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HIERARCHY OF STABILITY CONDITIONS FOR DISCRETE-TIME DELAYED NEURAL NETWORKS VIA GENERAL FREE-MATRIX SUMMATION INEQUALITIES

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Abstract. *In this paper, the stability of discrete-time neural networks with time-varying delays is further investigated. First, an N-dependent general free-weight-matrix summation inequality (NGFWMSI) is proposed to involve coupling information on additional state variables by additional free matrices and free vectors, which can relax the derived stability conditions. Second, a novel N-dependent Lyapunov-Krasovskii functional (LKF) is constructed, which contains not only the coupling information between the delay intervals and the state variables, but also the coupling information between the nonlinear terms and the delays and other state variables. Then, combining the N-dependent LKF with the NGFWMSI, the stability criteria derived reduce the conservatism of some published results. And the stability criteria are hierarchical, that is, the higher level of hierarchy, the less conservatism of the stability criteria. Finally, several numerical examples are presented to show the validity and superiority of the proposed methods.*

Keywords: Discrete-time, Lyapunov, Linear matrix inequality (LMI), Neural networks, Stability, Time-delayed system

1. **Introduction.** Time delay is regarded as the main cause of system instability, poor performance and oscillation [1]. Lyapunov stability theory is the main method to analyze the stability of time-delayed systems. The main purpose of this method is to derive stability criteria with less conservatism based on LMI, which mainly focuses on two aspects: i) In the stability range of the system, the maximum allowable delay upper bound (MADUB) is determined as large as possible; ii) The derived stability criterion reduces the conservatism and the number of decision variables in LMIs, so as to reduce the solution complexity. In order to achieve this purpose, a large number of mathematical techniques have developed along these two directions. It is mainly reflected in the following aspects. a) The corresponding LKF is augmented to include as much information on the state variable and the time-varying delay as possible. An LKF with a quadratic polynomial matrix is introduced for the stability of continuous-time and discrete-time systems with time-varying delays [2]. A delay-product-type LKF is constructed in [3], in which the information on time-varying delays and system states is taken into full consideration. The

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authors of [4] construct an augmented indefinite LKF, and prove that the LKF has indefinite derivatives by using simple integral inequalities. Beside, there are LKF based on delay partitioning technique [5], delay-square-dependent LKF [5], and so on. b) Derive tight integral summation inequalities as much as possible, so that the derivative terms of the LKF get a tight upper bound, for example, the delay-product-type integral inequality [6], the generalized integral inequalities based on free matrices [7], the Bessel summation inequalities [8, 9], the Jensen's summation inequality [10], the auxiliary-function-based inequality [11], the reciprocally convex matrix inequality [12], and the delay-dependent matrix-separation-based inequality [13].

Time-delayed neural networks can be used in image processing for image filtering, edge detection and feature extraction, and their stability characteristics can be utilized to improve the processing effect and speed [14]. Discrete-time delayed neural networks can idealize the physical reality of continuous-time delayed neural networks under hardware constraints, and have more complex dynamic characteristics and application prospects. Regarding the analysis of delay stability for extended dissipativity, there have been many excellent achievements, for example, exponential H_∞ filtering [15], finite-time dissipativeness [16], and event-triggered synchronization [17]. The main method to analyze the stability of discrete-time delayed neural networks is also to derive the delay-dependent stability criteria based on LMI by using Lyapunov stability theory [18, 19, 20, 21, 22, 23]. However, the stability criteria derived from Lyapunov stability theory are sufficient conditions and must have conservatism. In order to reduce this conservatism, in addition to constructing the LKF with as much coupling information between the delays and the system states as possible, it is necessary to select advanced summation inequalities to estimate the forward difference of the LKF. Recently, scholars have become more and more interested in the summation inequality techniques based on free matrices, such as discrete inequalities based on multiple auxiliary functions [11], discrete Wirtinger-based inequality [24], double summation inequality [25, 26], general free-matrix-based summation inequality [27], and bessel summation inequality [28, 29, 30]. Especially, in [28], hierarchical stability criteria based on the proposed arbitrary-order Legendre polynomials were given. In the numerical examples, it was shown that increasing the order of the polynomials can reduce the conservatism. Inspired by the idea in [27], since some free matrix variables are introduced, the conventional free-matrix-based summation inequalities have more freedom to estimate summation terms than the aforementioned ones. Therefore, for the bessel summation inequality, hierarchical stability criteria proposed in the literature, it is meaningful to generalize to the form of the general free weight matrix to further reduce the conservatism of the stability criterion for discrete-time delayed neural network.

Moreover, it is well known that one key to derive the stability criterion of time-delayed systems by using Lyapunov stability theory is the construction of LKF. The more state variables and time-varying delays information LKF contains, the less conservative the stability criteria obtained. Inspired by [31, 32], as a nonlinear system, the more information on nonlinear terms the constructed LKF contains, the less conservative the LKF is. However, most of the LKF constructed in the literature have little or no information on the nonlinear state variables. This increases the conservatism of LKF for nonlinear neural network systems. Hence, it is important to find a suitable way to add as much information as possible on nonlinearity to reduce the conservatism of stability criterion when constructing related LKF.

This paper further investigates the stability of discrete-time neural networks with an interval-like time-varying delay. A novel class of LKF, which takes full account of the nonlinear integral terms, is constructed, and the upper bound of the difference terms of the

LKF is determined by an N-dependent general free-weight-matrix summation inequality. New stability criteria for discrete-time delayed neural networks are obtained. The main contributions of this paper are as follows.

- Compared with the summation inequalities proposed in [19, 20, 21, 22, 23], an N-dependent general free-weight-matrix summation inequality is proposed to involve coupling information on additional state variables by additional free matrices and free vectors, which can relax the derived stability conditions.
- A novel N-dependent LKF is constructed in which some additional summation terms contain additional nonlinear integral terms. The additional nonlinear integral terms are ignored in [19, 20, 21, 22, 23]. According to the characteristics of nonlinear constraints, the nonlinear integral terms are introduced into the LKF, so that the whole LKF contains not only the coupling information between the delay intervals and the state variables, but also the coupling information between the nonlinear terms and the delays and other state variables.
- Combining the N-dependent LKF with the N-dependent general free-weight-matrix summation inequality, the stability criteria derived reduce the conservatism of some published results in [19, 20, 21, 22, 23]. And the stability criteria are hierarchical, that is, the higher level of hierarchy, the less conservatism of the stability criteria, which will be proved by the numerical examples.

This paper is organized as follows. Section 2 gives the problem statement and provides some useful definitions, assumptions and lemmas. Section 3 presents the stability criteria for the discrete-time delayed neural networks, including theorems and corollaries. Section 4 shows numerical examples and simulations. Conclusions are drawn in Section 5.

Notation: Throughout this paper, the notations are standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space; $\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices; For $P \in \mathbb{R}^{n\times n}$, $P > 0$ (respectively, $P < 0$) means that P is a positive (respectively, negative) definite matrix. $\text{diag}\{a_1, a_2, \ldots, a_n\}$ denotes an n-order diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$. e_i $(i = 1, \ldots, m)$ are block entry matrices. For example, $e_2^T = \begin{bmatrix} 0 & I & \underbrace{0 \cdots 0}_{m-2} \end{bmatrix}$. For a real matrix B and two real symmetric matrices A and C of appropriate dimensions, $\begin{bmatrix} A & B \\ * & C \end{bmatrix}$ denotes a real symmetric matrix, where $*$ denotes the entries implied by symmetry. $x^{\overline{m}}$ is a rising factorial given by $x(x+1)\cdots(x+m-1)$ and $x^{\underline{m}}$ is a falling factorial given by $x(x-1)\cdots(x-m+1)$. $\begin{pmatrix} n \\ k \end{pmatrix}$ denotes a binomial coefficient given by $\frac{n!}{k!(n-k)!}$.

2. Preliminaries.

Consider the following generalized discrete-time neural network in which the equilibrium point is assumed to be shifted to the origin:

$$\begin{cases} x(k+1) = Cx(k) + Af(z(k)) + A_d f(z(k-h(k))), \\ z(k) = A_0 x(k), \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the neural state vector associated with n neurons; $C = \text{diag}\{c_1, c_2, \ldots, c_n\}$ is the state feedback coefficient matrix, A, A_d and A_0 are known constant real matrices. The time-varying delay $h(k)$, abbreviated as h_k, satisfies $1 \le h_1 \le h_k \le h_2$, where h_1 and h_2 are known integers. The nonlinear function $f(z(k)) = \text{col}\{f_1(z_1(k)), \ldots, f_n(z_n(k))\}$ represents the neural activation function satisfying $f_i(0) = 0$ and [18]

$$0 \le k_j^- \le \frac{f_j(\sigma_2) - f_j(\sigma_1)}{\sigma_2 - \sigma_1} \le k_j^+, \quad \sigma_1 \ne \sigma_2,\ j \in \{1, 2, \ldots, n\}, \tag{2}$$

or [33]

$$d_j^- \le \frac{f_j(\sigma_2) - f_j(\sigma_1)}{\sigma_2 - \sigma_1} \le d_j^+, \quad \sigma_1 \ne \sigma_2,\ j \in \{1, 2, \ldots, n\}, \tag{3}$$

where k_j^-, k_j^+ are known positive constants, and d_j^-, d_j^+ are just known constants that may be positive, negative, or zero.

Remark 2.1. *The constants d_j^-, d_j^+ in* (3) *are allowed to be positive, negative or zero. Hence, the resulting activation functions could be non-monotonic. k_j^-, k_j^+ are positive, which shows the activation functions could be monotonically increasing. Obviously, the constraint condition* (2) *is included in* (3)*. Recently,* [31, 32] *proposed a tight nonlinear integral inequality to further reduce the conservatism of the stability criterion. Therefore, two types of nonlinear constraints will be considered separately in this paper.*

Lemma 2.1. [9] *Let $x : [a, b-1] \cap \mathbb{Z} \to \mathbb{R}^n$ be a vector function. For a positive definite matrix $R \in \mathbb{S}^n$, and integers $a, b \in \mathbb{Z}$, $N \in \mathbb{Z}_{\ge 0}$ satisfying $a \le b-1$, the following inequalities hold:*

$$\sum_{i=a}^{b-1} x^T(i) R x(i) \ge \frac{1}{b-a} \mathbb{T}_N^T(a,b) R_N \mathbb{T}_N(a,b), \tag{4}$$

$$\sum_{i=a}^{b-1} \Delta x^T(i) R \Delta x(i) \ge \frac{1}{b-a} \mathbb{Y}_N^T(a,b) R_N \mathbb{Y}_N(a,b) \tag{5}$$

with

$$\begin{aligned}
&\alpha_l^k = (-1)^{k+l} \left(\begin{array}{c} k \\ l \end{array} \right) \left(\begin{array}{c} k+l \\ l \end{array} \right),\ \rho(a,b,l) = \frac{l!}{(b-a+1)^{\bar{l}}}, \\
&\Delta x(i) = x(i+1) - x(i),\ R_N = \mathrm{diag}\left\{R, 3R, \ldots, (2N+1)R\right\}, \\
&\mathbb{T}_N(a,b) = \mathrm{col}\left\{T_0(a,b), \ldots, T_N(a,b)\right\},\ \mathbb{Y}_N(a,b) = \mathrm{col}\left\{Y_0(a,b), \ldots, Y_N(a,b)\right\}, \\
&T_k(a,b) = -x(b) + \sum_{l=0}^{k} \alpha_l^k \rho(a,b,l) \mathcal{S}_l(a,b),\ Y_k(a,b) = x(b) - \sum_{l=0}^{k} \alpha_l^k \rho(a,b,l) \mathcal{S}_{l-1}(a,b), \\
&\mathcal{S}_l(a,b) = \left\{ \begin{array}{ll} x(a), & \textit{if } l = -1 \\ \displaystyle\sum_{i_{l+1}=a}^{b} \cdots \sum_{i_2=i_3}^{b} \sum_{i_1=i_2}^{b} x(i_1), & \textit{if } l \ge 0 \end{array} \right. .
\end{aligned} \tag{6}$$

Remark 2.2. *In order to give the free-weighting matrix-based summation inequalities, the forms of summation inequalities* (4) *and* (5) *in Lemma 2.1 need to be further simplified as follows.*

$$\sum_{i=a}^{b-1} x^T(i) R x(i) \ge \frac{1}{b-a} \zeta_N^T(a,b) \Xi_N^T(a,b) R_N \Xi_N(a,b) \zeta_N(a,b), \tag{7}$$

$$\sum_{i=a}^{b-1} \Delta x^T(i) R \Delta x(i) \ge \frac{1}{b-a} \varphi_N^T(a,b) \Xi_N^T(a,b) R_N \Xi_N(a,b) \varphi_N(a,b) \tag{8}$$

with

$$\zeta_N(a,b) = \mathrm{col}\left\{x(b), \mathcal{S}_0, \ldots, \mathcal{S}_N\right\},\ \varphi_N(a,b) = \mathrm{col}\left\{x(b), \mathcal{S}_{-1}, \ldots, \mathcal{S}_{N-1}\right\},$$

$$\Xi_N(a,b) = \begin{bmatrix} -I & \alpha_0^0\rho(a,b,0) & 0 & \cdots & 0 \\ -I & \alpha_0^1\rho(a,b,0) & \alpha_1^1\rho(a,b,1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -I & \alpha_0^N\rho(a,b,0) & \alpha_1^N\rho(a,b,1) & \cdots & \alpha_N^N\rho(a,b,N) \end{bmatrix}.$$

Lemma 2.2. *(Affine Bessel Summation Inequalities, ABSI) Let* $x : [a, b-1] \cap \mathbb{Z} \to \mathbb{R}^n$ *be a vector function. For a positive definite matrix* $R \in \mathbb{S}^n$*,* M_1*,* M_2 *with appropriate dimensions and integers* $a, b \in \mathbb{Z}$*,* $N \in \mathbb{Z}_{\geq 0}$ *satisfying* $a \leq b-1$*, and vectors* β_N*,* ω_N*, the following inequalities hold:*

$$-\sum_{i=a}^{b-1} x^T(i)Rx(i) \leq \mathrm{Sym}\left\{\beta_N^T M_1^T \Xi_N(a,b)\zeta_N(a,b)\right\} + (b-a)\beta_N^T M_1^T R_N^{-1} M_1 \beta_N, \tag{9}$$

$$-\sum_{i=a}^{b-1} \Delta x^T(i)R\Delta x(i) \leq \mathrm{Sym}\left\{\omega_N^T M_2^T \Xi_N(a,b)\varphi_N(a,b)\right\} + (b-a)\omega_N^T M_2^T R_N^{-1} M_2 \omega_N. \tag{10}$$

Proof: Letting $A = \sum_{i=a}^{b-1} x^T(i)Rx(i)$, $B = \Xi_N(a,b)\zeta_N(a,b)$, $C = R_N^{-1}$, $S = (b-a)M_1\beta_N$, we can obtain the following two equivalent inequalities from Remark 2.2:

$$(b-a)A - B^T C^{-1} B > 0, \tag{11}$$

$$(b-a)A - B^T C^{-1} B + (B+CS)^T C^{-1}(B+CS) > 0, \tag{12}$$

and then we can obtain the following inequality based on (12):

$$(b-a)A + S^T B + B^T S + S^T C S > 0, \tag{13}$$

that is,

$$-A < \frac{1}{b-a}S^T B + \frac{1}{b-a}B^T S + \frac{1}{b-a}S^T C S, \tag{14}$$

which concludes the proof of (9). The proof of (10) is omitted due to the similarity to the proof of (9). □

