by (15) and (16), respectively, where $Q_1(s) = 1$ and $Q_2(s) = 1$. This yields

$$F_2(s) = F_1(s) = F_0(s). \tag{23}$$

In order to design the controller $C_2(s)$, we set $\hat{W}(s)$ satisfying (18) and (19) as

$$\hat{W}(s) = \frac{0.7(s+73.72)(s+16.28)}{(s+6.449)(s+1.551)}. \tag{24}$$

To confirm $\hat{W}(s)$ satisfies (18) and (19), the gain plot of $S_1(s)W(s)$, $W(s)$ and $\hat{W}(s)$ is shown in Figure 4. Here, the solid line shows the gain plot of $\hat{W}(s)$, the dash-dotteded line shows that of $S_1(s)W(s)$, and the dashed line shows that of $W(s)$. Figure 4 shows that $\hat{W}(s)$ satisfies both (18) and (19). The controller $C_2(s)$ satisfying (17) is designed by using LMI. The feedback controller $C_2(s)$ is given by

$$C_2(s) = \frac{7935379.0147(s+14.45)(s+4)(s+3)(s+2)}{(s+48030)(s+10.09)(s+6.354)(s+1.239)(s+0.5932)}. \tag{25}$$

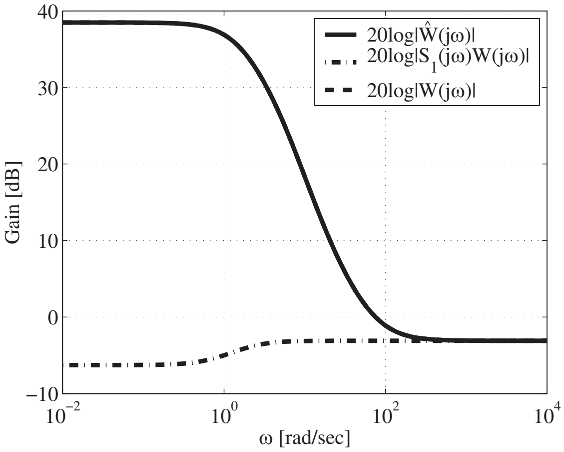

FIGURE 4. The gain plot of $\hat{W}(s)$, $S_1(s)W(s)$ and $W(s)$

Next, we confirm controllers $C_1(s)$ and $C_2(s)$ satisfy (5), (7) and (12). The gain plot of $S_1(s)$ and that of $1/W(s)$ are shown in Figure 5. Here, the solid line shows the gain plot of $S_1(s)$, and the dashed line shows that of $1/W(s)$. Figure 5 shows that the controller $C_1(s)$ satisfies (12). The gain plot of $S_2(s)$ and that of $1/W(s)$ are shown in Figure 6. Here, the solid line shows the gain plot of $S_2(s)$, and the dotted line shows that of $1/W(s)$. Figure 6 shows that the controller $C_1(s)$ satisfies (7). The gain plot of $S(s) = S_1(s)S_2(s)$ and that of $1/W(s)$ are shown in Figure 7. Here, the solid line shows the gain plot of $S(s) =$

$S_1(s)S_2(s)$, and the dashed line shows that of $1/W(s)$. Figure 7 shows that controllers $C_1(s)$ and $C_2(s)$ satisfies (12).

