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# A VIKOR-BASED MULTI-CRITERIA GROUP DECISION-MAKING METHOD USING HESITANT FUZZY $\beta$-COVERING ROUGH SET

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Abstract. *Hesitant fuzzy sets (HFSs) have demonstrated significant strength and efficiency in handling uncertainty and ambiguity in multi-criteria group decision-making (MCGDM). While numerous rough set (RS)-based decision methods have been developed for HFSs, few studies have explored hesitant fuzzy $\beta$-covering. To address this gap, this paper proposes a novel MCGDM method based on a hesitant fuzzy $\beta$-covering rough set (CHFRS). First, we introduce a CHFRS derived from hesitant fuzzy $\beta$-neighborhoods, investigate the properties and relationships among approximation operations, and further provide an axiomatic characterization of the approximation operators. Second, we define a consensus measure and develop an expert weight determination method to facilitate group preference aggregation. Building on these foundations, we integrate the VIKOR (VIseKriterijumska Optimizacija I Kompromisno Resenje) method with CHFRS to construct a VIKOR-CHFRS approach for MCGDM, specifically applied to research proposal evaluation. Finally, a numerical example with comparative analysis is presented to validate the effectiveness and rationality of the proposed VIKOR-CHFRS method.*
**Keywords:** Multi-criteria group decision-making, Hesitant fuzzy rough set, Hesitant fuzzy $\beta$-covering rough set, VIKOR

1. **Introduction.** Multi-criteria group decision-making (MCGDM) is a fundamental and widely applied process that involves ranking alternatives or identifying optimal solutions based on evaluations from multiple experts [1, 2, 3, 4]. The conventional MCGDM framework typically comprises three key stages: (1) collecting expert evaluations, (2) determining criterion weights, and (3) aggregating evaluations to form collective opinions for alternative ranking. In practical applications, decision-makers frequently encounter conflicting criteria, such as the trade-off between delivery speed and cost when selecting logistics providers. These inherent conflicts often preclude the existence of solutions that simultaneously optimize all criteria, thereby significantly complicating the decision-making

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process. To address this challenge, Yu [5] introduced the pioneering concept of compromise solutions and developed corresponding solution methodologies. This foundational work has spurred extensive research on compromise programming-based decision-making approaches. Among these developments, the VIKOR method [6, 7] has emerged as particularly prominent. VIKOR's distinctive strength lies in its ability to derive compromise solutions that balance maximum group utility against minimum individual regret. The method's effectiveness in handling conflicting and non-commensurable criteria has led to its adaptation across various uncertain environments, including Spherical fuzzy VIKOR [8, 9], Granular Z-VIKOR [10], Interval type-2 fuzzy VIKOR [11, 12], Probabilistic linguistic q-rung orthopair fuzzy VIKOR [13], Probabilistic linguistic term VIKOR [14], Stochastic VIKOR [15], Rough VIKOR [16]. These VIKOR extensions have demonstrated substantial practical value, finding successful applications across numerous domains where conflicting criteria must be reconciled.

Although significant theoretical and practical advances have been made in VIKOR-based decision-making methods, existing researches remain insufficient in handling multiple sources of uncertainty [17, 18]. This limitation arises from two key challenges. First, decision-makers often lack full confidence in expressing preferences using a single value. Due to inherent hesitation, they may consider multiple plausible evaluations more reflective of their uncertain judgment. Consequently, representing uncertain decision-making through a set of possible values provides a more natural and realistic approach. Second, in complex decision-making scenarios involving multiple experts, achieving unanimous agreement on evaluations is often unattainable. Divergent opinions among decision-makers lead to a range of plausible assessments, making it reasonable and necessary to incorporate multiple possible values to derive a robust final decision. This highlights the need for enhanced VIKOR-based methods capable of effectively capturing and processing such multi-faceted uncertainty in group decision-making contexts.

With the rapid development of hesitant fuzzy sets (HFSs) [19, 20, 21], they have gained significant attention and been widely applied in the process of MCGDM problems [22, 23, 24]. Consequently, various VIKOR methods have been extended to hesitant fuzzy environments, yielding extensive research outcomes. Particularly, Xu and Zhang [24] reviewed MCGDM methods utilizing HFSs theory recently. Liao and Xu [25] considered the service quality evaluation of the airlines using the hesitant fuzzy VIKOR method. Zhang and Wei [26] applied hesitant fuzzy VIKOR method to project evaluation. Dong et al. [27] gave a new order relationship of the linguistic hesitant fuzzy set by considering the weights of membership degrees, further defined some distance measures, and then extended VIKOR method. By defining the new score function and distance measure of dual hesitant fuzzy sets, Ren et al. [28] proposed a dual hesitant fuzzy VIKOR method to solve MCGDM problems. Zhang et al. [29] proposed a VIKOR method based on regret theory for the multi-attribute decision-making problem with completely unknown weight information under Pythagorean hesitate fuzzy environment. Çolak and Kaya [30] used hesitant fuzzy analytic hierarchy process (AHP) to determine the criteria weights, and then integrated VIKOR methods to rank energy storage technologies for Turkey. Wu et al. [31] defined a distance measure for hesitant fuzzy linguistic term sets to compute the consensus degrees, then determined the expert importance weights, and further gave the MCGDM method by combining with VIKOR. Throughout the previous reviews, the VIKOR methods and the variants under the hesitant fuzzy environment have made the way for solving uncertain MCGDM problems.

In general, MCGDM is to fuse the individual preference of different experts into a collective one, by which the ranking can be obtained. Therefore, two key issues should be solved during the process of information fusion: (1) How to use VIKOR methods to integrate all the hesitant fuzzy information on each criterion; (2) How to draw the consistent conclusion among different experts.

About the information fusion problems, many improved VIKOR methods using different techniques have been reported as stated in [24, 25, 26, 27, 28, 29, 30, 31]. However, these methods also suffer from the information loss and ineffective use of information. For example, the group utility measure, individual regret measure and compromise measure computed by a hesitant fuzzy Manhattan Lp-metric are converted into real values, which inevitably deviate from the true perception of decision-makers. To overcome these deficiencies, many great attempts have been made by using rough set theory. Rough set [32], characterized by the lower approximation and upper approximation of a set, is an effective mathematical tool for dealing with various vagueness caused by the granulation description of the universe. Some authors [33, 34, 35] suggested that rough set can be used to improve the information utilization and reduce the information loss with the help of lower approximation and upper approximation. Moreover, many combinations of hesitant fuzzy sets and rough sets have been researched to solve MCGDM [36, 37, 38, 39]. However, these methods may face limitations in specialized decision-making contexts like research proposals evaluation, and postgraduate admission. Usually, these decision-making problems should meet the following requirements: (1) The same criteria should be given the same scoring standards; (2) The decision-makers often predefine the lowest score of alternatives, only if it is superior to the minimum threshold, then this alternative can be possibly further evaluated. Considering these situations, a minimum threshold value $\beta$ for some criteria is necessary. Therefore, to extend the applications of rough sets, Ma [40] proposed the concept of fuzzy $\beta$-covering and defined different fuzzy rough set models. Based on the researches, Yang and Hu [41] further constructed three kinds of fuzzy $\beta$-coverings based fuzzy rough set models. After that, Zhang et al. [42] discussed four kinds of fuzzy $\beta$-coverings based fuzzy rough set models with the help of fuzzy implication operator and a triangular norm. Niu et al. [43] generalized the couple approximate operators of Ma [40] to the information system and proposed a new binary model-dyad fuzzy $\beta$-covering rough set model to integrate the multi-level fuzzy information. Yang and Atef [44] introduced the fuzzy $\beta$-minimal and $\beta$-maximal descriptions over two universes and then gave four types of fuzzy $\beta$-covering-based rough set models. Shi et al. [45] introduced two novel fuzzy $\beta$-covering rough sets that meet the inclusion property. Li et al. [46] proposed several intuitionistic fuzzy $\beta$-covering rough set models and designed an intuitionistic fuzzy VIKOR decision-making method. Li et al. [47] proposed the concept of dual hesitant fuzzy $\beta$-covering and established several dual hesitant fuzzy covering-based rough set models. Although there are some discussions on the combinations of hesitant fuzzy sets and fuzzy $\beta$-covering rough sets for multi-criteria group decision-making, they lack operability and have deficiencies such as high computations because of $\beta$ taking hesitant fuzzy value or dual hesitant fuzzy value, which is not suitable for practical applications. Motivated by this idea, this paper attempts to propose a novel hesitant fuzzy $\beta$-coverings rough set model to improve the information utilization and reduce the computations.