Remark 2.3. *The motivation of proposing Inequalities* (9) *and* (10) *is to involve coupling information for additional state variables by additional free matrices and free vectors* β_N *and* ω_N*, which can relax the derived conditions. Compared with the summation inequalities proposed in* [8, 9, 11, 22, 24, 25, 26, 27, 29, 30]*, Inequalities* (9) *and* (10) *have some advantages.*

- *Inequalities* (9) *and* (10) *are more general than those in* [8, 9, 11, 22, 24, 25, 26, 27, 29, 30]*, that is, Inequalities* (9) *and* (10) *include those in* [8, 9, 11, 22, 24, 25, 26, 27, 29, 30]*. For example, Inequality* (10) *reduces to Lemmas 1 and 3 (DWI) of* [8, 24] *with* $N = 1$*,* $\omega_1^T = \varphi_1^T(a,b)\Xi_1^T(a,b)$ *and* $M_2^T = -\mathrm{diag}\left\{\frac{R}{b-a}, \frac{3R}{b-a}\right\}$*; reduces to Lemma 4, Remarks 3 and 4 of* [11] *with* $N = 2$*,* $\omega_2^T = \varphi_2^T(a,b)\Xi_2^T(a,b)$ *and* $M_2^T = -\mathrm{diag}\left\{\frac{R}{b-a}, \frac{3R}{b-a}, \frac{5R}{b-a}\right\}$*; reduces to Lemma 1 of* [25, 26, 30] *with* $N = 2$*,* $\omega_2^T = \varphi_2^T(a,b)$ *and* $M_2^T = [N_1 \quad N_2 \quad N_3]$*. Inequality* (10) *is equivalent to Lemma 2 of* [27] *when* $N = 1$*. Inequalities* (9) *and* (10) *reduce to Lemmas 1 and 2 of* [22] *with* $N = 2$*,* $\omega_2^T = \mathrm{col}\{\omega_0, \omega_1, \omega_2\}$ *and* $M_2^T = \mathrm{diag}\{N_0, N_1, N_2\}$*; are equivalent to Lemma 6 of* [9] *when* $\beta_N^T = \varphi_N^T(a,b)\Xi_N^T(a,b)$*; are equivalent to Corollary 1 of* [29] *when* $N = 2$*;*
- *From the relation between Inequalities* (11) *and* (12)*, Inequalities* (7) *and* (8) *of Remark 2.2 are tighter than* (11) *and* (12) *of Lemma 2.2. Thus, if* $b-a$ *is a constant, Inequalities* (7) *and* (8) *are usually used to estimate the upper bounds of the summation terms. In this case, just take* $\beta_N^T = \varphi_N^T(a,b)\Xi_N^T(a,b)$ *and* $M_2^T = -\frac{1}{b-a}R_N$*,*

which will not increase the conservatism, but also reduce the decision variables of LMIs and reduce the complexity of solving the LMIs. However, how to choose the optimal vectors β_N and ω_N is still interesting and challenging. However, as stated in the work of Lee et al. [28]*, the ABSI might be less conservative than Inequalities* (7) *and* (8) *when $b-a$ is time varying;*
- *The vectors β_N and ω_N are arbitrary. Generally speaking, if more state information or time delay information is contained in β_N and ω_N, a more relaxed condition can be obtained* [22].

Therefore, in order to fully find a suboptimal vector for time varying delay, that is, to contain as much useful state information or time delay information as possible, the conservatism of the stability condition can be reduced with as little solving complexity as possible. The following result is given.

Lemma 2.3. *(General Free-Matrix Summation Inequalities, GFMSI) Let $x:[a,b-1]\to\mathbb{R}^n$ be a vector function with $a,b\in\mathbb{Z}$. For a positive definite matrix $R\in\mathbb{R}^{n\times n}$, any matrices L_1, L_2, H_1, H_2 with appropriate dimensions, a time-varying function $c(k)\triangleq c_k$ with $a\le c_k\le b$, and vectors β_N, ω_N, the following inequalities hold:*

$$
\begin{aligned}
-\sum_{i=a}^{b-1}x^T(i)Rx(i) \le{}& \mathrm{Sym}\left\{\beta_N^T\left[L_1^T\Xi_N(a,c_k)L_2^T\Xi_N(c_k,b)\right]\alpha_N\right\}\\
&+\beta_N^T\left\{(c_k-a)L_1^TR_N^{-1}L_1+(b-c_k)L_2^TR_N^{-1}L_2\right\}\beta_N,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
-\sum_{i=a}^{b-1}\Delta x^T(i)R\Delta x(i) \le{}& \mathrm{Sym}\left\{\omega_N^T\left[H_1^T\Xi_N(a,c_k)H_2^T\Xi_N(c_k,b)\right]\varpi_N\right\}\\
&+\omega_N^T\left\{(c_k-a)H_1^TR_N^{-1}H_1+(b-c_k)H_2^TR_N^{-1}H_2\right\}\omega_N,
\end{aligned}
\tag{16}
$$

where $\alpha_N=\mathrm{col}\{\zeta_N(a,c_k),\zeta_N(c_k,b)\}$ and $\varpi_N=\mathrm{col}\{\varphi_N(a,c_k),\varphi_N(c_k,b)\}$ are defined in Remark 2.2.

Proof: According to Inequality (9) and dividing $\sum_{i=a}^{b-1}x^T(i)Rx(i)$ into $\sum_{i=a}^{c_k-1}x^T(i)Rx(i)$ and $\sum_{i=c_k}^{b-1}x^T(i)Rx(i)$, the following inequalities can be obtained for vectors β_{1N}, β_{2N} and free matrices $\tilde{L}_1$, $\tilde{L}_2$ with appropriate dimensions:

$$
-\sum_{i=a}^{c_k-1}x^T(i)Rx(i)\le \mathrm{Sym}\left\{\beta_{1N}^T\tilde{L}_1^T\Xi_N(a,c_k)\zeta_N(a,c_k)\right\}+(c_k-a)\beta_{1N}^T\tilde{L}_1^T\bar{R}_N^{-1}\tilde{L}_1\beta_{1N}, \tag{17}
$$

$$
-\sum_{i=c_k}^{b-1}x^T(i)Rx(i)\le \mathrm{Sym}\left\{\beta_{2N}^T\tilde{L}_2^T\Xi_N(c_k,b)\zeta_N(c_k,b)\right\}+(b-c_k)\beta_{2N}^T\tilde{L}_2^T\bar{R}_N^{-1}\tilde{L}_2\beta_{2N}. \tag{18}
$$

Then, letting $\beta_N=\beta_{1N}=\beta_{2N}$, Inequality (15) can be obtained by adding (17) and (18), where L_1 and L_2 are the corresponding free matrices after $\tilde{L}_1$ and $\tilde{L}_2$ augmented by the appropriate number of columns. This completes the proof of (15). The proof of (16) is omitted due to the similarity to the proof of (15). □

Remark 2.4. *For time varying delay $c(k)\triangleq c_k$ with $a\le c_k\le b$, in general, various summation inequality techniques are used to estimate the upper bounds of the summation terms in time varying delay intervals $[a,c_k]$ and $[c_k,b]$. Lemma 2.3 uses the free vectors β_N and ω_N to combine the summation terms to estimate their upper bounds. Compared with the published methods* [8, 9, 11, 22, 24, 25, 26, 27, 34]*, the main advantages are summarized as follows. Take Inequality* (16) *for an example.*

- *Applying the summation inequalities in* [8, 11, 24] *to estimating the upper bounds of* $-\sum_{i=a}^{b-1}\Delta x^T(i)\times R\Delta x(i)$ *yields*

$$-\sum_{i=a}^{b-1}\Delta x^T(i)R\Delta x(i)\le \lambda_1\varphi^T(a,c_k)\Xi^T R_N\Xi\varphi(a,c_k)+\lambda_2\varphi^T(c_k,b)\Xi^T R_N\Xi\varphi(c_k,b). \quad (19)$$

From (19)*, it can be found that the obtained terms* $\Xi\varphi(a,c_k)$ *and* $\Xi\varphi(c_k,b)$ *are only connected with themselves via the coefficient matrix* R_N*, respectively. However, letting the free vector* $\omega_N=\mathrm{col}\{\Xi\varphi(a,c_k),\Xi\varphi(c_k,b)\}$ *in* (16) *of Lemma 2.3, two free matrices* H_1 *and* H_2 *are employed to make* $\Xi\varphi(a,c_k)$ *and* $\Xi\varphi(c_k,b)$ *connect to each other and themselves. Therefore, additional freedom and state or delay coupling information are involved in Inequality* (16)*, which can relax the derived stability conditions.*

- *Applying the summation inequalities in* [9, 25, 26, 27, 34] *to estimating the upper bounds of* $-\sum_{i=a}^{b-1}\Delta x^T(i)R\Delta x(i)$ *yields*

$$\begin{aligned}-\sum_{i=a}^{b-1}\Delta x^T(i)R\Delta x(i)\le{}&\varphi^T(a,c_k)\Xi^T\left[M_1+M_1^T+\lambda_1M_1^T\bar{R}^{-1}M_1\right]\Xi\varphi(a,c_k)\\&+\varphi^T(c_k,b)\Xi^T\left[M_2+M_2^T+\lambda_2M_2^T\bar{R}^{-1}M_2\right]\Xi\varphi(c_k,b). \quad (20)\end{aligned}$$

(20) *is a special case of Lemma 2.2 with* $N\le 2$ *and* $\beta=\Xi\varphi$*. However, it can be found that the terms* $\Xi\varphi(a,c_k)$ *and* $\Xi\varphi(c_k,b)$ *are also only connected with themselves via the matrices* M_1 *and* M_2*, respectively, even if* M_1 *and* M_2 *are free matrices. However, just set the free vector* $\omega_N=\mathrm{col}\{\Xi\varphi(a,c_k),\Xi\varphi(c_k,b)\}$ *in* (16) *of Lemma 2.3, the terms* $\Xi\varphi(a,c_k)$ *and* $\Xi\varphi(c_k,b)$ *are not only connected with themselves, but also connected with each other via two free matrices* H_1 *and* H_2*.*

- *Recently, a free-matrix-based summation inequality was given by* [22]*. Indeed, letting* $N=2$*,* $\beta_2=\mathrm{col}\{\omega_{05},\omega_{15},\omega_{25},\omega_{06},\omega_{16},\omega_{26}\}$*,* $H_1=\mathrm{col}\{\mathrm{diag}\{N_{05},N_{15},N_{25}\},0\}$ *and* $H_2=\mathrm{col}\{0,\mathrm{diag}\{N_{06},N_{16},N_{26}\}\}$*, the summation inequality* (16) *reduces to the one of* [22]*. Although* β_2 *of* [22] *is also a free vector, it lacks a lot of state or delay coupling information, for example, the coupling information of* ω_{i5} $(i=0,1,2)$ *and* $\Xi\varphi(a,c_k)$*, and the coupling information of* ω_{i6} $(i=0,1,2)$ *and* $\Xi\varphi(c_k,b)$*. The authors of* [22] *pointed out that generally speaking, if more part is contained in* β_2*, a more relaxed condition can be obtained. However, the selection method of* β_2 *is not certain. Two cases of* β_2 *are considered in* [22]*. Case I)* β_2 *contained only the state variables involved in estimating the upper bound of the summation terms by the summation inequality; Case II)* β_2 *contained all the state variables involved in the derivation of the stability criterion. Case II obtained less conservative conditions than Case I at the cost of increasing the amount of solving decision variables.*

- *Lemma 2.3 proposes a suboptimal method for* β_N *and* ω_N*: It not only guarantees the coupling information between the state variables, but also includes the state variables involved in estimating the upper bounds of the single summation term, and avoids the increase of the computational complexity caused by the selection of all state variables.*

Lemma 2.4. [31, 32] *Based on the slope restriction* (2) *for the nonlinearity* $\varphi(\cdot)$*, the upper and lower bounds of the integral term are generated as, for all* $j\in\{1,2,\dots,n\}$*,* $L\le\int_{\sigma_1}^{\sigma_2}\varphi_j(\sigma)d\sigma\le U$*, where* $L=\varphi_j(\sigma_1)(\sigma_2-\sigma_1)+\frac{1}{2k_j^+}[\varphi_j(\sigma_2)-\varphi_j(\sigma_1)]^2$*,* $U=\varphi_j(\sigma_2)(\sigma_2-\sigma_1)-\frac{1}{2k_j^+}[\varphi_j(\sigma_2)-\varphi_j(\sigma_1)]^2$*.*

3. **Main Results.** In order to directly compare the differences between Lemmas 2.2 and 2.3, two different stability criteria will be derived using Lemmas 2.2 and 2.3, respectively. Several different stability criteria will also be proposed for different β_N and ω_N to show the different conservatism of the stability criteria.

3.1. **A novel N-dependent LKF.** The derivation of stability criteria for discrete time-delayed systems based on Lyapunov stability theory requires not only a generalized summation inequality technique, but also an appropriate LKF. Generally speaking, as much information about system states and time delays as possible is included, and the more state information necessary for a summation inequality technique is included, the less conservative the stability criteria obtained are. Therefore, a novel N-dependent LKF based on the summation inequality technique Lemmas 2.1-2.3 is constructed as follows.

$$V_N(k) = \sum_{i=1}^{6} V_{iN}(k), \tag{21}$$

where

$$\begin{aligned}
V_{1N}(k) &= \varrho_N^T(k) P_N \varrho_N(k),\\
V_{2N}(k) &= \sum_{i=k-h_1}^{k-1} \chi_1^T(i) Q_1 \chi_1(i) + \sum_{i=k-h_2}^{k-h_1-1} \chi_1^T(i) Q_2 \chi_1(i),\\
V_{3N}(k) &= \sum_{j=-h_1}^{-1} \sum_{j=k+i}^{k-1} \chi_2^T(i) R_1 \chi_2(i) + \sum_{j=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \chi_2^T(i) R_2 \chi_2(i),\\
V_{4N}(k) &= \sum_{j=-h_1}^{-1} \sum_{j=k+i}^{k-1} \Delta x^T(i) Z_1 \Delta x(i) + \sum_{j=-h_2}^{-h_1-1} \sum_{j=k+i}^{k-1} \Delta x^T(i) Z_2 \Delta x(i),\\
V_{5N}(k) &= 2\sum_{j=1}^{n} \gamma_j^- \int_0^{z_j(k)} \left[f_j(\sigma) - k_j^- \sigma\right] d\sigma + 2\sum_{j=1}^{n} \gamma_j^+ \int_0^{z_j(k)} \left[k_j^+ \sigma - f_j(\sigma)\right] d\sigma\\
&\quad + 2\sum_{j=1}^{n} g_j^- \int_0^{z_j(k+1)} \left[f_j(\sigma) - k_j^- \sigma\right] d\sigma + 2\sum_{j=1}^{n} g_j^+ \int_0^{z_j(k+1)} \left[k_j^+ \sigma - f_j(\sigma)\right] d\sigma,\\
V_{6N}(k) &= 2\sum_{j=1}^{n} s_j^- \int_{z_j(k)}^{z_j(k+1)} \left[f_j(\sigma) - f_j(z_j(k)) - k_j^-(\sigma - z_j(k))\right] d\sigma\\
&\quad + 2\sum_{j=1}^{n} s_j^+ \int_{z_j(k)}^{z_j(k+1)} \left[k_j^+(\sigma - z_j(k)) - (f_j(\sigma) - f_j(z_j(k)))\right] d\sigma,\\
\varrho_N(k) &= \mathrm{col}\{\eta_0(k), \eta_1(k), \ldots, \eta_N(k)\},\\
\chi_1(i) &= \mathrm{col}\{x(i), f(z(i))\},\ \chi_2(i) = \mathrm{col}\{x(i), \Delta x(i)\},\\
\eta_r(k) &= \begin{cases} \mathrm{col}\left\{x(k), \displaystyle\sum_{i=k-h_2}^{k-h_1-1} x(i), z(k), z(k+1), f(z(k)), f(z(k+1))\right\}, & \text{if } r = 0;\\ \mathcal{S}_{r-1}(k-h_1, k) - \begin{pmatrix} h_1 + r - 1 \\ r-1 \end{pmatrix} x(k), & \text{if } 1 \le r \le N. \end{cases}\\
d_1(k) &= 0,\ d_2(k) = h_1,\ d_3(k) = h_k,\ d_4(k) = h_2,\ a_i(k) = x(k - d_i(k)), (i = 1,2,3,4),\\
b_{i,l}(k) &= \frac{l!}{(d_{i+1}(k) - d_i(k) + 1)^{\bar{l}}} \mathcal{S}_{l-1}(k - d_{i+1}(k), k - d_i(k)), (i = 1,2,3),
\end{aligned}$$

$$\xi_N(k) = \text{col}\{a_1(k), \ldots, a_4(k), b_{1,1}(k), b_{1,2}(k), \ldots, b_{1,N}(k), b_{2,1}(k), b_{2,2}(k), \ldots, b_{2,N}(k), b_{3,1}(k), b_{3,2}(k), \ldots, b_{3,N}(k), z(k+1), z(k+2), f(z(k)), f(z(k-h_1)), f(z(k-h_k)), f(z(k-h_2)), f(z(k+1)), f(z(k+2))\}.$$