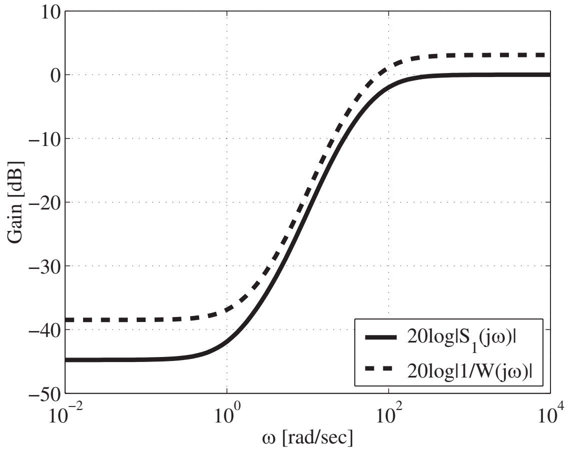

FIGURE 5. The gain plot of $S_1(s)$ and $1/W(s)$

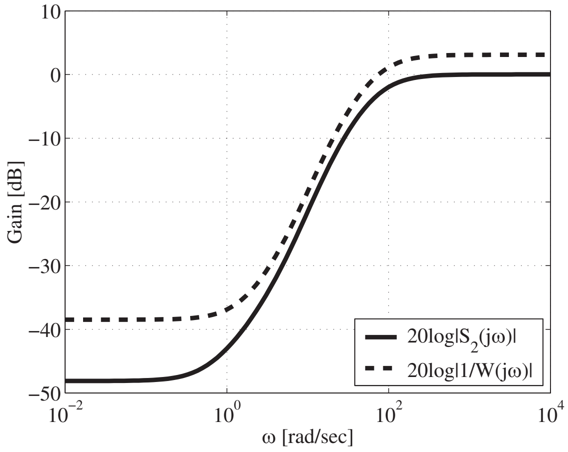

FIGURE 6. The gain plot of $S_2(s)$ and $1/W(s)$

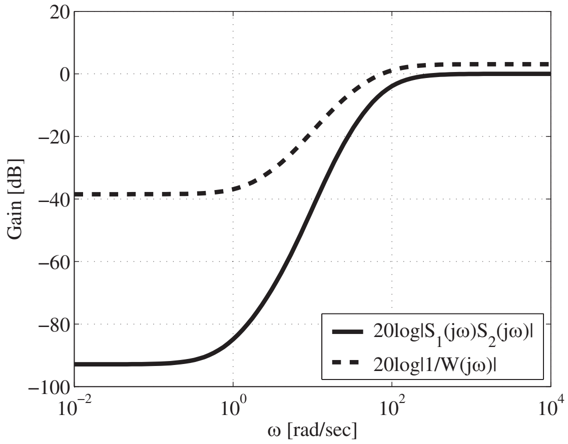

FIGURE 7. The gain plot of $S_1(s)S_2(s)$ and $1/W(s)$

Let $G(s)$ be

$$G(s) = \frac{0.45(s+10)(s+1)}{(s+0.15)(s-0.2)(s-3.5)}. \tag{26}$$

From (20) and (26), the number of poles of $G(s)$ in the closed right half plane is not equal to that of $F_0(s)$ and the relative degree of $G(s)$ is equal to that of $F_0(s)$. That is, $G(s)$ in (26) satisfies 1) in Definition 2.1. Then the fact that $G(s)$ in (26) is an element of Ω is confirmed by showing the gain plot of $\Delta(s)/(1+\Delta(s))$ and that of $W(s)$ in (21) as Figure 8. Here, the solid line shows the gain plot of $\Delta(s)/(1+\Delta(s))$ and the dashed line shows that of $W(s)$. Figure 8 shows that the uncertainty $\Delta(s)$ between $G(s)$ in (26) and $F_0(s)$ in (20) satisfies (3). Thus, the uncertainty $\Delta(s)$ is included in Ω.