At present, many studies on consistency in hesitant fuzzy group decision-making mainly focus on the consensus measure of hesitant fuzzy preference relation, and have made many useful results to MCGDM [22, 23, 48, 49]. However, these studies either assume equal expert weights or subjectively predefine them, resulting in unreasonable weight assignments that may deviate from real-world scenarios. Due to differences in knowledge levels and information availability, the same expert may hold varying degrees of importance across different decision-making problems. Thus, during the aggregation of individual preferences into a group decision, expert weights should be adjusted accordingly. Wu et al. [31]

argued that weights ought to be appropriately reduced when experts exhibit uncooperative behavior during the decision-making process. Therefore, the consistency measure of individual preferences offers a viable solution for determining expert weights rationally.

From the above discussions, it is clear that the current MCGDM techniques with VIKOR need to be improved. The contributions of our studies lie in the following four points.

(1) We establish a hesitant fuzzy $\beta$-coverings rough sets model (CHFRS), then probe the properties and explore the axiomatic characterization of two approximation operators.

(2) We extend VIKOR method for MCGDM by combining with CHFRS model (denoted by VIKOR-CHFRS). Firstly, the real values on group utility measure, individual regret measure and compromise measure are replaced by hesitant fuzzy numbers, which can avoid information loss during computation; Secondly, by synthesizing the lower approximation and upper approximation can help to improve the information utilization.

(3) We give the concept of consensus degree and further give a weight determination method of decision-makers, which can help to reduce the negative effect of the inconsistency on decision-making in the group aggregation so as to improve the decision quality.

(4) An application example demonstrates that the proposed VIKOR-CHFRS method is feasible and effective, and a detailed analysis and comparison highlights the advantages and characteristics of VIKOR-CHFRS method.

The rest of this article is displayed as follows. Section 2 surveys some basic concepts related to HFS, and further HF $\beta$-covering and HF $\beta$-neighborhood are proposed; In Section 3, we introduce CHFRS model and investigate the axiomatic systems. Based on the theoretical analysis of CHFRS model, we give a novel MCGDM method (VIKOR-CHFRS) by using VIKOR in Section 4. The research proposals evaluation problems are discussed to demonstrate the applications of VIKOR-CHFRS in Sections 5 and 6. Finally, some conclusions are given and potential directions are further presented in Section 7.

## 2. **Basic Knowledge.**

### 2.1. **Hesitant fuzzy set.**

In this section, we will mainly revisit the HFS and some operations. Among them, $U$ represents a nonempty finite set, and $P(U)$ represents the power set.

**Definition 2.1.** [17] *Let $U$ be a nonempty finite set, an HFS $\boldsymbol{A}$ on $U$ is a function $h_{\boldsymbol{A}}(a)$ with the following form:*

$$\boldsymbol{A} = \left\{ \langle a, h_{\boldsymbol{A}}(a) \rangle \,| a \in U \right\},$$

*where, $h_{\boldsymbol{A}}(a)$ is a subset of interval $[0,1]$, describing the possible membership degrees of $a \in U$ to $\boldsymbol{A}$. Usually, $h_{\boldsymbol{A}}(a)$ is named a hesitant fuzzy element (HFE). Following will use $HF(U)$ to represent the set of all HFSs on $U$.*

*For all $x \in U$, $\boldsymbol{A}$ is an empty HFS iff $h_{\boldsymbol{A}}(x) = \{0\}$, written as $\phi$; $\boldsymbol{A}$ is a full HFS iff $h_{\boldsymbol{A}}(x) = \{1\}$, written as $U$; $\boldsymbol{A}$ is a constant HFS iff $h_{\boldsymbol{A}}(x) = \{a_1 \cdots a_m\}$, and $a_i \in [0,1]$, $i = 1, \ldots, m$, written as $\{a_1 \cdots a_m\}$.*

*Because the length of HFE may vary, Xia et al. [22] gave the following assumptions to avoid mistakes in operation. Suppose $l(h_{\boldsymbol{A}}(a))$ represent the length of $h_{\boldsymbol{A}}(a)$.*

*(1) All values of $h_{\boldsymbol{A}}(a)$ are placed in growing order, $h_{\boldsymbol{A}}^{\sigma(k)}(a)$ is considered as the $k$th biggest element in $h_{\boldsymbol{A}}(a)$.*

*(2) Set $l = \max\left(l(h_{\boldsymbol{A}}(a)), l(h_{\boldsymbol{B}}(a))\right)$ if $l(h_{\boldsymbol{A}}(a)) \neq l(h_{\boldsymbol{B}}(a))$. The two HFEs $h_{\boldsymbol{A}}(a)$, $h_{\boldsymbol{B}}(a)$ should have the same length when comparing. The HFE with shorter length should extend its value by adding its maximal or minimal element until they possess the same length.*

**Definition 2.2.** [22, 24] *Let $h$, $h_1$, $h_2$ be three HFEs, $0 < \mu \leq 1$ is a real number, and then we have*

*(1)* $h^c = \bigcup_{k=1}^{l} \left\{1 - h^{\sigma(k)}\right\}$, *(2)* $h_1 \underline{\vee} h_2 = \bigcup_{k=1}^{l} \left\{h_1^{\sigma(k)} \vee h_2^{\sigma(k)}\right\}$,

*(3)* $h_1 \bar{\wedge} h_2 = \bigcup_{k=1}^{l} \left\{h_1^{\sigma(k)} \wedge h_2^{\sigma(k)}\right\}$, *(4)* $\mu h = \bigcup_{k=1}^{l} \left\{\mu h^{\sigma(k)}\right\}$,

*(5)* $h_1 \oplus h_2 = \bigcup_{k=1}^{l} \left\{h_1^{\sigma(k)} + h_2^{\sigma(k)}\right\}$, *(6)* $h_1 \ominus h_2 = \bigcup_{k=1}^{l} \left\{h_1^{\sigma(k)} - h_2^{\sigma(k)}\right\}$,

*(7)* $h_1 \otimes h_2 = \bigcup_{k=1}^{l} \left\{h_1^{\sigma(k)} \cdot h_2^{\sigma(k)}\right\}$, *(8)* $h_1 \oslash h_2 = \bigcup_{k=1}^{l} \left\{\bar{h}^{\sigma(k)}\right\}$,

*where,* $\bar{h}^{\sigma(k)} = \begin{cases} h_1/h_2, h_1 \leq h_2, h_2 \neq 0 \\ 1, \mathit{others} \end{cases}$.