Remark 3.1. *Inspired by [9, 22], we construct the N-dependent LKF (21). Some improvements are summarized below.*

- *$V_{1N}(k)$ is N-dependent quadratic term and contains all state variables necessary for the summation inequalities (5), (8), (10) and (16), avoiding the conservatism caused by the absence of some state or time delay information.*
- *The double summation term $V_{3N}(k)$ is introduced into the new N-dependent LKF, where the summation inequalities (4), (7), (9) and (15) can be fully used to estimate the upper bounds of the R_1- and R_2-dependent summation terms. Although $V_{3N}(k)$ contains $V_{4N}(k)$, the Z_1- and Z_2-dependent summation terms in $V_4(k)$ are estimated the upper bounds via the summation inequalities (5), (8), (10) and (16).*
- *V_{5N} and V_{6N} with nonlinear single-integrals are introduced into the LKF and contain more nonlinear information than some regular LKF. For the stability analysis of a nonlinear system, the conservatism can be reduced.*
- *Combining the N-dependent LKF (21) with the N-dependent summation inequality Lemmas 2.1-2.3, the stability criteria derived are hierarchical, that is, the higher level of hierarchy, the less conservatism of the stability criteria, which will be proved by the following result.*

Theorem 3.1. *For given h_1 and h_2, neural network (1) with $f(\sigma)$ satisfying (2) is asymptotically stable, if there exist matrices $P_N \in \mathbb{S}_+^{(N+6)n\times(N+6)n}$, $\left(Q_i, R_i \in \mathbb{S}_+^{2n\times 2n}\right)$, $\left(Z_i \in \mathbb{S}_+^{n\times n}\right)$, $(X_j \in \mathbb{S}^{n\times n})$, $\left(\Lambda_i, S_i, G_i, L_t, V_s \in \mathbb{D}_+^{n\times n}\right)$ $(i = 1, 2;\ j = 1, 2, 3;\ s = 1, 2, 3, 4, 5;\ t = 1, 2, 3, 4, 5, 6)$, and any matrices $\left(M, T \in \mathbb{R}^{2Nn\times(2N+3)n}\right)$, $\left(W, H \in \mathbb{R}^{(N+1)n\times(2N+3)n}\right)$ such that the following inequalities hold.*

$$\mathcal{R}_{1N} > 0, \tag{22}$$

$$\mathcal{F}_N(h_1) = \begin{bmatrix} \Pi_N(h_1) & h_{12}\beta_N^T T^T & h_{12}\beta_N^T H^T \\ * & -h_{12}\mathcal{R}_{3N} & 0 \\ * & * & -h_{12}Z_{2N} \end{bmatrix} < 0, \tag{23}$$

$$\mathcal{F}_N(h_2) = \begin{bmatrix} \Pi_N(h_2) & h_{12}\beta_N^T M^T & h_{12}\beta_N^T W^T \\ * & -h_{12}\mathcal{R}_{2N} & 0 \\ * & * & -h_{12}Z_{2N} \end{bmatrix} < 0 \tag{24}$$

with

$$\Pi_N(h_k) = \Pi_{1N} + \text{Sym}\{\Pi_{2N}(h_k) + \Theta_2\} + \Theta_1,$$

$$\begin{aligned}\Pi_{1N} = {} & U_N P_N U_N^T + [e_1\ e_{3N+7}]Q_1[e_1\ e_{3N+7}]^T + [e_2\ e_{3N+8}](Q_2 - Q_1)[e_2\ e_{3N+8}]^T \\ & - [e_4\ e_{3N+10}]Q_2[e_4\ e_{3N+10}]^T + e_1X_1e_1^T + e_2(X_2 - X_1)e_2^T + e_3(X_3 - X_2)e_3^T \\ & - e_4X_3e_4^T + [e_1\ \ e_s](h_1R_1 + h_{12}R_2)[e_1\ \ e_s]^T + e_s(h_1Z_1 + h_{12}Z_2)e_s^T \\ & - \frac{1}{h_1}\psi_{0(N-1)}^T \Xi_{N-1}^T \mathcal{R}_{1N} \Xi_{N-1}\psi_{0(N-1)} - \frac{1}{h_1}\phi_{0N}^T \Xi_N^T Z_{1N} \Xi_N \phi_{0N},\end{aligned}$$

$$\begin{aligned}\Pi_{2N}(h_k) = {} & \Gamma_N(h_k)P_N U_N^T + \beta_N^T \left[M^T\ \ T^T\right] \left[\psi_{1(N-1)}^T \Xi_{N-1}^T\ \ \psi_{2(N-1)}^T \Xi_{N-1}^T\right]^T \\ & + \beta_N^T \left[W^T\ \ H^T\right] \left[\phi_{1N}^T \Xi_N^T\ \ \phi_{2N}^T \Xi_N^T\right]^T,\end{aligned}$$

$$\Theta_1 = e_{3N+5}(\Lambda_2K_2 - \Lambda_1K_1 + G_1K_1 - G_2K_2)e_{3N+5}^T$$

$$
\begin{aligned}
&+ e_1 A_0^T(\mathit{\Lambda}_1 K_1 - \mathit{\Lambda}_2 K_2)A_0 e_1^T e_{3N+6}(G_2K_2 - G_1K_1)e_{3N+6}^T \\
&+ (e_{3N+6} - e_{3N+5})(S_2K_2 - S_1K_1)(e_{3N+6} - e_{3N+5})^T \\
&+ \left(e_{3N+5} - e_1A_0^T\right)(S_1K_1 - S_2K_2)\left(e_{3N+5} - e_1A_0^T\right)^T \\
&- (e_{3N+11} - e_{3N+7})(\Lambda_1 + \Lambda_2 + S_1 + S_2)K_2^{-1}(e_{3N+11} - e_{3N+7})^T \\
&- (e_{3N+12} - e_{3N+11})(G_1 + G_2 + S_1 + S_2)K_2^{-1}(e_{3N+12} - e_{3N+11})^T \\
&+ \mathrm{Sym}\Big\{\left(e_{3N+5} - e_1A_0^T\right)\left[(\Lambda_1 + S_1)e_{3N+11}^T - (\Lambda_2 - S_1 + 2S_2)e_{3N+7}^T\right] \\
&+ (e_{3N+6} - e_{3N+5})\left[(G_1 + S_1)e_{3N+12}^T - (G_2 - S_1)e_{3N+11}^T\right]\Big\} \\
&+ \left[e_{3N+10} - e_{3N+9} - (e_4 - e_3)A_0^TK_1^T\right]V_5\left[K_2A_0(e_4 - e_3)^T - e_{3N+10}^T + e_{3N+9}^T\right],
\end{aligned}
$$

$$
\begin{aligned}
\Theta_2 = &\left(e_{3N+7} - e_1A_0^TK_1^T\right)L_1\left(K_2A_0e_1^T - e_{3N+7}^T\right) + \left(e_{3N+11} - e_{3N+5}K_1^T\right)L_2\left(K_2e_{3N+5}^T\right. \\
&\left.- e_{3N+11}^T\right) + \left(e_{3N+12} - e_{3N+6}K_1^T\right)L_3\left(K_2e_{3N+6}^T - e_{3N+12}^T\right) + \left(e_{3N+8}\right. \\
&\left.- e_2A_0^TK_1^T\right)L_4\left(K_2A_0e_2^T - e_{3N+8}^T\right) + \left(e_{3N+9} - e_3A_0^TK_1^T\right)L_5\left(K_2A_0e_3^T - e_{3N+9}^T\right) \\
&+ \left(e_{3N+10} - e_4A_0^TK_1^T\right)L_6\left(K_2A_0e_4^T - e_{3N+10}^T\right) + \left(e_{3N+11} - e_{3N+7} - e_{3N+5}K_1^T\right. \\
&\left.+ e_1A_0^TK_1^T\right)V_1\left(K_2e_{3N+5}^T - K_2A_0e_1^T - e_{3N+11}^T + e_{3N+7}^T\right) + \left(e_{3N+12} - e_{3N+11}\right. \\
&\left.- e_{3N+6}K_1^T + e_{3N+5}K_1^T\right)V_2\left(K_2e_{3N+6}^T - K_2e_{3N+5}^T - e_{3N+12}^T + e_{3N+11}^T\right) \\
&+ \left[e_{3N+8} - e_{3N+7} - (e_2 - e_1)A_0^TK_1^T\right]V_3\left[K_2A_0(e_2 - e_1)^T - e_{3N+8}^T + e_{3N+7}^T\right] \\
&+ \left[e_{3N+9} - e_{3N+8} - (e_3 - e_2)A_0^TK_1^T\right]V_4\left[K_2A_0(e_3 - e_2)^T - e_{3N+9}^T + e_{3N+8}^T\right] \\
&+ \left[e_{3N+10} - e_{3N+9} - (e_4 - e_3)A_0^TK_1^T\right]V_5\left[K_2A_0(e_4 - e_3)^T - e_{3N+10}^T + e_{3N+9}^T\right],
\end{aligned}
$$

$$
U_N = [\mathcal{U}_0\ \ \mathcal{U}_1\ \ \cdots\ \ \mathcal{U}_N],\ \Gamma_N(h_k) = [\mathcal{G}_0(h_k)\ \ \mathcal{G}_1(h_k)\ \ \cdots\ \ \mathcal{G}_N(h_k)],
$$

$$
\underset{(j=0,1,2)}{\psi_{jN}^T} = \left[\Psi_0^j\ \ \Psi_1^j\ \ \cdots\ \ \Psi_N^j\right],\quad \underset{(j=0,1,2)}{\phi_{jN}^T} = \left[\Phi_{-1}^j\ \ \Phi_0^j\ \ \cdots\ \ \Phi_N^j\right],
$$

$$
\mathcal{U}_r = \begin{cases}
\begin{matrix}[e_s\ \ e_2 - e_4\ \ e_{3N+5} - A_0e_1\ \ e_{3N+6} - e_{3N+5} \\ e_{3N+11} - e_{3N+7}\ \ e_{3N+12} - e_{3N+11}]\end{matrix} & r = 0, \\
e_1 - e_2 & r = 1, \\
\begin{pmatrix} h_1 + r - 1 \\ r - 1 \end{pmatrix}(e_1 - e_{r+3}), & 2 \le r \le N,
\end{cases}
$$

$$
\mathcal{G}_r(h_k) = \begin{cases}
\begin{matrix}[e_1\ \ (h_{k1} + 1)e_{5+N} + (h_{k2} + 1)e_{5+2N} - e_2 - e_3\ \ A_0e_1 \\ e_{3N+5}\ \ e_{3N+7}\ \ e_{3N+11}]\end{matrix} & r = 0, \\
\begin{pmatrix} h_1 + r \\ r \end{pmatrix}e_{4+r} - \begin{pmatrix} h_1 + r - 1 \\ r - 1 \end{pmatrix}e_1, & 1 \le r \le N,
\end{cases}
$$

$$
\Psi_r^0 = \begin{cases}
[e_1\ \ 0] & r = 0, \\
[(h_1 + 1)e_5\ \ e_1 - e_2] & r = 1, \\
\left[\begin{pmatrix} h_1 + r \\ r \end{pmatrix}e_{r+4}\ \ \begin{pmatrix} h_1 + r - 1 \\ r - 1 \end{pmatrix}(e_1 - e_{r+3})\right], & 2 \le r \le N,
\end{cases}
$$

$$
\Phi_r^0 = \begin{cases}
e_1 & r = -1, \\
e_2 & r = 0, \\
\begin{pmatrix} h_1 + r \\ r \end{pmatrix}e_{r+4}, & 1 \le r \le N,
\end{cases}
$$

$$\underset{(i=1,2)}{\Psi_r^i} = \begin{cases} [e_{i+1} \quad -e_{i+1}] & r=0, \\ [(h_{ki}+1)e_{4+iN+r} \quad -e_{i+2}] & r=1, \\ \left[\begin{pmatrix} h_{ki}+r \\ r \end{pmatrix} e_{4+iN+r} \quad -\begin{pmatrix} h_{ki}+r-1 \\ r-1 \end{pmatrix} e_{3+iN+r}\right], & 2 \le r \le N, \end{cases}$$

$$\underset{(i=1,2)}{\Phi_r^i} = \begin{cases} e_{i+1} & r=-1, \\ e_{i+2} & r=0, \\ \begin{pmatrix} h_{ki}+r \\ r \end{pmatrix} e_{4+iN+r}, & 1 \le r \le N, \end{cases}$$

$$\beta_N^T = [e_2 \ \ e_3 \ \ e_4 \ \ e_{4+N+1} \ \ \cdots \ \ e_{4+2N} \ \ e_{4+2N+1} \ \ \cdots \ \ e_{4+3N}],$$

$$\underset{(i=1,2)}{\mathcal{R}_i} = R_i + \begin{bmatrix} 0 & X_i \\ X_i & X_i \end{bmatrix}, \ \mathcal{R}_3 = R_2 + \begin{bmatrix} 0 & X_3 \\ X_3 & X_3 \end{bmatrix},$$

$$\underset{(j=1,2,3)}{\mathcal{R}_{jN}} = \mathrm{diag}\{\mathcal{R}_j, 3\mathcal{R}_j, \ldots, (2N-1)\mathcal{R}_j\}, \ \underset{(i=1,2)}{Z_{iN}} = \mathrm{diag}\{Z_i, 3Z_i, \ldots, (2N+1)Z_i\},$$

$$e_s = e_1(C-I)^T + e_{3N+7}A^T + e_{3N+9}A_d^T = e_1C_0^T + e_{3N+7}A^T + e_{3N+9}A_d^T.$$