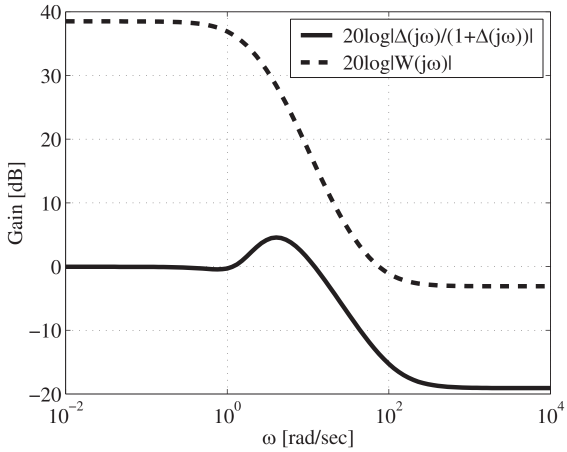

Figure 8. The gain plot of $\Delta(s)/(1+\Delta(s))$ and $W(s)$

Using above designed controllers, the step response from $r(t) = 1$ to $y(t)$ of the double feedback control system in Figure 1 that has a failure of making $u_{ck}(t)$ $(k = 1, 2)$ change to $\hat{u}_{ck} = 1$ after 6 seconds is shown in Figure 9. Here, the solid line in Figure 9 shows the response of the double feedback control system that has a failure of making $u_{c1}(t)$ change to $\hat{u}_{c1} = 1$, and the dashed line in Figure 9 shows the response of the double feedback control system that has a failure of making $u_{c2}(t)$ change to $\hat{u}_{c2} = 1$ after 6 seconds is shown in Figure 9, the dash-dotted in Figure 9 shows the response of the double feedback control system without a failure. Figure 9 shows that the double feedback control system in Figure 1 is stable even if $u_{ck}(t)$ $(k = 1, 2)$ changes $\hat{u}_{ck} = 1$. In addition, from the solid line in Figure 9, the response of the double feedback control system in Figure 1 after a failure of making $u_{c1}(t)$ change to $\hat{u}_{c1} = 1$ is close to the step response from $r(t) = 1$ to $y(t)$ of the double feedback control system without a failure. On the other hand, there exists the error between the response of the double feedback control system in Figure 1 after a failure of making $u_{c2}(t)$ change to $\hat{u}_{c2} = 1$ and the step response from $r(t) = 1$ to $y(t)$ of the double feedback control system without a failure. This implies that making $u_{c2}(t)$ change to $\hat{u}_{c2} = 1$, the input signal to the feedback loops becomes $F_2(s)/F_1(s)r(s) + 1/s$.

To show an influence of $d_2(t)$, which is usually considered as the noise, on the output $y(t)$ in the double feedback control system in Figure 1 when $u_{ck}(t)$ changes to $\hat{u}_{ck}$, the response from $d_2(t) = 1 + \sin t$ to $y(t)$ of the double feedback control system in Figure 1 that has a failure of making $u_{ck}(t)$ $(k = 1, 2)$ change to $\hat{u}_{ck} = 1$ after 6 seconds is shown in Figure 10 using above designed controllers. Here, the solid line in Figure 10 shows that the response from $d_2(t)$ to $y(t)$ of the double feedback control system in Figure 1 without a failure of making $u_{ck}(t)$ changes to $\hat{u}_{ck}(t) = 1$, the dash-dotted line shows that

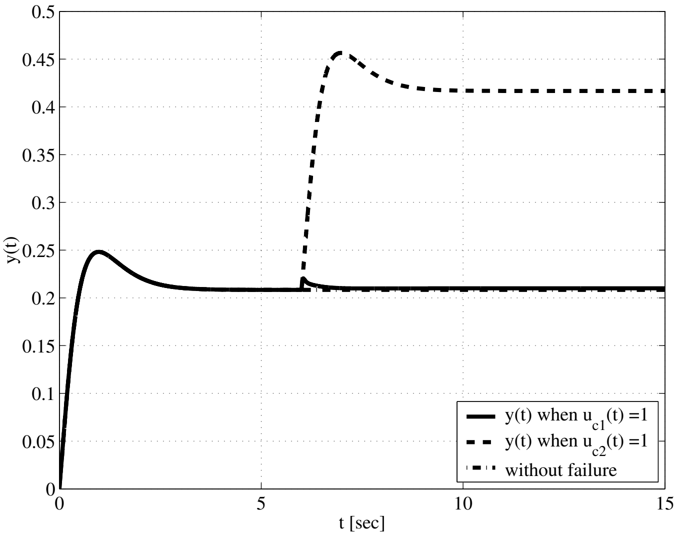

Figure 9. The step response from $r(t) = 1$ to $y(t)$ of the double feedback control system that has a failure of making $u_{ck}(t)$ $(k = 1, 2)$ change to $\hat{u}_{ck} = 1$ after 6 [sec]

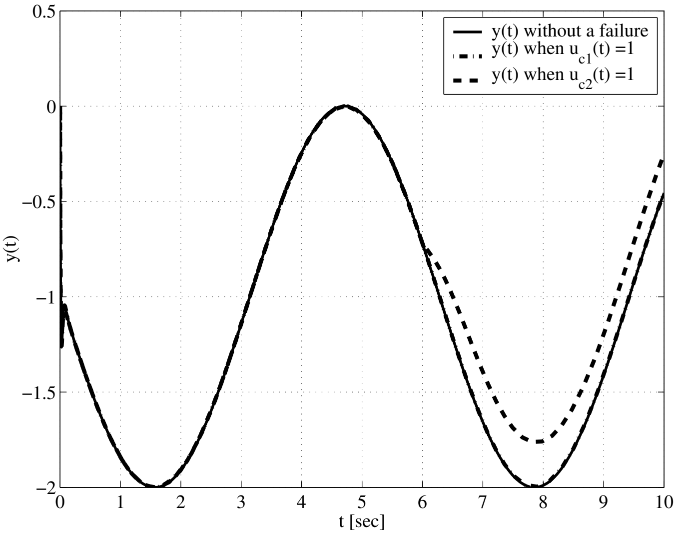

Figure 10. The response from $d_2(t) = 1 + \sin t$ to $y(t)$ of the double feedback control system that has a failure of making $u_{ck}(t)$ $(k = 1, 2)$ change to $\hat{u}_{ck} = 1$ after 6 [sec]