**Definition 2.3.** [18] *The score function of HFE $h$ is given by $s(h) = \sum_{r\in h} r \big/ l(h)$, and $l(h)$ is the length.*

*For two HFEs $h_1$ and $h_2$, $s(h_1) < s(h_2) \Rightarrow h_1 < h_2$; $s(h_1) = s(h_2) \Rightarrow h_1 \approx h_2$.*

**Definition 2.4.** [22] *Given $n$ HFEs $h_t$, $t = 1, \ldots, n$, a hesitant fuzzy weighted averaging (HFWA) operator is with the form as*

$$HFA\left(h_1, h_2, \ldots, h_n\right) = \oplus_{t=1}^{n} \left(w_t h_t\right) = \bigcup_{h_1^{\sigma(k)} \in h_1, h_2^{\sigma(k)} \in h_2, \ldots, h_t^{\sigma(k)} \in h_t} \left\{\sum_{t=1}^{n} w_t h_t^{\sigma(k)}\right\},$$

*$w_t$ is the weight of $h_t$ with $\sum_{t=1}^{n} w_t = 1$, $w_t \in [0, 1]$.*

In particular, when $w_t = 1/n$, a hesitant fuzzy averaging (HFA) operator is with the form as

$$HFA\left(h_1, h_2, \ldots, h_n\right) = \oplus_{t=1}^{n} \left(\frac{1}{n} h_t\right) = \bigcup_{h_1^{\sigma(k)} \in h_1, h_2^{\sigma(k)} \in h_2, \ldots, h_t^{\sigma(k)} \in h_t} \left\{\sum_{t=1}^{n} \frac{1}{n} h_t^{\sigma(k)}\right\}.$$

2.2. **Hesitant fuzzy $\boldsymbol{\beta}$-neighborhood.**

**Definition 2.5.** [38] *Let $U$ be a nonempty finite set, $\tilde{C}_i^H \in HF(U)$, $i = 1, 2, \ldots, m$, $\beta \in (0, 1]$. For each $x \in U$, if $\min\left(h_{\bigcup_{i=1}^{m} \tilde{C}_i^H}(x)\right) \geq \beta$, then we call $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \ldots, \tilde{C}_m^H\right\}$ an* HF *$\beta$-covering of $U$ and $\left(U, \tilde{C}^H\right)$ a hesitant fuzzy covering approximation space (HFCAS).*

**Definition 2.6.** [38] *Let $\left(U, \tilde{C}^H\right)$ be an HFCAS, where $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \ldots, \tilde{C}_m^H\right\}$ is an HF $\beta$-covering of $U$. For any $x \in U$, the HF $\beta$-neighborhood of $\tilde{H}_x^\beta$ of $x$ can be defined:*

$$\tilde{H}_x^\beta = \bigcap \left\{\tilde{C}_i^H \in \tilde{C}^H : \min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta\right\}.$$

**Example 2.1.** *Let $U = \bigcup_{i=1,2,3,4,5} \{x_i\}$, and a set of HFSs $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \tilde{C}_3^H, \tilde{C}_4^H\right\}$ is presented as Table 1.*

Then, $\tilde{C}^H$ is an HF $\beta$-covering of $U$, where, $0 < \beta \leq 0.7$. $\tilde{H}_{x_1}^{0.6} = \tilde{C}_3^H \cap \tilde{C}_4^H$, $\tilde{H}_{x_2}^{0.6} = \tilde{C}_1^H \cap \tilde{C}_3^H$, $\tilde{H}_{x_3}^{0.6} = \tilde{C}_1^H \cap \tilde{C}_2^H \cap \tilde{C}_4^H$, $\tilde{H}_{x_4}^{0.6} = \tilde{C}_1^H \cap \tilde{C}_3^H \cap \tilde{C}_4^H$, $\tilde{H}_{x_5}^{0.6} = \tilde{C}_2^H \cap \tilde{C}_4^H$. The $\tilde{H}_{x_i}^{0.6}$ of $x_i$ $(i = 1, 2, 3, 4, 5)$ are presented as Table 2.

**Remark 2.1.** *(1) HF $\beta$-neighborhood of $\tilde{H}_x^\beta$ is still an HFS. (2) $\forall x, y \in U$, the $\tilde{H}_x^\beta$ of $y$ (denoted by $\tilde{H}_x^\beta(y)$) can be thought as an HF relation (HFR) over $U$ reflecting the*

Table 1. A set of hesitant fuzzy sets

| | $\tilde{C}_1^H$ | $\tilde{C}_2^H$ | $\tilde{C}_3^H$ | $\tilde{C}_4^H$ |
|---|---|---|---|---|
| $x_1$ | $\{.5, .6\}$ | $\{.5, .6\}$ | $\{.7, .9\}$ | $\{.6, .8\}$ |
| $x_2$ | $\{.7, .8\}$ | $\{.3, .6\}$ | $\{.6, .7\}$ | $\{.5, .7\}$ |
| $x_3$ | $\{.6, .7\}$ | $\{.7, .9\}$ | $\{.5, .9\}$ | $\{.6, .8\}$ |
| $x_4$ | $\{1\}$ | $\{.5, .8\}$ | $\{.6, .7\}$ | $\{.8, .9\}$ |
| $x_5$ | $\{.5, .6\}$ | $\{1\}$ | $\{.5, .7\}$ | $\{.6, .8\}$ |

Table 2. The result of $\tilde{H}_{x_i}^{0.6}$ of $x_i$ $(i = 1, 2, 3, 4, 5)$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $\tilde{H}_{x_1}^{0.6}$ | $\{.6, .8\}$ | $\{.5, .7\}$ | $\{.5, .8\}$ | $\{.6, .7\}$ | $\{.5, .7\}$ |
| $\tilde{H}_{x_2}^{0.6}$ | $\{.5, .6\}$ | $\{.6, .7\}$ | $\{.5, .7\}$ | $\{.6, .7\}$ | $\{.5, .6\}$ |
| $\tilde{H}_{x_3}^{0.6}$ | $\{.5, .6\}$ | $\{.3, .6\}$ | $\{.6, .7\}$ | $\{.5, .8\}$ | $\{.5, .6\}$ |
| $\tilde{H}_{x_4}^{0.6}$ | $\{.5, .6\}$ | $\{.5, .7\}$ | $\{.5, .7\}$ | $\{.6, .7\}$ | $\{.5, .6\}$ |
| $\tilde{H}_{x_5}^{0.6}$ | $\{.5, .6\}$ | $\{.3, .6\}$ | $\{.6, .8\}$ | $\{.5, .8\}$ | $\{.6, .8\}$ |