Proof: The forward differences of $V(k)$ along the trajectory of (21) are computed as

$$\begin{aligned} \Delta V_{1N}(k) &= \varrho_N^T(k+1)P_N\varrho_N(k+1) - \varrho_N^T(k)P_N\varrho_N(k) \\ &= [\varrho_N(k+1)-\varrho_N(k)]^T P_N[\varrho_N(k+1)-\varrho_N(k)] \\ &\quad + \mathrm{Sym}\left\{[\varrho_N(k+1)-\varrho_N(k)]^T P_N\varrho_N(k)\right\} \\ &= \xi_N^T(k)\left[U_NP_NU_N^T + \mathrm{Sym}\left\{\Gamma_N(h_k)P_NU_N^T\right\}\right]\xi_N(k), \end{aligned} \tag{25}$$

$$\begin{aligned} \Delta V_{2N}(k) &= \chi_1^T(k)Q_1\chi_1(k) + \chi_1^T(k-h_1)(Q_2-Q_1)\chi_1(k-h_1) \\ &\quad - \chi_1^T(k-h_2)Q_2\chi_1(k-h_2) \\ &= \xi_N^T(k)\left([e_1 \ e_{3N+7}]Q_1[e_1 \ e_{3N+7}]^T + [e_2 \ e_{3N+8}](Q_2-Q_1)[e_2 \ e_{3N+8}]^T \right. \\ &\quad \left. - [e_4 \ e_{3N+10}]Q_2[e_4 \ e_{3N+10}]^T\right)\xi_N(k), \end{aligned} \tag{26}$$

$$\begin{aligned} \Delta V_{3N}(k) &= \chi_2^T(k)\left(h_1R_1 + h_{12}R_2\right)\chi_2(k) - \sum_{i=k-h_1}^{k-1}\chi_2^T(i)R_1\chi_2(i) \\ &\quad - \sum_{i=k-h_2}^{k-h_1-1}\chi_2^T(i)R_2\chi_2(i), \end{aligned} \tag{27}$$

$$\begin{aligned} \Delta V_{4N}(k) &= \Delta x^T(k)\left(h_1Z_1 + h_{12}Z_2\right)\Delta x(k) - \sum_{i=k-h_1}^{k-1}\Delta x^T(i)Z_1\Delta x(i) \\ &\quad - \sum_{i=k-h_2}^{k-h_1-1}\Delta x^T(i)Z_2\Delta x(i), \end{aligned} \tag{28}$$

$$\begin{aligned} \Delta V_{5N}(k) &= z^T(k+1)(\Lambda_2K_2 - \Lambda_1K_1 + G_1K_1 - G_2K_2)z(k+1) \\ &\quad + z^T(k)(\Lambda_1K_1 - \Lambda_2K_2)z(k) + z^T(k+2)(G_2K_2 - G_1K_1)z(k+2) \\ &\quad - 2\sum_{j=1}^{n}\left(\gamma_j^+ - \gamma_j^-\right)\int_{z_j(k)}^{z_j(k+1)} f_j(\sigma)d\sigma - 2\sum_{j=1}^{n}\left(g_j^+ - g_j^-\right)\int_{z_j(k+1)}^{z_j(k+2)} f_j(\sigma)d\sigma, \end{aligned} \tag{29}$$

$$\begin{aligned} \Delta V_{6N}(k) &= [z(k+2)-z(k+1)]^T(S_2K_2 - S_1K_1)[z(k+2)-z(k+1)] \\ &\quad + [z(k+1)-z(k)]^T(S_1K_1 - S_2K_2)[z(k+1)-z(k)] \\ &\quad 2[z(k+2)-z(k+1)]^T(S_2-S_1)f(z(k+1)) \end{aligned}$$

$$+ 2[z(k+1) - z(k)]^T(S_1 - S_2)f(z(k))$$
$$- 2\sum_{j=1}^{n}\left(s_j^+ - s_j^-\right)\int_{z_j(k+1)}^{z_j(k+2)} f_j(\sigma)d\sigma + 2\sum_{j=1}^{n}\left(s_j^+ - s_j^-\right)\int_{z_j(k)}^{z_j(k+1)} f_j(\sigma)d\sigma. \quad (30)$$

The following equations are obvious for symmetric matrices X_1, X_2 and X_3.

$$0 = x^T(k)X_1x(k) - x^T(k-h_1)X_1x(k-h_1) - \sum_{i=k-h_1}^{k-1}\left[\Delta x^T(i)X_1\Delta x(i) + 2\Delta x^T(i)X_1x(i)\right], \quad (31)$$
$$0 = x^T(k-h_1)X_2x(k-h_1) - x^T(k-h_k)X_2x(k-h_k) - \sum_{i=k-h_k}^{k-h_1-1}\left[\Delta x^T(i)X_2\Delta x(i) + 2\Delta x^T(i)X_2x(i)\right], \quad (32)$$
$$0 = x^T(k-h_k)X_3x(k-h_k) - x^T(k-h_2)X_3x(k-h_2) - \sum_{i=k-h_2}^{k-h_k-1}\left[\Delta x^T(i)X_3\Delta x(i) + 2\Delta x^T(i)X_3x(i)\right]. \quad (33)$$

It follows that from $\Delta V_{3N}(k)$ and the above zero equations (31)-(33):

$$\Delta V_3(k) = \chi_2^T(k)\left(h_1R_1 + h_{12}R_2\right)\chi_2(k) + \xi_N^T(k)\left[e_1X_1e_1^T + e_2(X_2 - X_1)e_2^T + e_3(X_3 - X_2)e_3^T - e_4X_3e_4\right]\xi_N(k) - \sum_{i=k-h_1}^{k-1}\chi_2^T(i)\mathcal{R}_1\chi_2(i) - \sum_{i=k-h_k}^{k-h_1-1}\chi_2^T(i)\mathcal{R}_2\chi_2(i) - \sum_{i=k-h_2}^{k-h_k-1}\chi_2^T(i)\mathcal{R}_3\chi_2(i). \quad (34)$$

The following $\mathcal{R}_1$- and Z_1-dependent summation inequalities in $\Delta V_{3N}(k)$ and $\Delta V_{4N}(k)$ can be obtained according to (22), Lemma 2.1 and Remark 2.2.

$$-\sum_{i=k-h_1}^{k-1}\chi_2^T(i)\mathcal{R}_1\chi_2(i) \le -\frac{1}{h_1}\xi_N^T(k)\psi_{0N}\Xi_{N-1}^T\mathcal{R}_{1N}\Xi_{N-1}\psi_{0N}^T\xi_N(k), \quad (35)$$
$$-\sum_{i=k-h_1}^{k-1}\Delta x^T(i)Z_1\Delta x(i) \le -\frac{1}{h_1}\xi_N^T(k)\phi_{0N}\Xi_N^TZ_{1N}\Xi_N\phi_{0N}^T\xi_N(k). \quad (36)$$

$\mathcal{R}_2 > 0$ and $\mathcal{R}_3 > 0$ can be obtained from (23) and (24). Thus, the improved GFMSI Lemma 2.3 can be used to estimate the following $\mathcal{R}_2$-, $\mathcal{R}_3$- and Z_2-dependent summation inequalities.

$$-\sum_{i=k-h_k}^{k-h_1-1}\chi_2^T(i)\mathcal{R}_2\chi_2(i) - \sum_{i=k-h_2}^{k-h_k-1}\chi_2^T(i)\mathcal{R}_3\chi_2(i)$$
$$\le \xi_N^T(k)\mathrm{Sym}\left\{\beta_N^T\left[M^T\Xi_{N-1}\ \ T^T\Xi_{N-1}\right]\left[\psi_{1N}\ \psi_{2N}\right]^T\right\}\xi_N(k) + h_{k1}\xi_N^T(k)\beta_N^TM^T\mathcal{R}_{2N}^{-1}M\beta_N\xi_N(k) + h_{k2}\xi_N^T(k)\beta_N^TT^T\mathcal{R}_{3N}^{-1}T\beta_N\xi_N(k), \quad (37)$$

$$-\sum_{i=k-h_1-1}^{k-h_2}\Delta x^T(i)Z_2\Delta x(i)$$
$$= -\sum_{i=k-h_k}^{k-h_1-1}\Delta x^T(i)Z_2\Delta x(i) - \sum_{i=k-h_2}^{k-h_k-1}\Delta x^T(i)Z_2\Delta x(i)$$

$$\le \xi_N^T(k)\mathrm{Sym}\left\{\beta_N^T\left[W^T\Xi_N \;\; H^T\Xi_N\right]\left[\phi_{1N}\;\phi_{2N}\right]^T\right\}\xi_N(k)$$
$$+h_{k1}\xi_N^T(k)\beta_N^TW^TZ_{2N}^{-1}W\beta_N\xi_N(k)+h_{k2}\xi_N^T(k)\beta_N^TH^TZ_{2N}^{-1}H\beta_N\xi_N(k). \quad (38)$$

Using Lemma 2.4, the bounds of the resulting integration terms of $f_j(\sigma)$ in ΔV_{5N} and ΔV_{6N} are obtained as

$$\int_{z_j(k)}^{z_j(k+1)} f_j(\sigma)d\sigma$$
$$\le f_j(z_j(k+1))[z_j(k+1)-z_j(k)]-\frac{1}{2k_j^+}[f_j(z_j(k+1))-f_j(z_j(k))]^2, \quad (39)$$
$$-\int_{z_j(k)}^{z_j(k+1)} f_j(\sigma)d\sigma$$
$$\le -f_j(z_j(k))[z_j(k+1)-z_j(k)]-\frac{1}{2k_j^+}[f_j(z_j(k+1))-f_j(z_j(k))]^2, \quad (40)$$
$$\int_{z_j(k+1)}^{z_j(k+2)} f(\sigma)d\sigma$$
$$\le f_j(z_j(k+2))[z_j(k+2)-z_j(k+1)]-\frac{1}{2k_j^+}[f_j(z_j(k+2))-f_j(z_j(k+1))]^2, \quad (41)$$
$$-\int_{z_j(k+1)}^{z_j(k+2)} f_j(\sigma)d\sigma$$
$$\le -f_j(z_j(k+1))[z_j(k+2)-z_j(k+1)]-\frac{1}{2k_j^+}[f_j(z_j(k+2))-f_j(z_j(k+1))]^2. \quad (42)$$

According to (39)-(42), ΔV_{5N} and ΔV_{6N} have the following upper bound

$$\Delta V_{5N}+\Delta V_{6N}\le \xi_N^T(k)\Theta_1\xi_N(k). \quad (43)$$

And, the following inequalities hold from (2):

$$l_i(s)=2[f(z(s))-K_1z(s)]^TL_i[K_2z(s)-f(z(s))]\ge 0,\quad i\in\{1,2,3,4,5,6\},$$
$$t_j(s_1,s_2)=2[f(z(s_2))-f(z(s_1))-K_1(z(s_2)-z(s_1))]^TV_j$$
$$\times[K_2(z(s_2)-z(s_1))-(f(z(s_2))-f(z(s_1)))]\ge 0,\quad j\in\{1,2,3,4,5\},$$

where $L_i=\mathrm{diag}\{l_{i1},l_{i2},\ldots,l_{im}\}\ge 0$, $V_j=\mathrm{diag}\{v_{j1},v_{j2},\ldots,v_{jm}\}\ge 0$. As a result, the following inequalities hold

$$l_1(k)+l_2(k+1)+l_3(k+2)+t_1(k,k+1)+t_2(k+1,k+2)\ge 0 \quad (44)$$
$$l_4(k-h_1)+l_5(k-h_k)+l_6(k-h_2)+t_3(k,k-h_1)+t_4(k-h_1,k-h_k)$$
$$+t_5(k-h_k,k-h_2)\ge 0. \quad (45)$$

From Equations (25)-(30) and Equations (34)-(45), we have

$$\Delta V_N(k)\le \xi_N^T(k)\left[\Pi_N(h_k)+\beta_N^T\left(h_{k1}M^T\mathcal{R}_{2N}^{-1}M+h_{k2}T^T\mathcal{R}_{3N}^{-1}T^T\right)\beta_N\right]\xi_N(k)$$
$$+\xi_N^T(k)\beta_N^T\left(h_{k1}W^TZ_{2N}^{-1}W+h_{k2}H^TZ_{2N}^{-1}H\right)\beta_N\xi_N(k), \quad (46)$$

which together with Schur complement and (23) and (24) imply that $\Delta V_N(k)<0$. Therefore, by Lyapunov stability theorem, it can be guaranteed that the neural network (1) is asymptotically stable. □

Remark 3.2. *When the nonlinear activation function $f(\sigma)$ satisfies* (2)*, Theorem 3.1 combines LKF* (21) *and Lemmas 2.1-2.4 to propose a sufficient condition for the asymptotic stability of neural network* (1)*. In order to reduce the conservatism of the stability* *criterion, the main contributions are as follows: I) To reduce the conservatism caused by the construction of LKF, six additional nonlinear integral summation terms are introduced into the LKF* (21)*. When estimating the upper bounds of* ΔV_{5N} *and* ΔV_{6N}*, the tight nonlinear integral inequality lemma proposed in* [31, 32] *is utilized; II) When estimating the upper bounds of* $\mathcal{R}_2$*-,* $\mathcal{R}_3$*- and* Z_2*-dependent summation terms in* ΔV_{3N} *and* ΔV_{4N}*, Lemma 2.3 is used to estimate the summation terms as a whole in the delay interval* $[h_1, h_k]$ *and* $[h_k, h_2]$ *instead of Lemma 2.2, which increases the coupling information between state variables; III) As mentioned in* [22]*, the choice of free vector* β_N *has an important influence on the conservatism of the stability criterion. Therefore, the selection principle of* β_N *in Theorem 3.1 is to include the state and time delay information involved in the summation inequality as much as possible, which can ensure the full application of the GFMSI.*

In order to show the difference between Lemma 2.2 and Lemma 2.3, based on Theorem 3.1, Lemma 2.2 is used to give the following corresponding result.