the response from $d_2(t)$ to $y(t)$ of the double feedback control system in Figure 1 with a failure of making $u_{c1}(t)$ change to $\hat{u}_{c1}(t) = 1$ after $t = 6$ [sec], and the dashed line in Figure 10 shows the response from $d_2(t)$ to $y(t)$ of the double feedback control system in Figure 1 with a failure of making $u_{c2}(t)$ change to $\hat{u}_{c1} = 1$ after $t = 6$ [sec]. From Figure 10, causing the failure of making $u_{c1}(t)$ change to $\hat{u}_{c1} = 1$ after $t = 6$ [sec], the response from $d_2(t)$ to $u(t)$ does not change. On the other hand, causing the failure of making $u_{c2}(t)$ change to $\hat{u}_{c2} = 1$ after $t = 6$ [sec], the response from $d_2(t)$ to $y(t)$ differs from that without a failure of making $u_{ck}(t)$ change to $\hat{u}_{ck} = 1$. This difference between the response from $d_2(t)$ to $y(t)$ of the double feedback control system in Figure 1 with a failure of making $u_{c1}(t)$ change to $\hat{u}_{c1} = 1$ and that of the double feedback control system in Figure 1 with a failure of making $u_{c2}(t)$ change to $\hat{u}_{c2} = 1$ is caused so that the double feedback control system cannot attenuate the influence of $d_2(t)$ to $y(t)$, effectively by making $u_{c2}(t)$ change to $\hat{u}_{c2} = 1$.

Finally, for comparison with conventional control system design methods, the response of the double feedback control system designed by a method in [25] from $r(t) = 1$ to $y(t)$ for the nominal plant $F_0(s)$ given by

$$F_0(s) = \frac{0.5(s+10)(s+1)}{(s+3)(s+4)(s-2)}, \tag{27}$$

and the upper bound $W(s)$ given by (21) will be shown. Controllers $C_1(s)$, $C_2(s)$, $F_1(s)$ and $F_2(s)$ are designed using a method in [25]. Then the gain plot of $S(s) = S_1(s)S_2(s)$ is shown in Figure 11. Here, the solid line shows the gain plot of $S(s) = S_1(s)S_2(s)$, and the dashed line shows that of $1/W(s)$. Figure 11 shows the double feedback control system in Figure 1 satisfies (5). Then the fact that $G(s)$ in (26) is an element of Ω is confirmed by showing the gain plot of $\Delta(s)/(1+\Delta(s))$ and that of $W(s)$ in (21) as Figure 12 when $F_0(s)$ is given as (27). Here, the solid line shows the gain plot of $\Delta(s)/(1+\Delta(s))$ and the dashed line shows that of $W(s)$. Figure 12 shows that the uncertainty $\Delta(s)$ between $G(s)$ in (26) and $F_0(s)$ in (27) satisfies (3). Thus, the uncertainty $\Delta(s)$ is included in Ω.

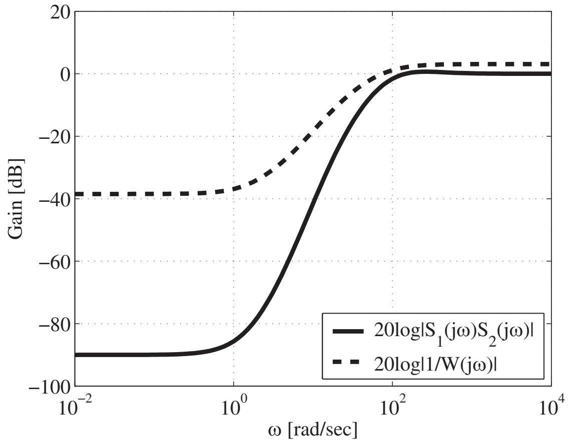

FIGURE 11. The gain plot of $S_1(s)S_2(s)$ in the double feedback control system designed by a method in [25] and $1/W(s)$