*membership degree of the relationships between $x$ and $y$. In general, $\tilde{H}_x^\beta(y)$ is not reflexive, symmetric and transitive.*

**Proposition 2.1.** [38] *$\forall x \in U$, $\min\left(h_{\tilde{H}_x^\beta}(x)\right) \geq \beta$.*

**Proposition 2.2.** [38] *$\forall x, y, z \in U$, if $\min\left(h_{\tilde{H}_x^\beta}(y)\right) \geq \beta$ and $\min\left(h_{\tilde{H}_y^\beta}(z)\right) \geq \beta$, then $\min\left(h_{\tilde{H}_x^\beta}(z)\right) \geq \beta$.*

**Proposition 2.3.** *$\forall x, y \in U$, $\min\left(h_{\tilde{H}_x^\beta}(y)\right) \geq \beta$ is equivalent to $\tilde{H}_y^\beta \subseteq \tilde{H}_x^\beta$.*

**Proof:** "$\Rightarrow$" $\min\left(h_{\tilde{H}_x^\beta}(y)\right) = \min\left(h_{\left(\bigcap_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} \tilde{C}_i^H\right)}(y)\right) = \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} h_{\tilde{C}_i^H}(y)\right)$ $\geq \beta$, so $\left\{\tilde{C}_i^H \in \tilde{C}^H : \min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta\right\} \subseteq \left\{\tilde{C}_i^H \in \tilde{C}^H : \min\left(h_{\tilde{C}_i^H}(y)\right) \geq \beta\right\}$.

For any $z \in U$, we have

$$\begin{aligned}
\min\left(h_{\tilde{H}_x^\beta}(z)\right) &= \min\left(h_{\left(\bigcap_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} \tilde{C}_i^H\right)}(z)\right) \\
&\geq \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} h_{\tilde{C}_i^H}(z)\right) = \min\left(h_{\tilde{H}_\eta^\beta}(z)\right),
\end{aligned}$$

then $\tilde{H}_y^\beta \subseteq \tilde{H}_x^\beta$.

"$\Leftarrow$" $\forall x, y \in U$, if $\tilde{H}_y^\beta \subseteq \tilde{H}_x^\beta$, then $\beta \leq \min\left(h_{\tilde{H}_y^\beta}(y)\right) \leq \min\left(h_{\tilde{H}_x^\beta}(y)\right)$.

**Proposition 2.4.** [38] *For any* $\beta \in (0,1]$, *we have* $\tilde{C}_i^H \supset \bigcup \left\{ \tilde{H}_x^\beta : \min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta \right\}$, $x \in U$, $i \in \{1, \ldots, m\}$.

**Proposition 2.5.** [38] *If* $\beta_1 \leq \beta_2$, *then* $\tilde{H}_x^{\beta_1} \subset \tilde{H}_x^{\beta_2}$ *for any* $x \in U$.

**Proposition 2.6.** *Let* $\left(U, \tilde{C}^H\right)$ *be an HFCAS, where* $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \ldots, \tilde{C}_m^H\right\}$, $\beta \in (0,1]$, *and then the following two expressions are equivalent.*

*(1)* $\forall x \in U$, *if* $\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta$, *then there exists* $y \in U$ *satisfying* $\min\left(h_{\tilde{C}_i^H}(y)\right) = 1$, *i.e.,* $h_{\tilde{C}_i^H}(y) = \{1\}$, $i \in \{1, \ldots, m\}$.

*(2)* $\forall x \in U$, $\underset{x \in U}{\vee}\left(h_{\hat{H}_x^\beta}(z)\right) = \{1\}$.

**Proof:** (1) $\Rightarrow$ (2) $\forall y \in U$, if (1) is true, then

$$\min\left(h_{\tilde{H}_x^\beta}(y)\right) = \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} h_{\tilde{C}_i^H}(y)\right) = \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} \{1\}\right) = 1,$$

therefore,

$$1 \geq \underset{z \in U}{\vee} \min\left(h_{\tilde{H}_x^\beta}(z)\right) \geq \min\left(h_{\tilde{H}_x^\beta}(y)\right) = 1,$$

so we have $\underset{z \in U}{\vee}\left(h_{\tilde{H}_x^\beta}(z)\right) = \{1\}$.

(2) $\Rightarrow$ (1) If (2) is true, then there exists $y \in U$ satisfying $h_{\tilde{H}_x^\beta}(y) = \{1\}$, from $h_{\tilde{H}_x^\beta}(y) = \wedge_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} h_{\tilde{C}_i^H}(y)$, we have that, if $\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta$, then $\min\left(h_{\tilde{C}_i^H}(y)\right) = 1$, i.e., $h_{\tilde{C}_i^H}(y) = \{1\}$, $i \in \{1, 2, \ldots, m\}$.

**Corollary 2.1.** *Let* $\left(U, \tilde{C}^H\right)$ *be an HFCAS, where* $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \ldots, \tilde{C}_m^H\right\}$, $\beta \in (0,1]$, *and then the following two expressions are equivalent.*

*(1)* $\forall x \in U$, *if* $\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta$, *then* $\min\left(h_{\tilde{C}_i^H}(x)\right) = 1$, *that is,* $h_{\tilde{C}_i^H}(x) = \{1\}$, $i \in \{1, \ldots, m\}$.

*(2)* $\forall x \in U$, $h_{\tilde{H}_x^\beta}(x) = \{1\}$.

**Proposition 2.7.** *Let* $\left(U, \tilde{C}^H\right)$ *be an HFCAS,* $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \ldots, \tilde{C}_m^H\right\}$, $\beta \in (0,1]$, *and* $\tilde{C}^{H'}, \tilde{C}^{H''} \in \tilde{C}^H$, $\tilde{C}^{H'} \subseteq \tilde{C}^{H''} \Rightarrow \min\left(h_{\tilde{C}^{H'}}(x)\right) = \min\left(h_{\tilde{C}^{H''}}(x)\right)$, *the following two expressions are equivalent.*

*(1)* $\forall x, y \in U$, *if* $\left\{\tilde{C}^{H'} \in \tilde{C}^H \middle| \min\left(h_{\tilde{C}^{H'}}(x)\right) \geq \beta\right\} = \left\{\tilde{C}^{H'} \in \tilde{C}^H \middle| \min\left(h_{\tilde{C}^{H'}}(y)\right) \geq \beta\right\}$, *then we have* $\min\left(h_{\tilde{C}_i^H}(x)\right) = \min\left(h_{\tilde{C}_i^H}(y)\right)$ *for any*

$$\tilde{C}_i^H \in \left\{\tilde{C}^{H'} \in \tilde{C}^H \middle| \min\left(h_{\tilde{C}^{H'}}(x)\right) \geq \beta\right\}.$$

*(2)* $\forall x, y \in U$, *we have* $\min\left(h_{\tilde{H}_x^\beta}(y)\right) = \min\left(h_{\tilde{H}_y^\beta}(x)\right)$.