Corollary 3.1. *For given* h_1 *and* h_2*, neural network* (1) *with* $f(\sigma)$ *satisfying* (2) *is asymptotically stable, if there exist matrices* $P_N \in \mathbb{S}_+^{(N+6)n\times(N+6)n}$*,* $\left(Q_i, R_i \in \mathbb{S}_+^{2n\times 2n}\right)$*,* $\left(Z_i \in \mathbb{S}_+^{n\times n}\right)$*,* $(G_j \in \mathbb{S}^{n\times n})$*,* $\left(\Lambda_i, S_i, G_i, L_t, V_s \in \mathbb{D}_+^{n\times n}\right)$ $(i = 1, 2;\ j = 1, 2, 3;\ s = 1, 2, 3, 4, 5;\ t = 1, 2, 3, 4, 5, 6)$*, and any matrices* $\left(\tilde{M}, \tilde{T} \in \mathbb{R}^{Nn\times(N+3)n}\right)$*,* $\left(\tilde{W}, \tilde{H} \in \mathbb{R}^{(N+1)n\times(N+3)n}\right)$ *such that LMI* (22) *and the following inequalities hold.*

$$\tilde{\mathcal{F}}_N(h_1) = \begin{bmatrix} \tilde{\Pi}_N(h_1) & h_{12}\beta_{2N}^T\tilde{T}^T & h_{12}\beta_{2N}^T\tilde{H}^T \\ * & -h_{12}\mathcal{R}_{3N} & 0 \\ * & * & -h_{12}Z_{2N} \end{bmatrix} < 0, \tag{47}$$

$$\tilde{\mathcal{F}}_N(h_2) = \begin{bmatrix} \tilde{\Pi}_N(h_2) & h_{12}\beta_{1N}^T\tilde{M}^T & h_{12}\beta_{2N}^T\tilde{W}^T \\ * & -h_{12}\mathcal{R}_{2N} & 0 \\ * & * & -h_{12}Z_{2N} \end{bmatrix} < 0 \tag{48}$$

with

$$\begin{aligned} \tilde{\Pi}_N(h_k) &= \Pi_{1N} + \mathrm{Sym}\left\{\tilde{\Pi}_{2N}(h_k) + \Theta_2\right\} + \Theta_1, \\ \tilde{\Pi}_{2N}(h_k) &= \Gamma_N(h_k)P_N U_N^T + \beta_{1N}^T\tilde{M}^T\varXi_{N-1}\psi_{1N}^T + \beta_{2N}^T\tilde{T}^T\varXi_{N-1}\psi_{2N}^T \\ &\quad + \beta_{1N}^T\tilde{W}^T\varXi_N\phi_{1N}^T + \beta_{2N}^T\tilde{H}^T\varXi_N\phi_{2N}^T, \\ \beta_{1N}^T &= [e_2\ \ e_3\ \ e_4\ \ e_{4+N+1}\ \ \cdots\ \ e_{4+2N}],\ \beta_{2N}^T = [e_2\ \ e_3\ \ e_4\ \ e_{4+2N+1}\ \ \cdots\ \ e_{4+3N}]. \end{aligned}$$

Here, Π_{1N}*,* $\Gamma_N(h_k)$*,* U_N*,* $\varXi_{N-1}$*,* $\varXi_N$*,* ψ_{iN}*,* ϕ_{iN}*,* $\mathcal{R}_{jN}$*,* Z_{2N}*,* Θ_i $(i = 1, 2;\ j = 2, 3)$ *are given in Theorem 3.1.*

Proof: We use Lemma 2.2 instead of the improved Lemma 2.3 to estimate the following $\mathcal{R}_2$-, $\mathcal{R}_3$- and Z_2-dependent summation inequalities in $\Delta V_{3N}(k)$ and $\Delta V_{4N}(k)$.

$$\begin{aligned} &-\sum_{i=k-h_k}^{k-h_1-1}\chi_2^T(i)\mathcal{R}_2\chi_2(i) - \sum_{i=k-h_2}^{k-h_k-1}\chi_2^T(i)\mathcal{R}_3\chi_2(i) \\ &\le \xi_N^T(k)\Big\{\mathrm{Sym}\left\{\beta_{1N}^T\tilde{M}^T\varXi_{N-1}\psi_{1N}^T + \beta_{2N}^T\tilde{T}^T\varXi_{N-1}\psi_{2N}^T\right\} \\ &\quad + h_{k1}\beta_{1N}^T\tilde{M}^T\mathcal{R}_{2N}^{-1}\tilde{M}\beta_{1N} + h_{k2}\beta_{2N}^T\tilde{T}^T\mathcal{R}_{3N}^{-1}\tilde{T}\beta_{2N}\Big\}\xi_N(k), \end{aligned} \tag{49}$$

$$-\sum_{i=k-h_1-1}^{k-h_2}\Delta x^T(i)Z_2\Delta x(i)$$

$$
\begin{aligned}
&= -\sum_{i=k-h_k}^{k-h_1-1} \Delta x^T(i) Z_2 \Delta x(i) - \sum_{i=k-h_2}^{k-h_k-1} \Delta x^T(i) Z_2 \Delta x(i) \\
&\le \xi_N^T(k) \Big\{ \mathrm{Sym}\left\{ \beta_{1N}^T \tilde{W}^T \varXi_N \phi_{1N}^T + \beta_{2N}^T \tilde{H}^T \varXi_N \phi_{2N}^T \right\} \\
&\quad + h_{k1} \beta_{1N}^T \tilde{W}^T Z_{2N}^{-1} \tilde{W} \beta_{1N} + h_{k2} \beta_{2N}^T \tilde{H}^T Z_{2N}^{-1} \tilde{H} \beta_{2N} \Big\} \xi_N(k). \qquad (50)
\end{aligned}
$$

The proof of the rest of the Corollary 3.1 is similar to Theorem 3.1, which is omitted. □

Remark 3.3. *When the neural activation function $f(\sigma)$ satisfies (3), d_j^- and d_j^+ may be negative, that is, the activation function $f(\sigma)$ could be non-monotonic. In this case, Lemma 2.4 based on monotonically increasing nonlinearity cannot be directly used to estimate the upper bounds of the nonlinear integral terms. Although the positive definiteness of $V_{5N}(k)$ and $V_{6N}(k)$ can still be guaranteed, the upper bounds of the summation terms in $\Delta V_{5N}(k)$ and $\Delta V_{6N}(k)$ will not be estimated. Therefore, Theorem 3.1 will reduce to the following result.*

Corollary 3.2. *For given h_1 and h_2, neural network (1) with $f(\sigma)$ satisfying (3) is asymptotically stable, if there exist matrices $\bar{P}_N \in \mathbb{S}_+^{(N+3)n\times(N+3)n}$, $\left(Q_i, R_i \in \mathbb{S}_+^{2n\times 2n}\right)$, $(Z_i \in \mathbb{S}_+^{n\times n})$, $(G_j \in \mathbb{S}^{n\times n})$, $(L_t, V_s \in \mathbb{D}_+^{n\times n})$ $(i = 1, 2;\ j = 1, 2, 3;\ s = 3, 4, 5;\ t = 1, 4, 5, 6)$, and any matrices $\left(M, T \in \mathbb{R}^{2Nn\times(2N+3)n}\right)$, $\left(W, H \in \mathbb{R}^{(N+1)n\times(2N+3)n}\right)$ such that LMI (22) and the following inequalities hold.*

$$
\bar{\mathcal{F}}_N(h_1) = \begin{bmatrix} \bar{\Pi}_N(h_1) & h_{12}\beta_N^T T^T & h_{12}\beta_N^T H^T \\ * & -h_{12}\mathcal{R}_{3N} & 0 \\ * & * & -h_{12} Z_{2N} \end{bmatrix} < 0, \qquad (51)
$$

$$
\bar{\mathcal{F}}_N(h_2) = \begin{bmatrix} \bar{\Pi}_N(h_2) & h_{12}\beta_N^T M^T & h_{12}\beta_N^T W^T \\ * & -h_{12}\mathcal{R}_{2N} & 0 \\ * & * & -h_{12} Z_{2N} \end{bmatrix} < 0 \qquad (52)
$$

with

$$
\begin{aligned}
&\bar{\Pi}_N(h_k) = \bar{\Pi}_{1N} + \mathrm{Sym}\left\{ \bar{\Pi}_{2N}(h_k) + \bar{\Theta}_2 \right\}, \\
&\bar{\Pi}_{1N} = \bar{U}_N \bar{P}_N \bar{U}_N^T + [\bar{e}_1\ \bar{e}_{3N+5}] Q_1 [\bar{e}_1\ \bar{e}_{3N+5}]^T + [\bar{e}_2\ \bar{e}_{3N+6}](Q_2 - Q_1)[\bar{e}_2\ e_{3N+6}]^T \\
&\qquad - [\bar{e}_4\ \bar{e}_{3N+8}] Q_2 [\bar{e}_4\ \bar{e}_{3N+8}]^T + \bar{e}_1 X_1 \bar{e}_1^T + \bar{e}_2 (X_2 - X_1) \bar{e}_2^T \\
&\qquad + \bar{e}_3 (X_3 - X_2) \bar{e}_3^T - \bar{e}_4 X_3 \bar{e}_4^T + [\bar{e}_1\ \ \bar{e}_s](h_1 R_1 + h_{12} R_2)[\bar{e}_1\ \ \bar{e}_s]^T \\
&\qquad + \bar{e}_s (h_1 Z_1 + h_{12} Z_2) \bar{e}_s^T - \frac{1}{h_1} \psi_{0N}^T \varXi_{N-1}^T \mathcal{R}_{1N} \varXi_{N-1} \psi_{0N} - \frac{1}{h_1} \phi_{0N}^T \varXi_N^T Z_{1N} \varXi_N \phi_{0N}, \\
&\bar{\Pi}_{2N}(h_k) = \bar{\Gamma}_N(h_k) \bar{P}_N \bar{U}_N^T + \beta_N^T \left[M^T\ \ T^T \right] \left[\psi_{1N}^T \varXi_{N-1}^T\ \ \psi_{2N}^T \varXi_{N-1}^T \right]^T \\
&\qquad\qquad + \beta_N^T \left[W^T\ \ H^T \right] \left[\phi_{1N}^T \varXi_N^T\ \ \phi_{2N}^T \varXi_N^T \right]^T, \\
&\bar{U}_N = \left[\bar{\mathcal{U}}_0\ \ \mathcal{U}_1\ \ \cdots\ \ \mathcal{U}_N \right], \ \bar{\Gamma}_N(h_k) = \left[\bar{\mathcal{G}}_0(h_k)\ \ \mathcal{G}_1(h_k)\ \ \cdots\ \ \mathcal{G}_N(h_k) \right], \\
&\bar{\mathcal{U}}_0 = \left[\bar{e}_s\ \ \bar{e}_2 - \bar{e}_4\ \ \bar{e}_s A_0^T \right], \\
&\bar{\mathcal{G}}_0(h_k) = \left[\bar{e}_1\ \ (h_{k1}+1)\bar{e}_{5+N} + (h_{k2}+1)\bar{e}_{5+2N} - \bar{e}_2 - \bar{e}_3\ \ \bar{e}_1 A_0^T \right], \\
&\bar{\Theta}_2 = \left(\bar{e}_{3N+5} - \bar{e}_1 A_0^T K_1^T \right) L_1 \left(K_2 A_0 \bar{e}_1^T - \bar{e}_{3N+5}^T \right) \\
&\qquad + \left(\bar{e}_{3N+6} - \bar{e}_2 A_0^T K_1^T \right) L_4 \left(K_2 A_0 \bar{e}_2^T - \bar{e}_{3N+6}^T \right) \\
&\qquad + \left(\bar{e}_{3N+7} - \bar{e}_3 A_0^T K_1^T \right) L_5 \left(K_2 A_0 \bar{e}_3^T - \bar{e}_{3N+7}^T \right)
\end{aligned}
$$

$$+\left(\bar{e}_{3N+8}-\bar{e}_4A_0^TK_1^T\right)L_6\left(K_2A_0\bar{e}_4^T-\bar{e}_{3N+8}^T\right)$$
$$+\left[\bar{e}_{3N+6}-\bar{e}_{3N+5}-(\bar{e}_2-\bar{e}_1)A_0^TK_1^T\right]V_3\left[K_2A_0(\bar{e}_2-\bar{e}_1)^T-\bar{e}_{3N+6}^T+\bar{e}_{3N+5}^T\right]$$
$$+\left[\bar{e}_{3N+7}-\bar{e}_{3N+6}-(\bar{e}_3-\bar{e}_2)A_0^TK_1^T\right]V_4\left[K_2A_0(\bar{e}_3-\bar{e}_2)^T-\bar{e}_{3N+7}^T+\bar{e}_{3N+6}^T\right]$$
$$+\left[\bar{e}_{3N+8}-\bar{e}_{3N+7}-(\bar{e}_4-\bar{e}_3)A_0^TK_1^T\right]V_5\left[K_2A_0(\bar{e}_4-\bar{e}_3)^T-\bar{e}_{3N+8}^T+\bar{e}_{3N+7}^T\right].$$

Here, β_N, Ξ_{N-1}, Ξ_N, ψ_{iN}, ϕ_{iN}, $\mathcal{R}_{iN}$, Z_{jN}, $\mathcal{U}_s$, $\mathcal{G}_s(h_k)$ $(i=1,2,3;\ j=1,2;\ s=1,2,\ldots,N)$ *are given in Theorem 3.1.*

Proof: We just have to simplify LKF (21) to

$$\bar{V}_N(k)=\bar{\varrho}_N^T(k)P_N\bar{\varrho}_N(k)+\sum_{i=2}^{4}V_{iN}, \tag{53}$$

where $\bar{\varrho}_N(k)=\mathrm{col}\{\bar{\eta}_0(k),\eta_1(k),\ldots,\eta_N(k)\}$, $\bar{\eta}_0(k)=\mathrm{col}\left\{x(k),\sum_{i=k-h_2}^{k-h_1-1}x(i),z(k)\right\}$, $\bar{\xi}_N(k)=\mathrm{col}\{a_1(k),\ldots,a_4(k),b_{1,1}(k),\ldots,b_{1,N}(k),b_{2,1}(k),\ldots,b_{2,N}(k),b_{3,1}(k),\ldots,b_{3,N}(k),f(z(k)),f(z(k-h_1)),f(z(k-h_k)),f(z(k-h_2))\}$ where the dimension of block entry matrices $\bar{e}_i$ $(i=1,2,\ldots,3N+8)$ is determined by $\bar{\xi}_N(k)$. The proof of the rest of the Corollary 3.2 is similar to Theorem 3.1, which is omitted. □

Similar to Corollary 3.1, in order to show the difference between Lemma 2.2 and Lemma 2.3, based on Corollary 3.2, Lemma 2.2 is used to give the following corresponding result.

Corollary 3.3. *For given* h_1 *and* h_2*, neural network* (1) *with* $f(\sigma)$ *satisfying* (3) *is asymptotically stable, if there exist matrices* $\bar{P}_N\in\mathbb{S}_+^{(N+2)n\times(N+2)n}$, $\left(Q_i,R_i\in\mathbb{S}_+^{2n\times2n}\right)$, $\left(Z_i\in\mathbb{S}_+^{n\times n}\right)$, $\left(G_j\in\mathbb{S}^{n\times n}\right)$, $\left(L_t,V_s\in\mathbb{D}_+^{n\times n}\right)$ $(i=1,2;\ j=1,2,3;\ s=3,4,5;\ t=4,5,6)$, *and any matrices* $\left(\tilde{M},\tilde{T}\in\mathbb{R}^{Nn\times(N+3)n}\right)$, $\left(\tilde{W},\tilde{H}\in\mathbb{R}^{(N+1)n\times(N+3)n}\right)$ *such that LMI* (22) *and the following inequalities hold.*

$$\hat{\mathcal{F}}_N(h_1)=\begin{bmatrix}\hat{\Pi}_N(h_1) & h_{12}\beta_{2N}^T\hat{T}^T & h_{12}\beta_{2N}^T\hat{H}^T\\ * & -h_{12}\mathcal{R}_{3N} & 0\\ * & * & -h_{12}Z_{2N}\end{bmatrix}<0, \tag{54}$$

$$\hat{\mathcal{F}}_N(h_2)=\begin{bmatrix}\hat{\Pi}_N(h_2) & h_{12}\beta_{1N}^T\hat{M}^T & h_{12}\beta_{2N}^T\hat{W}^T\\ * & -h_{12}\mathcal{R}_{2N} & 0\\ * & * & -h_{12}Z_{2N}\end{bmatrix}<0 \tag{55}$$

with $\hat{\Pi}_N(h_k)=\bar{\Pi}_{1N}+\mathrm{Sym}\left\{\hat{\Pi}_{2N}(h_k)+\bar{\Theta}_2\right\}$, $\hat{\Pi}_{2N}(h_k)=\bar{\Gamma}_N(h_k)\bar{P}_N\bar{U}_N^T+\beta_{1N}^T\hat{M}^T\Xi_{N-1}\psi_{1N}^T+\beta_{2N}^T\hat{T}^T\Xi_{N-1}\psi_{2N}^T+\beta_{1N}^T\hat{W}^T\Xi_N\phi_{1N}^T+\beta_{2N}^T\hat{H}^T\Xi_N\phi_{2N}^T$. β_{1N}, β_{2N}, ψ_{1N} *and* ψ_{2N} *are given in Corollary 3.1, and* $\bar{\Pi}_{1N}$, $\bar{\Theta}_2$, $\bar{\Gamma}_N(h_k)$, $\bar{U}_N$ *are given in Corollary 3.2, and* Ξ_{N-1}, Ξ_N, $\mathcal{R}_{2N}$, $\mathcal{R}_{3N}$ *and* Z_{2N} *are given in Theorem 3.1.*

Remark 3.4. *Theorem 3.1 and Corollaries 3.1-3.3 respectively derive four* N*-dependent global asymptotic stability criteria for neural network* (1) *based on the LKF* V_N, $\bar{V}_N$ *and summation inequality Lemmas 2.1-2.4. For the nonlinear constraint condition* (2)*, the important factor of Theorem 3.1 and Corollary 3.1 that reduce the conservatism of the stability criteria is the application of* V_{5N}, V_{6N} *and Lemmas 2.2 and 2.3. For the nonlinear constraint condition* (3)*, although the LKF is just an* N*-dependent augment inspired by* [22]*, the application of summation inequalities Lemmas 2.2 and 2.3 is the main factor that reduces the conservatism of the stability criteria.*

Remark 3.5. *When the summation inequalities* (9), (10), (15) *and* (16) *are, respectively, utilized to estimate the* $\mathcal{R}_2$-, $\mathcal{R}_3$- *and* Z_2*-dependent terms summation in the results* *above, β_N, β_{1N} and β_{2N} can be chosen as the whole $\xi_N(k)$, $\bar{\xi}_N(k)$ or other parts of them, respectively. Generally speaking, if more part is contained in β_N, β_{1N} and β_{2N}, more relaxed conditions can be obtained. However, it is obvious that choosing different expressions of β_N, β_{1N} and β_{2N} leads to different complexity in solving LMIs. Indeed, the number of decision variables involved in the above results can be respectively calculated as $(13N^2 + 33.5N + 15.5)n^2 + (26.5 + 0.5N)n$ for Theorem 3.1, $(5N^2 + 25.5N + 15.5)n^2 + (26.5 + 0.5N)n$ for Corollary 3.1, $(13N^2 + 30.5N + 2.5)n^2 + (25.5 + 0.5N)n$ for Corollary 3.2, $(5N^2 + 22.5N + 2.5)n^2 + (25.5 + 0.5N)n$ for Corollary 3.3. The stability criteria of different conservatism will be shown in the numerical examples of Section 4.*

3.2. **Hierarchy of stability criterion.** In the previous subsection, four stability criteria, i.e., Theorem 3.1 and Corollaries 3.1-3.3, are derived in terms of LMIs, which are all N-dependent. In general, the larger N is, the less conservative the stability criteria are. The following result will show that the stability criteria form a hierarchy of LMI conditions. Based on Theorem 3.1 and Corollaries 3.1-3.3, the following theorem is obtained.