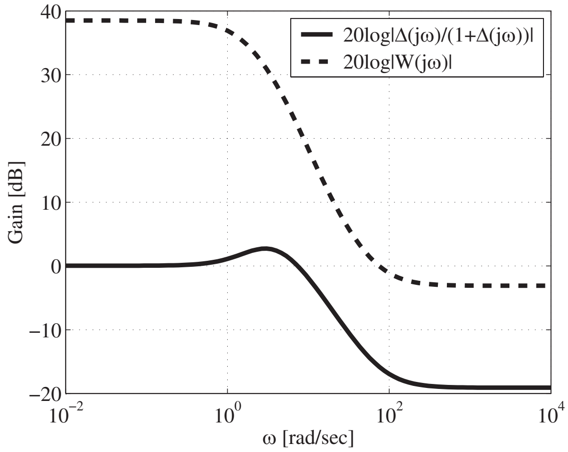

FIGURE 12. The gain plot of $\Delta(s)/(1+\Delta(s))$ and $W(s)$ when $F_0(s)$ is given as (27)

For $F_0(s)$ given as (27), the step response from $r(t) = 1$ to $y(t)$ of the double feedback control system in Figure 1 designed by a method in [25] that has a failure of making $u_{ck}(t)$ $(k = 1, 2)$ change to $\hat{u}_{ck} = 1$ after 6 seconds is shown in Figure 13. Here, the solid line in Figure 13 shows the response of the double feedback control system that has a failure of making $u_{c1}(t)$ change to $\hat{u}_{c1} = 1$, and the dashed line in Figure 13 shows the response of the double feedback control system that has a failure of making $u_{c2}(t)$ change to $\hat{u}_{c2} = 1$ after 6 seconds. Figure 13 shows that the double feedback control system in Figure 1 that does not satisfy Theorems 2.1, 3.1 and 4.1 is unstable for a failure of making $u_{c1}(t)$ change to $\hat{u}_{c1} = 1$.

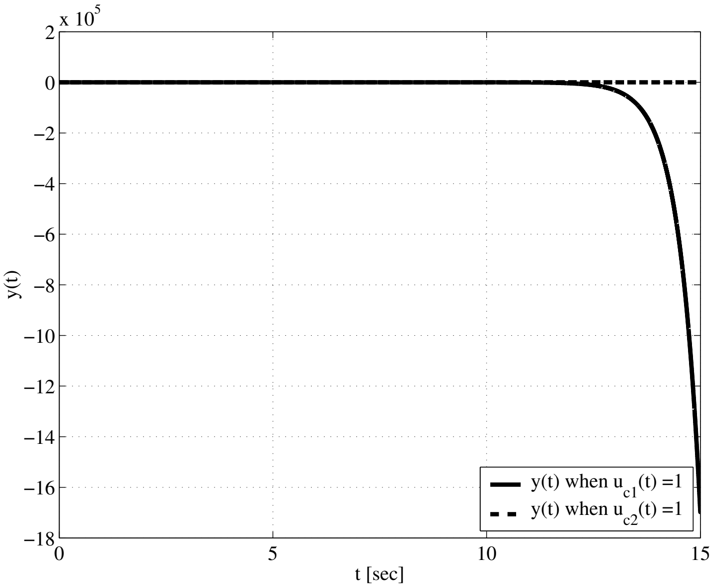

FIGURE 13. The step response from $r(t) = 1$ to $y(t)$ of the double feedback control system designed by a method in [25] that has a failure of making $u_{ck}(t)$ $(k = 1, 2)$ change to $\hat{u}_{ck} = 1$ after 6 [sec]

7. **Conclusion.** In this paper, we have examined a design method for the double feedback control system to maintain stability against a failure of making the output signal from the controller a constant value. However, the double feedback control system changes to a different control structure depending on the case of failure. This paper only considers the failure of making the output signal from a controller in the double feedback control system a constant value due to space limitations. To maintain stability against the other case of failure, we should examine the stabilization problem for the double feedback control system in the other case of failure. The problem will be considered in another paper as future work. In addition, the application of the result in this paper for actual systems will also be discussed in another article.

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