**Proof:** (1) $\Rightarrow$ (2) $\forall x, y \in U$, if (1) is true, then

$$\min\left(h_{\tilde{H}_x^\beta}(y)\right) = \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(x)\right) \geq \beta} h_{\tilde{C}_i^H}(y)\right)$$

$$= \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(y)\right)\geq\beta} h_{\tilde{C}_i^H}(x)\right) = \min\left(h_{\tilde{H}_y^\beta}(x)\right).$$

(2) $\Rightarrow$ (1) Assuming that there exists $\tilde{C}_i^H \in \left\{\tilde{C}^{H'} \in \tilde{C}^H \middle| \min\left(h_{\tilde{C}^{H'}}(x)\right) \geq \beta\right\}$ such that $\min\left(h_{\tilde{C}_i^H}(x)\right) \neq \min\left(h_{\tilde{C}_i^H}(y)\right)$, then we have that

$$\begin{aligned}\min\left(h_{\tilde{H}_x^\beta}(y)\right) &= \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(x)\right)\geq\beta} h_{\tilde{C}_i^H}(y)\right)\\ &\neq \min\left(\wedge_{\min\left(h_{\tilde{C}_i^H}(y)\right)\geq\beta} h_{\tilde{C}_i^H}(x)\right) = \min\left(h_{\tilde{H}_y^\beta}(x)\right),\end{aligned}$$

which is contradictory with $\min\left(h_{\tilde{H}_x^\beta}(y)\right) = \min\left(h_{\tilde{H}_y^\beta}(x)\right)$ in (2).

Therefore, for any $\tilde{C}_i^H \in \left\{\tilde{C}^{H'} \in \tilde{C}^H \middle| \min\left(h_{\tilde{C}^{H'}}(x)\right) \geq \beta\right\}$, we have $\min\left(h_{\tilde{C}_i^H}(x)\right) = \min\left(h_{\tilde{C}_i^H}(y)\right)$.

## 3. CHFRS and the Axiomatic Characterization.

In this section, we firstly construct an HF $\beta$-CRS model and investigate the properties, and then discuss the axiomatic representation of the lower (upper) HF rough operators.

### 3.1. CHFRS based on hesitant fuzzy $\beta$-neighborhood.

**Definition 3.1.** *Let $\left(U, \tilde{C}^H\right)$ be an HFCAS with $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \ldots, \tilde{C}_m^H\right\}$ being an HF $\beta$-covering of $U$ for some $\beta \in (0, 1]$. The lower and upper approximations of $\boldsymbol{A} \in HF(U)$ on $\tilde{H}_x^\beta$, denoted as $\underline{C_\beta^H}(\boldsymbol{A})$ and $\overline{C_\beta^H}(\boldsymbol{A})$, are expressed as*

$$\underline{C_\beta^H}(\boldsymbol{A}) = \left\{\left\langle x, h_{\underline{C_\beta^H}}(\boldsymbol{A})(x)\right\rangle | x \in U\right\}, \quad \overline{C_\beta^H}(\boldsymbol{A}) = \left\{\left\langle x, h_{\overline{C_\beta^H}(\boldsymbol{A})}(x)\right\rangle | x \in U\right\},$$

*where* $h_{\underline{C_\beta^H}(\boldsymbol{A})}(x) = \bar{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y) \veebar h_{\boldsymbol{A}}(y)\right\}$, $h_{\overline{C_\beta^H}(\boldsymbol{A})}(x) = \underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^\beta}(y) \wedge h_{\boldsymbol{A}}(y)\right\}$.

The pair $\left(\underline{C_\beta^H}(\boldsymbol{A}), \overline{C_\beta^H}(\boldsymbol{A})\right)$ is termed as hesitant fuzzy covering rough set of $\boldsymbol{A}$ on $\tilde{H}_x^\beta$ in $\left(U, \tilde{C}^H\right)$, denoted as CHFRS.

**Example 3.1.** *Let $\left(U, \tilde{C}^H\right)$ be an HFCAS in Example 2.1. For*

$$\begin{aligned}\boldsymbol{A} &= \{\{0.3, 0.4, 0.6\}, \{0.5, 0.7\}, \{0.2, 0.4, 0.8\}, \{0.4, 0.5\}, \{0.6, 0.7, 0.8\}\};\\ \boldsymbol{A}^c &= \{\{0.4, 0.6, 0.7\}, \{0.3, 0.5\}, \{0.2, 0.6, 0.8\}, \{0.5, 0.6\}, \{0.2, 0.3, 0.4\}\}.\end{aligned}$$

*Then, for $\beta = 0.6$, $h_{\underline{C_\beta^H}(\boldsymbol{A})}(x_1) = \{0.2, 0.4, 0.5\}$, $h_{\underline{C_\beta^H}(\boldsymbol{A})}(x_2) = \{0.3, 0.5\}$, $h_{\underline{C_\beta^H}(\boldsymbol{A})}(x_3) = \{0.3, 0.4, 0.5\}$, $h_{\underline{C_\beta^H}(\boldsymbol{A})}(x_4) = \{0.3, 0.5\}$, $h_{\underline{C_\beta^H}(\boldsymbol{A})}(x_5) = \{0.2, 0.4, 0.5\}$.*

*Then, we can conclude that*

$$\begin{aligned}\underline{C_\beta^H}(\boldsymbol{A}) &= \{\{0.2, 0.4, 0.5\}, \{0.3, 0.4, 0.5\}, \{0.3, 0.4, 0.5\}, \{0.3, 0.4, 0.5\}, \{0.2, 0.4, 0.5\}\};\\ \overline{C_\beta^H}(\boldsymbol{A}) &= \{\{0.5, 0.7, 0.8\}, \{0.5, 0.7\}, \{0.5, 0.6, 0.7\}, \{0.5, 0.7\}, \{0.6, 0.7, 0.8\}\};\\ \underline{C_\beta^H}(\boldsymbol{A}^c) &= \{\{0.2, 0.3, 0.5\}, \{0.3, 0.5\}, \{0.3, 0.4, 0.5\}, \{0.3, 0.5\}, \{0.2, 0.3, 0.4\}\};\\ \overline{C_\beta^H}(\boldsymbol{A}^c) &= \{\{0.5, 0.6, 0.8\}, \{0.5, 0.6, 0.7\}, \{0.5, 0.6, 0.7\}, \{0.5, 0.6, 0.7\}, \{0.5, 0.6, 0.8\}\}.\end{aligned}$$

*In general,* $\underline{C_\beta^H}(\boldsymbol{A}) \subseteq \boldsymbol{A} \subseteq \overline{C_\beta^H}(\boldsymbol{A})$ *may not hold.*

**Theorem 3.1.** *Let* $\left(U, \tilde{C}^H\right)$ *be an HFCAS with* $\tilde{C}^H = \left\{\tilde{C}_1^H, \tilde{C}_2^H, \ldots, \tilde{C}_m^H\right\}$ *being an HF* $\beta$*-covering of* $U$ *for some* $\beta \in (0,1]$*.* $\forall \boldsymbol{A}, \boldsymbol{B} \in HF(U)$*, the lower and upper approximation operators on* $\tilde{H}_x^\beta$ *have some properties:*

*(1)* $\underline{C_\beta^H}(\boldsymbol{A}^c) = \left(\overline{C_\beta^H}(\boldsymbol{A})\right)^c$, $\left(\underline{C_\beta^H}(\boldsymbol{A})\right)^c = \overline{C_\beta^H}(\boldsymbol{A}^c)$;