Theorem 3.2. *For given h_1 and h_2, neural network* (1) *with the delay h_k satisfying* (2) *is asymptotically stable if any of the following sets is true.*

$$\begin{aligned}\Omega_N(h_1,h_2) = \Big\{ & h_1,h_2,N\in\mathbb{Z}_{\geq 0}:\exists P_N\in\mathbb{S}_+^{(N+6)n\times(N+6)n},\left(Q_i,R_i\in\mathbb{S}_+^{2n\times 2n}\right),\\ & Z_i\in\mathbb{S}_+^{n\times n},G_j\in\mathbb{S}^{n\times n},\left(\Lambda_i,S_i,G_i,L_t,V_s\in\mathbb{D}_+^{n\times n}\right),(i=1,2;j=1,2,3;\\ & s=1,2,3,4,5;t=1,2,3,4,5,6),\left(M,T\in\mathbb{R}^{2Nn\times(2N+3)n}\right),\\ & \left(W,H\in\mathbb{R}^{(N+1)n\times(2N+3)n}\right), s.t.\ \mathcal{R}_{1N}>0,\ \mathcal{F}_N(h_1)<0,\ \mathcal{F}_N(h_2)<0\Big\},\end{aligned} \tag{56}$$

$$\begin{aligned}\tilde{\Omega}_N(h_1,h_2) = \Big\{ & h_1,h_2,N\in\mathbb{Z}_{\geq 0}:\exists P_N\in\mathbb{S}_+^{(N+6)n\times(N+6)n},\left(Q_i,R_i\in\mathbb{S}_+^{2n\times 2n}\right),\\ & Z_i\in\mathbb{S}_+^{n\times n},G_j\in\mathbb{S}^{n\times n},\left(\Lambda_i,S_i,G_i,L_t,V_s\in\mathbb{D}_+^{n\times n}\right),(i=1,2;j=1,2,3;\\ & s=1,2,3,4,5;t=1,2,3,4,5,6),\ \left(\tilde{M},\tilde{T}\in\mathbb{R}^{Nn\times(N+3)n}\right),\\ & \left(\tilde{W},\tilde{H}\in\mathbb{R}^{(N+1)n\times(N+3)n}\right), s.t.\ \mathcal{R}_{1N}>0,\tilde{\mathcal{F}}_N(h_1)<0,\tilde{\mathcal{F}}_N(h_2)<0\Big\},\end{aligned} \tag{57}$$

$$\begin{aligned}\bar{\Omega}_N(h_1,h_2) = \Big\{ & h_1,h_2,N\in\mathbb{Z}_{\geq 0}:\exists P_N\in\mathbb{S}_+^{(N+2)n\times(N+2)n},\left(Q_i,R_i\in\mathbb{S}_+^{2n\times 2n}\right),\\ & Z_i\in\mathbb{S}_+^{n\times n},G_j\in\mathbb{S}^{n\times n},\left(L_t,V_s\in\mathbb{D}_+^{n\times n}\right),(i=1,2;j=1,2,3;s=3,4,5;\\ & t=4,5,6),\left(M,T\in\mathbb{R}^{2Nn\times(2N+3)n}\right),\left(W,H\in\mathbb{R}^{(N+1)n\times(2N+3)n}\right),\\ & s.t.\ \mathcal{R}_{1N}>0,\bar{\mathcal{F}}_N(h_1)<0,\bar{\mathcal{F}}_N(h_2)<0\Big\},\end{aligned} \tag{58}$$

$$\begin{aligned}\hat{\Omega}_N(h_1,h_2) = \Big\{ & h_1,h_2,N\in\mathbb{Z}_{\geq 0}:\exists P_N\in\mathbb{S}_+^{(N+2)n\times(N+2)n},\left(Q_i,R_i\in\mathbb{S}_+^{2n\times 2n}\right),\\ & Z_i\in\mathbb{S}_+^{n\times n},G_j\in\mathbb{S}^{n\times n},\left(L_t,V_s\in\mathbb{D}_+^{n\times n}\right),(i=1,2;j=1,2,3;s=3,4,5;\\ & t=4,5,6),\left(\tilde{M},\tilde{T}\in\mathbb{R}^{Nn\times(N+3)n}\right),\left(\tilde{W},\tilde{H}\in\mathbb{R}^{(N+1)n\times(N+3)n}\right),\\ & s.t.\ \mathcal{R}_{1N}>0,\hat{\mathcal{F}}_N(h_1)<0,\hat{\mathcal{F}}_N(h_2)<0\Big\}.\end{aligned} \tag{59}$$

Then, the following relation is true.

$$\Omega_N(h_1,h_2) \subset \Omega_{N+1}(h_1,h_2), \quad \forall N \in \mathbb{Z}_{\geq 0}, \tag{60}$$
$$\tilde{\Omega}_N(h_1,h_2) \subset \tilde{\Omega}_{N+1}(h_1,h_2), \quad \forall N \in \mathbb{Z}_{\geq 0}, \tag{61}$$
$$\bar{\Omega}_N(h_1,h_2) \subset \bar{\Omega}_{N+1}(h_1,h_2), \quad \forall N \in \mathbb{Z}_{\geq 0}, \tag{62}$$
$$\hat{\Omega}_N(h_1,h_2) \subset \hat{\Omega}_{N+1}(h_1,h_2), \quad \forall N \in \mathbb{Z}_{\geq 0}. \tag{63}$$

Proof: Without loss generality, suppose that $\Omega_N(h_1,h_2)$, $\tilde{\Omega}_N(h_1,h_2)$, $\bar{\Omega}_N(h_1,h_2)$ and $\hat{\Omega}_N(h_1,h_2)$ are not empty. The N-dependent matrices in Theorem 3.1 and Corollaries 3.1-3.3 are variables to be chosen. Without loss of generality, we give the proof of (56). Let us define

$$P_{N+1} = \begin{bmatrix} P_N & 0 \\ 0 & \lambda I \end{bmatrix},\ (N+1) = \begin{bmatrix} M(N) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ T(N+1) = \begin{bmatrix} T(N) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$W(N+1) = \begin{bmatrix} W(N) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ H(N+1) = \begin{bmatrix} H(N) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$
$$\begin{aligned}\xi_{N+1}(k) = \operatorname{col}\{&a_1(k),\ldots,a_4(k),b_{1,1}(k),b_{1,2}(k),\ldots,b_{1,N}(k),b_{2,1}(k),b_{2,2}(k),\ldots,b_{2,N}(k),\\ &b_{3,1}(k),b_{3,2}(k),\ldots,b_{3,N}(k),z(k+1),z(k+2),f(z(k)),f(z(k-h_1)),\\ &f(z(k-h_k)),f(z(k-h_2)),f(z(k+1)),f(z(k+2)),b_{1,N+1}(k),b_{2,N+1}(k),\\ &b_{3,N+1}(k)\}\\ = \operatorname{col}\{&\xi_N(k),\xi_+(k)\},\end{aligned}$$

where $\lambda > 0$ is a sufficiently small scalar to be chosen.

Then, we have the following relationship from (37) and (38):

$$\begin{aligned}\beta_{N+1}^T = \operatorname{col}\big\{&e_2^T\xi_N(k),e_3^T\xi_N(k),e_4^T\xi_N(k),e_{N+5}^T\xi_N(k),\ldots,e_{3N+4}^T\xi_N(k),\tilde{e}_{3N+13}^T\xi_{N+1}(k),\\ &\tilde{e}_{3N+14}^T\xi_{N+1}(k),\tilde{e}_{3N+15}^T\xi_{N+1}(k)\big\}^T,\end{aligned} \tag{64}$$

where the dimensions of e_i and $\tilde{e}_j$ ($i = 1,\ldots,3N+12$; $j = 1,\ldots,3N+15$) correspond to the dimensions of $\xi_N(k)$ and $\xi_{N+1}(k)$, respectively.

Then, the following LKF relationship holds $V_{N+1}(k) = V_N(k) + \lambda\eta_{N+1}^T(k)\eta_{N+1}(k) > 0$. In this situation, similar to the proof of Theorem 3.1, rewrite $\mathcal{F}_N(h_1)$ and $\mathcal{F}_N(h_2)$ as Inequality (46) according to Schur complement lemma.

$$\begin{aligned}\Delta V_N(k) &= \xi_N^T(k)\mathcal{F}_N(h_k)\xi_N(k)\\ &= \xi_N^T(k)\Omega\xi_N(k) + \xi_N^T(k)\bigg\{U_NP_NU_N^T - \frac{1}{h_1}\psi_{0N}^T\Xi_{N-1}^T\mathcal{R}_{1N}\Xi_{N-1}\psi_{0N}\\ &\quad - \frac{1}{h_1}\phi_{0N}^T\Xi_N^TZ_{1N}\Xi_N\phi_{0N} + \operatorname{Sym}\{\Pi_{2N}(h_k)\}\\ &\quad + \beta_N^T\left(h_{k1}M^T(N)\mathcal{R}_{2N}^{-1}M(N) + h_{k2}T^T(N)\mathcal{R}_{3N}^{-1}T^T(N)\right)\beta_N\\ &\quad + \beta_N^T\left(h_{k1}W^T(N)Z_{2N}^{-1}W(N) + h_{k2}H^T(N)Z_{2N}^{-1}H(N)\right)\beta_N\bigg\}\xi_N(k),\end{aligned} \tag{65}$$

where

$$\begin{aligned}\Omega = {}&[e_1\ \ e_{3N+7}]Q_1[e_1\ \ e_{3N+7}]^T + [e_2\ \ e_{3N+8}](Q_2-Q_1)[e_2\ \ e_{3N+8}]^T\\ &- [e_4\ \ e_{3N+10}]Q_2[e_4\ \ e_{3N+10}]^T + e_1X_1e_1^T + e_2(X_2-X_1)e_2^T + e_3(X_3-X_2)e_3^T\\ &- e_4X_3e_4^T + [e_1\ \ e_s](h_1R_1 + h_{12}R_2)[e_1\ \ e_s]^T + e_s(h_1Z_1 + h_{12}Z_2)e_s^T + \Theta_1 + \Theta_2.\end{aligned}$$

Then, according to the proof of Theorem 3.1 and the matrix definitions above, $\mathcal{F}_{N+1}(h_1)$ and $\mathcal{F}_{N+1}(h_2)$ can also be rewritten as

$$
\begin{aligned}
\Delta V_{N+1}(k) &= \xi_{N+1}^T(k)\mathcal{F}_{N+1}(h_k)\xi_{N+1}(k) \\
&= \xi_N^T(k)\Omega\xi_N(k) + \xi_{N+1}^T(k)\bigg\{U_{N+1}\begin{bmatrix} P_N & 0 \\ 0 & \lambda I \end{bmatrix}U_{N+1}^T \\
&\quad + \mathrm{Sym}\left\{\Gamma_{N+1}(h_k)\begin{bmatrix} P_N & 0 \\ 0 & \lambda I \end{bmatrix}U_{N+1}^T\right\}\bigg\}\xi_{N+1}(k) \\
&\quad + \xi_N^T(k)\bigg\{-\frac{1}{h_1}\psi_{0N}^T\Xi_{N-1}^T\mathcal{R}_{1N}\Xi_{N-1}\psi_{0N} - \frac{1}{h_1}\phi_{0N}^T\Xi_N^T Z_{1N}\Xi_N\phi_{0N} \\
&\quad + \mathrm{Sym}\left\{\Pi_{2N}(h_k) - \Gamma_N(h_k)P_N U_N^T\right\} + \beta_N^T\left(h_{k1}M^T(N)\mathcal{R}_{2N}^{-1}M(N)\right. \\
&\quad \left.+ h_{k2}T^T(N)\mathcal{R}_{3N}^{-1}T^T(N)\right)\beta_N + \beta_N^T\left(h_{k1}W^T(N)Z_{2N}^{-1}W(N)\right. \\
&\quad \left.+ h_{k2}H^T(N)Z_{2N}^{-1}H(N)\right)\beta_N\bigg\}\xi_N(k) - \xi_{N+1}^T(k)\bigg\{\frac{2N+1}{h_1}\theta_{N+1}^T\mathcal{R}_1\theta_{N+1} \\
&\quad + \frac{2N+3}{h_1}\tilde{\theta}_{N+1}^T Z_1\tilde{\theta}_{N+1}\bigg\}\xi_{N+1}(k). \qquad (66)
\end{aligned}
$$

Obviously, the following difference holds with

$$
\begin{aligned}
\theta_{N+1} &= \mathrm{col}\left\{-I, \alpha_N^0\rho(k-h_1,k,0), \alpha_N^1\rho(k-h_1,k,1), \ldots, \alpha_N^N\rho(k-h_1,k,N)\right\}^T\psi_{0(N+1)}, \\
\tilde{\theta}_{N+1} &= \mathrm{col}\left\{-I, \alpha_{N+1}^0\rho(k-h_1,k,0), \alpha_{N+1}^1\rho(k-h_1,k,1), \ldots, \alpha_{N+1}^{N+1}\rho(k-h_1,k,\right. \\
&\qquad \left. N+1)\right\}^T\varphi_{0(N+1)}.
\end{aligned}
$$