*(2)* $\underline{C_\beta^H}(U) = U$, $\overline{C_\beta^H}(\phi) = \phi$;

*(3)* $\underline{C_\beta^H}(\boldsymbol{A} \cap \boldsymbol{B}) = \underline{C_\beta^H}(\boldsymbol{A}) \cap \underline{C_\beta^H}(\boldsymbol{B})$, $\overline{C_\beta^H}(\boldsymbol{A} \cup \boldsymbol{B}) = \overline{C_\beta^H}(\boldsymbol{A}) \cup \overline{C_\beta^H}(\boldsymbol{B})$;

*(4)* $\boldsymbol{A} \subseteq \boldsymbol{B} \Rightarrow \underline{C_\beta^H}(\boldsymbol{A}) \subseteq \underline{C_\beta^H}(\boldsymbol{B})$, $\boldsymbol{A} \subseteq \boldsymbol{B} \Rightarrow \overline{C_\beta^H}(\boldsymbol{A}) \subseteq \overline{C_\beta^H}(\boldsymbol{B})$;

*(5)* $\underline{C_\beta^H}(\boldsymbol{A} \cup \boldsymbol{B}) \supseteq \underline{C_\beta^H}(\boldsymbol{A}) \cup \underline{C_\beta^H}(\boldsymbol{B})$, $\overline{C_\beta^H}(\boldsymbol{A} \cap \boldsymbol{B}) \subseteq \overline{C_\beta^H}(\boldsymbol{A}) \cap \overline{C_\beta^H}(\boldsymbol{B})$;

*(6)* $\forall x \in U$*, if* $h_{\left(\tilde{H}_x^\beta\right)^c}(x) \le h_{\boldsymbol{A}}(x) \le h_{\tilde{H}_x^\beta}(x)$*, then* $\underline{C_\beta^H}(\boldsymbol{A}) \subseteq \boldsymbol{A} \subseteq \overline{C_\beta^H}(\boldsymbol{A})$;

*(7) If* $0 < \beta_1 \le \beta_2 < 1$*, then* $\underline{C_{\beta_1}^H}(\boldsymbol{A}) \supseteq \underline{C_{\beta_2}^H}(\boldsymbol{A})$, $\overline{C_{\beta_1}^H}(\boldsymbol{A}) \subseteq \overline{C_{\beta_2}^H}(\boldsymbol{A})$.

*(8)* $\forall x \in U$, $\min\left(h_{\overline{C_\beta^H}(U)}(x)\right) \ge \beta$, $\max\left(h_{\underline{C_\beta^H}}(x)\right) \le 1 - \beta$.

**Proof:** (1) $\forall x \in U$, $\boldsymbol{A} \in HF(U)$,

$$\begin{aligned} h_{\underline{C_\beta^H}(\boldsymbol{A}^c)}(x) &= \overline{\wedge}_{y \in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y) \underline{\vee}\, h_{\boldsymbol{A}^c}(y)\right\} = \overline{\wedge}_{y \in U}\left\{\left(\sim h_{\tilde{H}_x^\beta}(y)\right) \underline{\vee} (\sim h_{\boldsymbol{A}}(y))\right\} \\ &= \overline{\wedge}_{y \in U}\left\{\sim \left(h_{\tilde{H}_x^\beta}(y) \,\overline{\wedge}\, h_{\boldsymbol{A}}(y)\right)\right\} = \sim \left(\underline{\vee}_{y \in U}\left\{h_{\tilde{H}_x^\beta}(y) \,\overline{\wedge}\, h_{\boldsymbol{A}}(y)\right\}\right) \\ &= h_{\left(\overline{C_\beta^H}(\boldsymbol{A})\right)^c}(x). \end{aligned}$$

Hence, $\underline{C_\beta^H}(\boldsymbol{A}^c) = \left(\overline{C_\beta^H}(\boldsymbol{A})\right)^c$. Similarly, $\left(\underline{C_\beta^H}(\boldsymbol{A})\right)^c = \overline{C_\beta^H}(\boldsymbol{A}^c)$ can be proved.

(2) $\forall x \in U$, $h_U(x) = \{1\}$, $h_\phi(x) = \{0\}$,

$$h_{\underline{C_\beta^H}(U)}(x) = \bar{\wedge}_{y \in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y) \underline{\vee}\, h_U(y)\right\} = \bar{\wedge}_{y \in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y) \underline{\vee} \{1\}\right\} = \{1\} = h_U(x).$$

$$h_{\overline{C_\beta^H}(\phi)}(x) = \underline{\vee}_{y \in U}\left\{h_{\tilde{H}_x^\beta}(y) \,\overline{\wedge}\, h_\phi(y)\right\} = \underline{\vee}_{y \in U}\left\{h_{\tilde{H}_x^\beta}(y) \,\overline{\wedge}\, \{0\}\right\} = \{0\} = h_\phi(x).$$

Hence, we have $\underline{C_\beta^H}(U) = U$, $\overline{C_\beta^H}(\phi) = \phi$.

(3) $\forall x \in U$, $k = 1, 2, \ldots, l$,

$$\begin{aligned} h_{\underline{C_\beta^H}(\boldsymbol{A} \cap \boldsymbol{B})}(x) &= \overline{\wedge}_{y \in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y) \underline{\vee}\, h_{\boldsymbol{A} \cap \boldsymbol{B}}(y)\right\} \\ &= \overline{\wedge}_{y \in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y) \underline{\vee}\, (h_{\boldsymbol{A}}(y) \,\overline{\wedge}\, h_{\boldsymbol{B}}(y))\right\} \\ &= \left\{\wedge_{y \in U}\left(h^{\sigma(k)}_{\left(\tilde{H}_x^\beta\right)^c}(y)\right) \vee \left(h^{\sigma(k)}_{\boldsymbol{A}}(y) \wedge h^{\sigma(k)}_{\boldsymbol{B}}(y)\right)\right\} \\ &= \left\{\wedge_{y \in U}\left(h^{\sigma(k)}_{\left(\tilde{H}_x^\beta\right)^c}(y) \vee h^{\sigma(k)}_{\boldsymbol{A}}(y)\right)\right\} \wedge \left\{\wedge_{y \in U}\left(h^{\sigma(k)}_{\left(\tilde{H}_x^\beta\right)^c}(y) \vee h^{\sigma(k)}_{\boldsymbol{B}}(y)\right)\right\} \\ &= h_{\underline{C_\beta^H}(\boldsymbol{A})}(x) \,\overline{\wedge}\, h_{\underline{C_\beta^H}(\boldsymbol{B})}(x). \end{aligned}$$

Hence, we have $\underline{C_\beta^H}(\boldsymbol{A} \cap \boldsymbol{B}) = \underline{C_\beta^H}(\boldsymbol{A}) \cap \underline{C_\beta^H}(\boldsymbol{B})$.

Similarly, $\overline{C_\beta^H}(\boldsymbol{A} \cup \boldsymbol{B}) = \overline{C_\beta^H}(\boldsymbol{A}) \cup \overline{C_\beta^H}(\boldsymbol{B})$ is obtained.

(4) If $\boldsymbol{A} \subseteq \boldsymbol{B}$, then $h_{\boldsymbol{A}}^{\sigma(k)}(x) \preceq h_{\boldsymbol{B}}^{\sigma(k)}(x)$ for any $x, y \in U$, $k = 1, 2, \ldots, l$. It follows that

$$h_{\underline{C_\beta^H}(\boldsymbol{A})}(x) = \overline{\wedge}_{y \in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y) \underline{\vee}\, h_{\boldsymbol{A}}(y)\right\}$$

$$= \wedge_{y\in U}\left(h^{\sigma(k)}_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(y) \vee h^{\sigma(k)}_{\boldsymbol{A}}(y)\right)$$

$$\leq \wedge_{y\in U}\left(h^{\sigma(k)}_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(y) \vee h^{\sigma(k)}_{\boldsymbol{B}}(y)\right)$$

$$= \overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(y) \underline{\vee}\, h_{\boldsymbol{B}}(y)\right\}$$

$$= h_{\underline{C^{H}_{\beta}}(\boldsymbol{B})}(x).$$

Therefore, $\boldsymbol{A} \subseteq \boldsymbol{B} \Rightarrow \underline{C^{H}_{\beta}}(\boldsymbol{A}) \subseteq \underline{C^{H}_{\beta}}(\boldsymbol{B})$. Similarly, we have $\boldsymbol{A} \subseteq \boldsymbol{B} \Rightarrow \overline{C^{H}_{\beta}}(\boldsymbol{A}) \subseteq \overline{C^{H}_{\beta}}(\boldsymbol{B})$.

(5) Since $\boldsymbol{A} \subseteq \boldsymbol{A}\cup\boldsymbol{B}$, $\boldsymbol{B} \subseteq \boldsymbol{A}\cup\boldsymbol{B}$, $\boldsymbol{A}\cap\boldsymbol{B} \subseteq \boldsymbol{A}$, $\boldsymbol{A}\cap\boldsymbol{B} \subseteq \boldsymbol{B}$, according to the conclusion of (4), $\underline{C^{H}_{\beta}}(\boldsymbol{A}\cup\boldsymbol{B}) \supseteq \underline{C^{H}_{\beta}}(\boldsymbol{A}) \cup \underline{C^{H}_{\beta}}(\boldsymbol{B})$, $\overline{C^{H}_{\beta}}(\boldsymbol{A}\cap\boldsymbol{B}) \subseteq \overline{C^{H}_{\beta}}(\boldsymbol{A}) \cap \overline{C^{H}_{\beta}}(\boldsymbol{B})$ can be obtained immediately.

(6) $\forall x \in U$, if there exists $h_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(x) \leq h_{\boldsymbol{A}}(x) \leq h_{\tilde{H}^{\beta}_{x}}(x)$, then

$$h_{\boldsymbol{A}}(x) = h_{\tilde{H}^{\beta}_{x}}(x) \overline{\wedge}\, h_{\boldsymbol{A}}(x) \leq \underline{\vee}_{y\in U}\left\{h_{\tilde{H}^{\beta}_{x}}(y) \overline{\wedge}\, h_{\boldsymbol{A}}(y)\right\} = h_{\overline{C^{H}_{\beta}}(\boldsymbol{A})}(x),$$

$$h_{\underline{C^{H}_{\beta}}(\boldsymbol{A})}(x) = \overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(y) \underline{\vee}\, h_{\boldsymbol{A}}(y)\right\} \leq h_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(x) \underline{\vee}\, h_{\boldsymbol{A}}(x) = h_{\boldsymbol{A}}(x).$$

Hence, $\underline{C^{H}_{\beta}}(\boldsymbol{A}) \subseteq \boldsymbol{A} \subseteq \overline{C^{H}_{\beta}}(\boldsymbol{A})$.

(7) $\forall x \in U$, if $0 < \beta_1 \leq \beta_2 < 1$, then $\tilde{H}^{\beta_1}_{x} \subseteq \tilde{H}^{\beta_2}_{x}$. According to Definition 3.1, we can easily have that $\underline{C^{H}_{\beta_1}}(\boldsymbol{A}) \supseteq \underline{C^{H}_{\beta_2}}(\boldsymbol{A})$, $\overline{C^{H}_{\beta_1}}(\boldsymbol{A}) \subseteq \overline{C^{H}_{\beta_2}}(\boldsymbol{A})$.

(8) $\forall x \in U$,

$$\begin{aligned}
\min\left(h_{\overline{C^{H}_{\beta}}(U)}(x)\right) &= \min\left(\underline{\vee}_{y\in U}\left\{h_{\tilde{H}^{\beta}_{x}}(y) \overline{\wedge}\, h_{U}(y)\right\}\right)\\
&= \min\left(\underline{\vee}_{y\in U} h_{\tilde{H}^{\beta}_{x}}(y)\right) \geq \min\left(h_{\tilde{H}^{\beta}_{x}}(x)\right) \geq \beta,\\
\max\left(h_{\underline{C^{H}_{\beta}}(\phi)}(x)\right) &= \max\left(\overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(y) \underline{\vee}\, h_{\phi}(y)\right\}\right)\\
&= \max\left(\overline{\wedge}_{y\in U} h_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(y)\right) \leq \max\left(h_{\left(\tilde{H}^{\beta}_{x}\right)^{c}}(x)\right) \leq 1-\beta.
\end{aligned}$$

**Remark 3.1.** *If $0 < \beta_1 \leq \beta_2 \leq \cdots \leq \beta_n \leq 1$, then*

$$\underline{C^{H}_{\beta_1}}(\boldsymbol{A}) \supseteq \underline{C^{H}_{\beta_2}}(\boldsymbol{A}) \supseteq \cdots \supseteq \underline{C^{H}_{\beta_n}}(\boldsymbol{A}) \text{ and } \overline{C^{H}_{\beta_1}}(\boldsymbol{A}) \subseteq \overline{C^{H}_{\beta_2}}(\boldsymbol{A}) \subseteq \cdots \subseteq \overline{C^{H}_{\beta_n}}(\boldsymbol{A}).$$

**Remark 3.2.** *For some HFEs $\beta = \{\beta_1, \beta_2, \ldots, \beta_n\}$, suppose the HF $\beta$-neighborhood to be $N^{\beta}_{x}$, the lower and upper approximations of Zhou and Li [39] to be $\underline{N^{\beta}_{C}}(\boldsymbol{A})$, $\overline{N^{\beta}_{C}}(\boldsymbol{A})$ respectively. If taking $\beta_0 = \min\beta = \min\{\beta_1, \beta_2, \ldots, \beta_n\}$, then $\tilde{H}^{\beta_0}_{x} \subseteq N^{\beta}_{x}$; therefore, $\underline{C^{H}_{\beta_0}}(\boldsymbol{A}) \supseteq \underline{N^{\beta}_{C}}(\boldsymbol{A})$, $\overline{C^{H}_{\beta_0}}(\boldsymbol{A}) \subseteq \overline{N^{\beta}_{C}}(\boldsymbol{A})$.*