We can obtain the following equation by $\Delta V_{N+1}(k) - \Delta V_N(k)$:

$$
\begin{aligned}
&\Delta V_{N+1}(k) - \Delta V_N(k) \\
&= \xi_{N+1}^T(k)\bigg\{U_{N+1}\begin{bmatrix} P_N & 0 \\ 0 & \lambda I \end{bmatrix}U_{N+1}^T - U_{N+1}\begin{bmatrix} P_N & 0 \\ 0 & 0 \end{bmatrix}U_{N+1}^T \\
&\quad + \mathrm{Sym}\left\{\Gamma_{N+1}(h_k)\begin{bmatrix} P_N & 0 \\ 0 & \lambda I \end{bmatrix}U_{N+1}^T - \Gamma_{N+1}(h_k)\begin{bmatrix} P_N & 0 \\ 0 & 0 \end{bmatrix}U_{N+1}^T\right\} \\
&\quad - \frac{2N+1}{h_1}\theta_{N+1}^T\mathcal{R}_1\theta_{N+1} + \frac{2N+3}{h_1}\tilde{\theta}_{N+1}^T Z_1\tilde{\theta}_{N+1}\bigg\}\xi_{N+1}(k), \\
&= \xi_+^T(k)\lambda\left(\mathcal{U}_{N+1}\mathcal{U}_{N+1}^T + \mathrm{Sym}\left\{\mathcal{G}_{N+1}(h_k)\mathcal{U}_{N+1}^T\right\}\right)\xi_+(k) \\
&\quad - \xi_{N+1}^T(k)\left\{\frac{2N+1}{h_1}\theta_{N+1}^T\mathcal{R}_1\theta_{N+1} - \frac{2N+3}{h_1}\tilde{\theta}_{N+1}^T Z_1\tilde{\theta}_{N+1}\right\}\xi_{N+1}^T(k). \qquad (67)
\end{aligned}
$$

This equation ensures that the upper bound of $\Delta V_{N+1}(k)$ is represented as follows:

$$
\begin{aligned}
\Delta V_{N+1}(k) &\le \xi_{N+1}^T(k)\mathcal{F}_{N+1}(h_k)\xi_{N+1}(k) \\
&= \xi_N^T(k)\mathcal{F}_N(h_k)\xi_N(k) \\
&\quad - \frac{1}{h_1}\xi_{N+1}^T(k)\left[(2N+1)\theta_{N+1}^T\mathcal{R}_1\theta_{N+1} + (2N+3)\tilde{\theta}_{N+1}^T Z_1\tilde{\theta}_{N+1}\right]\xi_{N+1}(k) \\
&\quad + \lambda\xi_+^T(k)\left(\mathcal{U}_{N+1}\mathcal{U}_{N+1}^T + \mathrm{Sym}\left\{\mathcal{G}_{N+1}(h_k)\mathcal{U}_{N+1}^T\right\}\right)\xi_+(k). \qquad (68)
\end{aligned}
$$

Therefore, for a sufficiently small $\lambda > 0$, $\mathcal{F}_N(h_k) < 0$ also guarantees $\Delta V_{N+1}(k) < 0$, that is, $\mathcal{F}_{N+1}(h_k) < 0$, which together with Schur complement imply that if $\mathcal{F}_N(h_1) < 0$ and $\mathcal{F}_N(h_2) < 0$ are true, $\mathcal{F}_{N+1}(h_1) < 0$ and $\mathcal{F}_{N+1}(h_2) < 0$ are also true.

This completes the proof of (60), and we can also obtain (61)-(63) via a similar process. □

4. **Numerical Examples.** In this section, three widely used numerical examples are taken to show the effectiveness of the proposed method. The numerical example parameters based on the discrete-time neural network system (1) are as follows.

Example 4.1.

$$C = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}, \; A = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.004 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.01 & 0.01 \\ -0.02 & -0.01 \end{bmatrix},$$
$$A_0 = \text{diag}\{1,1\}, \; K_1 = \text{diag}\{0,0\}, \; K_2 = \text{diag}\{1,1\}.$$

Example 4.2.

$$C = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.9 \end{bmatrix}, \; A = \begin{bmatrix} 0.001 & 0 \\ 0 & 0.005 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.1 & 0.01 \\ -0.2 & -0.1 \end{bmatrix},$$
$$A_0 = \text{diag}\{1,1\}, \; K_1 = \text{diag}\{0,0\}, \; K_2 = \text{diag}\{1,1\}.$$

Example 4.3.

$$C = \begin{bmatrix} 0.4 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \; A = \begin{bmatrix} 0.2 & -0.2 & 0.1 \\ 0 & -0.3 & 0.2 \\ -0.2 & -0.1 & -0.2 \end{bmatrix}, \; A_d = \begin{bmatrix} -0.2 & 0.1 & 0 \\ -0.2 & 0.3 & 0.1 \\ 0.1 & -0.2 & 0.3 \end{bmatrix},$$
$$A_0 = \text{diag}\{1,1,1\}, \; K_1 = \text{diag}\{0,-0.4,-0.2\}, \; K_2 = \text{diag}\{0.6,0,0\}.$$

4.1. **Conservativeness comparison.** The nonlinear activation functions of Examples 4.1 and 4.2 satisfy the constraint condition (2), and then Theorem 3.1 and Corollaries 3.1-3.3 can be used to calculate the corresponding MADUB. In Table 1 and Table 2, the MADUBs obtained by Theorem 3.1 and Corollaries 3.1-3.3 are listed and compared with some published results for given different h_1. The nonlinear activation function of Example 4.3 satisfies the constraint condition (3), then Corollaries 3.2 and 3.3 can still calculate the MADUB, but Theorem 3.1 and Corollary 3.1 fail. In Table 3, the MADUBs obtained by Corollaries 3.2 and 3.3 are listed and compared with some published results for given different h_1. The results of the tables are discussed and summarized below. In

TABLE 1. The MADUBs h_2 for different h_1 (Example 4.1)

Methods \ h_1	2	4	6	8	10	20
Co. 3.3 [19]	32	34	36	38	40	52
Th. 1 [20]	99	101	103	105	107	117
Co. 1 [21]	102	104	106	108	110	120
Th. 1 [21]	104	106	108	110	112	122
Th. 1(Case I) [22]	3119	3121	3123	3125	3127	3137
Th. 1(Case II) [22]	3120	3122	3124	3126	3128	3138
Co. 2 [23]	3120	3122	3124	3126	3128	3138
Th. 1 [35]	–	5285	3681	3190	3129	3137
Co. 3.3 ($N = 2$)	5285	5011	3144	3161	3172	3150
Co. 3.1 ($N = 2$)	5285	5157	3154	3165	3183	3166
Co. 3.2 ($N = 2$)	5285	5285	3681	3190	3192	3273
Th. 3.1 ($N = 2$)	5763	5769	3896	3897	3897	3897
Th. 3.1 ($N = 3$)	5776	5779	3915	3919	3923	3925
Th. 3.1 ($N = 4$)	5779	5784	3933	3935	3937	3937
Th. 3.1 ($N = 5$)	5788	5788	3937	3938	3938	3938

Table 2. The MADUBs h_2 for different h_1 (Example 4.2)

Methods \ h_1	4	6	8	10	12	15
Th. 1 [20]	20	20	21	22	21	24
Th. 1 [21]	20	21	21	22	22	24
Th. 1(Case II) [22]	20	21	21	22	23	24
Pro. 1 [36]	21	21	22	–	–	25
Co. 2 [23]	22	22	22	23	24	25
Th. 1 [35]	27	22	22	22	24	24
Co. 3.3 ($N = 2$)	25	22	22	22	24	24
Co. 3.1 ($N = 2$)	26	22	22	22	24	24
Co. 3.2 ($N = 2$)	27	22	22	22	24	24
Th. 3.1 ($N = 2$)	30	35	39	41	41	43
Th. 3.1 ($N = 3$)	31	36	40	42	42	43
Th. 3.1 ($N = 4$)	32	37	41	42	42	44
Th. 3.1 ($N = 5$)	33	38	42	43	44	46

Table 3. The MADUBs h_2 for different h_1 (Example 4.3)

Methods \ h_1	2	4	6	10	15	20
Th. 1 [20]	11	12	14	18	22	27
Th. 1 [37]	18	20	22	26	31	36
Co. 2 [23]	19	21	23	27	32	37
Th. 1 [35]	31	32	33	35	38	42
Co. 3.3 ($N = 2$)	28	29	30	33	36	40
Co. 3.2 ($N = 2$)	31	32	33	35	38	42
Co. 3.2 ($N = 3$)	32	33	34	35	38	42
Co. 3.2 ($N = 4$)	33	34	35	36	39	43
Co. 3.2 ($N = 5$)	34	35	36	37	40	44

tables, Th. and Co. denote Theorem and Corollary, and "–" denotes the corresponding results are not given.

- The MADUBs obtained by the results proposed in this paper are larger than some existing results [19, 20, 21, 22, 23, 35, 36, 37], which indicates that both the augment of an LKF and the improvement of the summation inequality techniques are effective in reducing the conservatism of stability criteria, especially Theorem 3.1 which combines an augmented LKF and improvement inequality techniques. The results of Corollary 3.2 ($N = 2$) are similar to these of [35], because Th. 1 [35] is a special case of $N = 2$ in Corollary 3.2. However, the results increase as N increases, which demonstrates the advantages of the hierarchical structure;
- The MADUBs obtained by Corollary 3.2 are larger than those obtained by Corollary 3.3, showing that the general summation inequality Lemma 2.2 with high degrees of freedom β_N, ω_N, M_1 and M_2 can reduce the conservatism of the stability criteria, which matches the explanation in Remark 2.3;
- Comparing Corollaries 3.1 and 3.3, Theorem 3.1 and Corollary 3.2, the MADUBs obtained by Corollary 3.1 and Theorem 3.1 are respectively larger than those obtained

by Corollaries 3.3 and 3.2. It is obvious that in the case of the same summation inequality technique, the more nonlinear information included in the construction of LKF, the less conservative the stability criteria, that is, the introduction of $V_{5N}(k)$ and $V_{6N}(k)$ can reduce the conservatism of the stability criteria, which matches the explanation in Remark 3.1;

- The MADUBs calculated by Corollary 3.2 are larger than those calculated by Corollary 3.1. Corollary 3.2 is derived without introducing $V_{5N}(k)$ and $V_{6N}(k)$ in LKF, whereas Corollary 3.1 is the opposite. It shows that the main contribution of Corollary 3.1 to reducing the conservatism of the stability criterion is to estimate the upper bounds of the difference summation terms using Lemma 2.3 instead of Lemma 2.2. Moreover, when estimating the upper bounds of the summation terms in the delay intervals $[h_1, h_k]$ and $[h_k, h_2]$, the overall estimation is less conservative than the separate estimation;
- The MADUBs calculated by Theorem 3.1 are the largest in the results proposed in this paper, indicating that in order to reduce the conservatism of the stability criterion, the augment of the LKF and the upper bound estimation technique for the LKF difference need to be considered simultaneously, so as to minimize the conservatism of the stability criterion;
- The hierarchy of the stability criteria proposed in this paper is also verified, that is, the larger N is, the less conservative the stability criteria are, and the larger MADUB is. However, according to the decision variable formulas in Remark 3.5, the decision variables of the corresponding LMIs increase with the increase of N, that is, the complexity of solving LMIs increases with the increase of N. For example, in Theorem 3.1, the decision variables for LMIs (22)-(24) are $86.5n^2 + 27.5n$ when $N = 2$, $137n^2 + 28n$ when $N = 3$, $197.5n^2 + 28.5n$ when $N = 4$, and $268n^2 + 29n$ when $N = 5$.

4.2. **Simulation verification.** To confirm the obtained result from Tables 1-3, the simulation results are shown in Figures 1-3. Obviously, the discrete time-varying delay in Example 4.1 satisfies $20 = h_1 \leq h_k \leq h_2 = 3938$, and the nonlinearity satisfies the constraint condition (1). The discrete time-varying delay in Example 4.2 satisfies $15 = h_1 \leq h_k \leq h_2 = 46$, and the nonlinearity satisfies the constraint condition (1). The discrete time-varying delay in Example 4.3 satisfies $20 = h_1 \leq h_k \leq h_2 = 44$, and the nonlinearity satisfies the constraint condition (2). As you can see from Figures 1-3, the state responses of the discrete-time neural network (1) converge to zero, which verifies the neural network (1) is stable at the equilibrium points under the MADUB given in this paper. Simulation conditions are described as follows, where "INT" denotes round number.

$$\text{Example 4.1: } f(z(k)) = \begin{bmatrix} \tanh(z_1(k)) \\ \tanh(z_2(k)) \end{bmatrix},\ x(0) = \mathrm{col}\{0.7, 0.1\},\ t \in [-3938, 0],$$
$$h_k = \mathrm{INT}\left[\frac{3958}{2} + \frac{3918}{2}\sin\left(\frac{k\pi}{50}\right)\right];$$
$$\text{Example 4.2: } f(z(k)) = \begin{bmatrix} \tanh(z_1(k)) \\ \tanh(z_2(k)) \end{bmatrix},\ x(t) = \mathrm{col}\{0.7, 0.1\},\ t \in [-46, 0],$$
$$h_k = \mathrm{INT}\left[\frac{61}{2} + \frac{31}{2}\sin\left(\frac{k\pi}{50}\right)\right];$$

Example 4.3: $f(z(k)) = \begin{bmatrix} 0.6\tanh(z_1(k)) \\ -0.4\tanh(z_2(k)) \\ -0.2\tanh(z_2(k)) \end{bmatrix}$, $x(t) = \text{col}\{0.05, 0.01, -0.05\}$,

$$t \in [-44, 0],\ h_k = \text{INT}\left[\frac{64}{2} + \frac{24}{2}\sin\left(\frac{k\pi}{50}\right)\right].$$

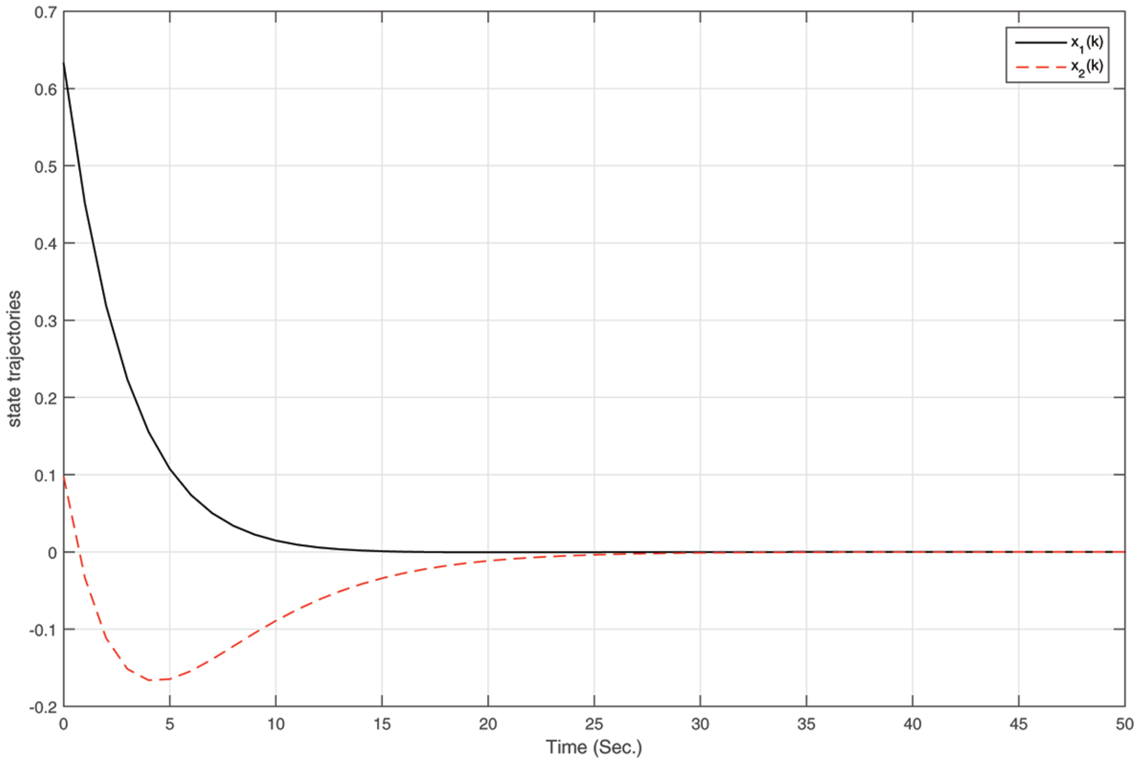

FIGURE 1. The state response under the conditions given in Example 4.1

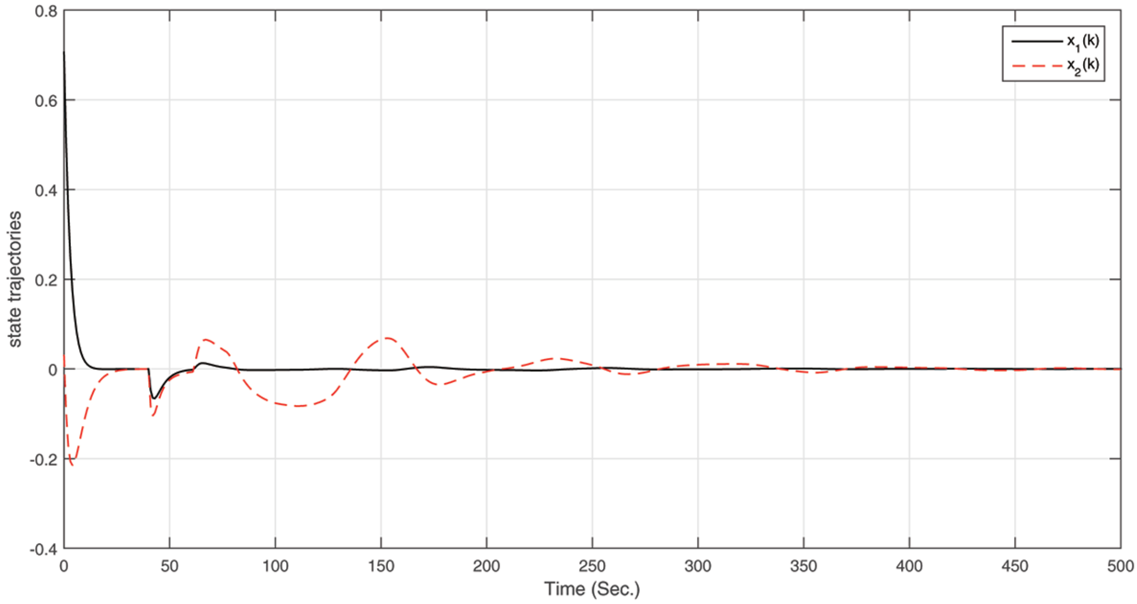

FIGURE 2. The state response under the conditions given in Example 4.2

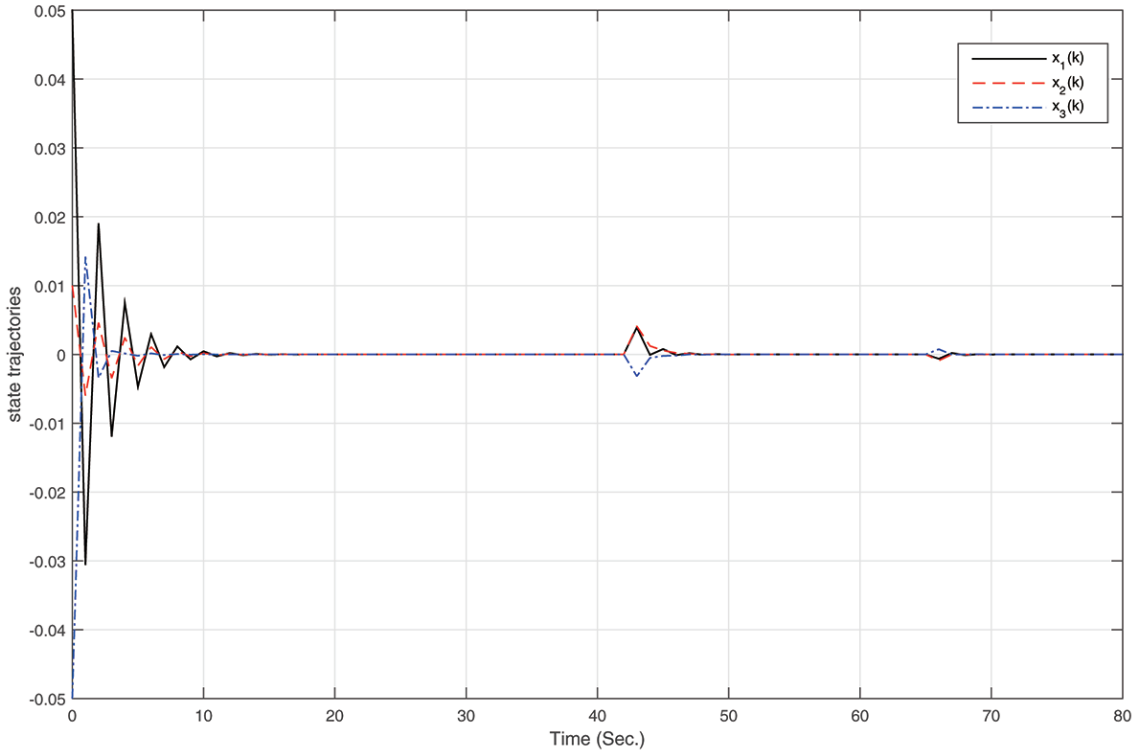

Figure 3. The state response under the conditions given in Example 4.3

5. **Conclusion.** In this paper, the stability of discrete-time neural networks with an interval-like time-varying delay is further studied. A novel N-dependent LKF is constructed in which some additional summation terms contain additional nonlinear integral terms. According to the characteristics of nonlinear constraints, the nonlinear integral terms are introduced into the LKF, so that the whole LKF contains not only the coupling information between the delay intervals and the state variables, but also the coupling information between the nonlinear terms and the delays and other state variables. An N-dependent general free-weight-matrix summation inequality is proposed to involve coupling information on additional state variables by additional free matrices and free vectors, which can relax the derived stability conditions. Combining the N-dependent LKF with the N-dependent general free-weight-matrix summation inequality, the stability criteria derived are hierarchical, that is, the higher level of hierarchy, the less conservatism of the stability criteria. Finally, the effectiveness of the proposed method is illustrated by comparison and discussion in numerical examples.

The conservativeness of our criteria is reduced at the cost of increasing decision variables compared with those in the literature. The main reasons include augmented vectors in the LKF and the application of the general free-matrix summation inequalities. Developing tighter summation inequality techniques without adding additional free-weight matrices, or improving the LKF by making full use of the time-delay information without increasing the dimension of the quadratic term vector will be always one of our team's future research topics.

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REFERENCES

[1] K. Agrawal, R. Negi and V. Pal, An anti-windup control strategy for discrete time-delay system based on triple Lyapunov functional, *Transactions of the Institute of Measurement and Control*, vol.45, DOI: 10.1177/01423312221137621, 2023.

[2] X. Zhang, Q. Han, X. Ge and C. Peng, Stability analysis of delayed discrete-time systems based on a delay-square-dependent Lyapunov functional, *Automatica*, vol.147, 110592, 2023.

[3] B. Ganesan and M. Annamalai, Anti-synchronization analysis of chaotic neural networks using delay product type looped-Lyapunov functional, *Chaos, Solitons & Fractals*, vol.174, 113898, 2023.

[4] Y. Sun, Y. Bo, H. Xu, H. Yang and W. Duan, Enhanced robust stability for uncertain Lur'e systems with time-varying delays, *International Journal of Innovative Computing, Information and Control*, vol.19, no.4, pp.1297-1310, 2023.

[5] A. Komal, N. Richa, P. Chandra and S. Nehal, Delay partitioning approach to the delay-dependent stability of discrete-time systems with anti-windup, *Smart Science*, vol.11, pp.1-22, 2023.

[6] X. Peng and Y. He, Consensus of multiagent systems with time-varying delays and switching topologies based on delay-product-type functionals, *IEEE Transactions on Cybernetics*, vol.54, pp.101-110, 2024.

[7] N. Padmaja and P. Balasubramaniam, Design of H_∞ passive state feedback control for delayed fractional order gene regulatory networks via new improved integral inequalities, *Communications in Nonlinear Science and Numerical Simulation*, vol.111, 106507, 2022.

[8] A. Seuret, F. Gouaisbaut and E. Fridman, Stability of discrete-time systems with time-varying delays via a novel summation inequality, *IEEE Transactions on Automatic and Control*, vol.60, pp.2740-2745, 2015.

[9] S. Lee, J. Park and P. Park, Bessel summation inequalities for stability analysis of discrete-time systems with time-varying delays, *International Journal of Robust and Nonlinear Control*, vol.29, pp.473-491, 2019.

[10] J. Thipcha, P. Tangsiridamrong, T. Botmart, B. Meesuptong, M. Ali, P. Srisilp and K. Mukdasai, Robust stability and passivity analysis for discrete-time neural networks with mixed time-varying delays via a new summation inequality, *AIMS Mathematics*, vol.8, pp.4973-5006, 2023.

[11] P. Nam, H. Trinh and P. Pathirana, Discrete inequalities based on multiple auxiliary functions and their applications to stability analysis of time-delay systems, *Journal of the Franklin Institute*, vol.352, pp.5810-5831, 2015.

[12] G. Soundararajan and G. Nagamani, Non-fragile output-feedback synchronization for delayed discrete-time complex-valued neural networks with randomly occurring uncertainties, *Neural Networks*, vol.159, pp.70-83, 2023.

[13] C. Zhang, W. Chen, C. Zhu, Y. He and M. Wu, Stability analysis of discrete-time systems with time-varying delay via a delay-dependent matrix-separation-based inequality, *Automatica*, vol.156, 111192, 2023.

[14] X. Zhang, Q. Han and X. Ge, An overview of neuronal state estimation of neural networks with time-varying delays, *Information Sciences*, vol.478, pp.83-99, 2019.

[15] G. Soundararajan, G. Nagamani and A. Kashkynbayev, Exponential H_∞ filtering for complex-valued uncertain discrete-time neural networks with time-varying delays, *Communications in Nonlinear Science and Numerical Simulation*, vol.128, 107595, 2024.

[16] B. Adhira and G. Nagamani, Exponential H_∞ filtering for complex-valued uncertain discrete-time neural networks with time-varying delays, *Chaos, Solitons & Fractals*, vol.176, 114162, 2023.

[17] N. Rong, Y. Jing, S. Ding and X. Xie, Event-triggered synchronization for discrete-time delayed neural networks via aperiodic detection, *Expert Systems with Applications*, vol.244, 123048, 2024.

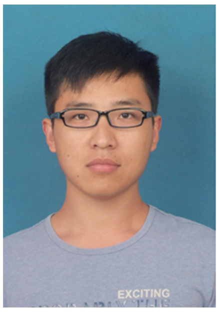
[18] X. Liao, K. Wong and C. Li, Global exponential stability for a class of generalized neural networks with distributed delays, *Nonlinear Analysis: Real World Applications*, vol.5, pp.527-547, 2004.

[19] L. Banu and P. Balasubramaniam, Robust stability analysis for discrete-time neural networks with time-varying leakage delays and random parameter uncertainties, *Neurocomputing*, vol.179, pp.126-134, 2016.

[20] C. Zhang, Y. He, L. Jiang, Q. Wang and M. Wu, Stability analysis of discrete-time neural networks with time-varying delay via an extended reciprocally convex matrix inequality, *IEEE Transactions on Cybernetics*, vol.47, pp.3040-3049, 2017.

[21] J. Chen, J. Park and S. Xu, Improved stability criteria for discrete-time delayed neural networks via novel Lyapunov-Krasovskii functionals, *IEEE Transactions on Cybernetics*, vol.52, pp.11885-11892, 2022.
[22] J. Chen, J. H. Park and S. Xu, Stability analysis of discrete-time neural networks with an interval-like time-varying delay, *Neurocomputing*, vol.329, pp.248-254, 2019.
[23] K. Xie, W. Chen, L. Jin, C. Zhang and Y. He, Novel stability criteria for discrete-time delayed neural networks via extended negative-definiteness approaches of matrix-valued quadratic function, *IET Control Theory & Applications*, vol.17, pp.580-590, 2023.
[24] P. Nam, P. Pathirana and H. Trinh, Discrete Wirtinger-based inequality and its application, *Journal of the Franklin Institute*, vol.352, pp.1893-1905, 2015.
[25] C. Hua, S. Wu and X. Guan, New robust stability condition for discrete-time recurrent neural networks with time-varying delays and nonlinear perturbations, *Neurocomputing*, vol.219, pp.203-209, 2017.
[26] D. Liao, S. Zhong, J. Chen, J. Luo, X. Zhang and Q. Zhong, New stability criteria of discrete systems with time-varying delays, *IEEE Access*, vol.7, pp.1677-1684, 2019.
[27] S. Qiu, X. Liu, F. Wang and Q. Chen, Stability and passivity analysis of discrete-time linear systems with time-varying delay, *Systems & Control Letters*, vol.134, 104543, 2019.
[28] S. Lee, W. Lee and P. Park, Polynomials-based summation inequalities and their applications to discrete-time systems with time-varying delays, *International Journal of Robust and Nonlinear Control*, vol.27, pp.3604-3619, 2017.
[29] L. Zhu and C. Zhu, Enhanced stability criteria for discrete-time systems with time-varying delay, *International Journal of Control, Automation and Systems*, vol.19, pp.2385-2394, 2021.
[30] X. Zhang, Q. Han, X. Ge and B. Zhang, Delay-variation-dependent criteria on extended dissipativity for discrete-time neural networks with time-varying delay, *IEEE Transactions on Neural Networks and Learning Systems*, vol.34, pp.1578-1587, 2023.
[31] B. Park, P. Park and N. Kwon, An improved stability criterion for discrete-time Lur'e systems with sector- and slope-restrictions, *Automatica*, vol.51, pp.255-258, 2015.
[32] J. Park, S. Lee and P. Park, A less conservative stability criterion for discrete-time Lur'e systems with sector and slope restrictions, *IEEE Transactions on Automatic Control*, vol.64, pp.4391-4395, 2019.
[33] Y. Liu, Z. Wang and X. Liu, Global exponential stability of generalized recurrent neural networks with discrete and distributed delays, *Neural Networks*, vol.19, pp.667-675, 2006.
[34] D. Liao, S. Zhong, J. Chen, C. Zhao, X. Zhang and Y. Yu, A new result on stability analysis for discrete system with interval time-varying delays, *Advances in Difference Equations*, vol.123, pp.1-12, 2019.
[35] X. Bo, Y. Sun and W. Duan, Improved delay-dependent stability conditions for discrete-time neural networks with time-varying delays, *International Journal of Dynamics and Control*, vol.12, pp.1034-1043, 2024.
[36] X. Zhang, Q. Han, X. Ge and B. Zhang, Delay-variation-dependent criteria on extended dissipativity for discrete-time neural networks with time-varying delay, *IEEE Transactions on Neural Networks and Learning Systems*, vol.34, pp.1578-1587, 2023.
[37] L. Jin, Y. He and M. Wu, Improved delay-dependent stability analysis of discrete-time neural networks with time-varying delay, *Journal of Franklin Institute*, vol.354, pp.1922-1936, 2017.

Author Biography



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