**Theorem 3.2.** *Let $\left(U, \tilde{C}^{H}\right)$ be an HFCAS with $\tilde{C}^{H} = \left\{\tilde{C}^{H}_{1}, \tilde{C}^{H}_{2}, \ldots, \tilde{C}^{H}_{m}\right\}$ being an HF $\beta$-covering of $U$ for some $\beta \in (0,1]$. For all $\boldsymbol{A} \in HF(U)$, $\{a_1 \cdots a_m\}$ is a constant HFS, and then we have the following conclusions:*

*(1) $\overline{C^{H}_{\beta}}\left(\{a_1 \cdots a_m\} \cap \boldsymbol{A}\right) = \{a_1 \cdots a_m\} \cap \overline{C^{H}_{\beta}}(\boldsymbol{A})$, $\underline{C^{H}_{\beta}}\left(\{a_1 \cdots a_m\} \cup \boldsymbol{A}\right) = \{a_1 \cdots a_m\} \cup \underline{C^{H}_{\beta}}(\boldsymbol{A})$;*

*(2) $\overline{C^{H}_{\beta}}\left(\{a_1 \cdots a_m\}\right) \subseteq \{a_1 \cdots a_m\}$, $\{a_1 \cdots a_m\} \subseteq \underline{C^{H}_{\beta}}\left(\{a_1 \cdots a_m\}\right)$;*

*(3)* $\overline{C_{\beta}^{H}}\left(\{a_1\cdots a_m\}\right)=\{a_1\cdots a_m\}\Leftrightarrow\overline{C_{\beta}^{H}}(U)=U$, $\underline{C_{\beta}^{H}}\left(\{a_1\cdots a_m\}\right)=\{a_1\cdots a_m\}\Leftrightarrow\underline{C_{\beta}^{H}}(\phi)=\phi$.

**Proof:** (1) $\forall x\in U$,

$$
\begin{aligned}
h_{\overline{C_{\beta}^{H}}(\{a_1\cdots a_m\}\cap\boldsymbol{A})}(x)&=\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\,\overline{\wedge}_{(\{a_1\cdots a_m\}\cap\boldsymbol{A})}(y)\right\}\\
&=\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\,\overline{\wedge}\,\left(\{a_1\cdots a_m\}\,\overline{\wedge}h_{\boldsymbol{A}}(y)\right)\right\}\\
&=\{a_1\cdots a_m\}\,\overline{\wedge}\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\,\overline{\wedge}\,h_{\boldsymbol{A}}(y)\right\}\\
&=\{a_1\cdots a_m\}\,\overline{\wedge}\,h_{\overline{C_{\beta}^{H}}(\boldsymbol{A})}(x)=h_{\left(\{a_1\cdots a_m\}\cap\overline{C_{\beta}^{H}}(\boldsymbol{A})\right)}(x),\\
h_{\underline{C_{\beta}^{H}}(\{a_1\cdots a_m\}\cap\boldsymbol{A})}(x)&=\overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^{\beta}\right)^c}(y)\,\underline{\vee}_{(\{a_1\cdots a_m\}\cap\boldsymbol{A})}(y)\right\}\\
&=\overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^{\beta}\right)^c}(y)\,\underline{\vee}\,\left(\{a_1\cdots a_m\}\,\underline{\vee}\,h_{\boldsymbol{A}}(y)\right)\right\}\\
&=\{a_1\cdots a_m\}\,\underline{\vee}\overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^{\beta}\right)^c}(y)\,\underline{\vee}\,h_{\boldsymbol{A}}(y)\right\}\\
&=\{a_1\cdots a_m\}\,\underline{\vee}\,h_{\underline{C_{\beta}^{H}}(\boldsymbol{A})}(x)=h_{\left(\{a_1\cdots a_m\}\cap\underline{C_{\beta}^{H}}(\boldsymbol{A})\right)}(x).
\end{aligned}
$$

So we have

$$
\begin{aligned}
\overline{C_{\beta}^{H}}\left(\{a_1\cdots a_m\}\cap\boldsymbol{A}\right)&=\{a_1\cdots a_m\}\cap\overline{C_{\beta}^{H}}(\boldsymbol{A}),\\
\underline{C_{\beta}^{H}}\left(\{a_1\cdots a_m\}\cup\boldsymbol{A}\right)&=\{a_1\cdots a_m\}\cup\underline{C_{\beta}^{H}}(\boldsymbol{A}).
\end{aligned}
$$

(2) $\forall x\in U$,

$$
\begin{aligned}
h_{\overline{C_{\beta}^{H}}(\{a_1\cdots a_m\})}(x)&=\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\,\overline{\wedge}\,h_{(\{a_1\cdots a_m\})}(y)\right\}\\
&=\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\,\underline{\triangle}\,\{a_1\cdots a_m\}\right\}\\
&\le\underline{\vee}_{y\in U}\left\{\{1\cdots 1\}\,\overline{\wedge}\,\{a_1\cdots a_m\}\right\}\\
&=\underline{\vee}_{y\in U}\{a_1\cdots a_m\}=\{a_1\cdots a_m\}=h_{(\{a_1\cdots a_m\})}(x),\\
h_{\underline{C_{\beta}^{H}}(\{a_1\cdots a_m\})}(x)&=\overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^{\beta}\right)^c}(y)\,\underline{\vee}\,h_{(\{a_1\cdots a_m\})}(y)\right\}\\
&=\overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^{\beta}\right)^c}(y)\,\underline{\vee}\,\{a_1\cdots a_m\}\right\}\\
&\ge\overline{\wedge}_{y\in U}\left\{\{0\cdots 0\}\,\underline{\vee}\,\{a_1\cdots a_m\}\right\}\\
&=\overline{\wedge}_{y\in U}\{a_1\cdots a_m\}=\{a_1\cdots a_m\}=h_{(\{a_1\cdots a_m\})}(x).
\end{aligned}
$$

So we have

$$
\overline{C_{\beta}^{H}}\left(\{a_1\cdots a_m\}\right)\subseteq\{a_1\cdots a_m\},\quad \{a_1\cdots a_m\}\subseteq\underline{C_{\beta}^{H}}\left(\{a_1\cdots a_m\}\right).
$$

(3) "$\Rightarrow$" If $a_1=a_2=\cdots=a_m=1$ or $a_1=a_2=\cdots=a_m=0$, it is apparent that $\overline{C_{\beta}^{H}}(U)=U$, $\underline{C_{\beta}^{H}}(\phi)=\phi$.

"$\Leftarrow$" $\forall x,y\in U$, since

$$
h_{\overline{C_{\beta}^{H}}(U)}(x)=\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\,\overline{\wedge}\,\{1\cdots 1\}\right\}=\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\right\}=h_U(x)=\{1\cdots 1\},
$$

we can get

$$
h_{\overline{C_{\beta}^{H}}(\{a_1\cdots a_m\})}(x)=\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^{\beta}}(y)\,\overline{\wedge}\,h_{(\{a_1\cdots a_m\})}(y)\right\}
$$

$$= \underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^\beta}(y)\,\underline{\wedge}\,\{a_1\cdots a_m\}\right\}$$
$$= \{a_1\cdots a_m\}\,\overline{\wedge}\,\underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^\beta}(y)\right\}$$
$$= \{a_1\cdots a_m\}\,\overline{\wedge}\,\{1\cdots 1\} = \{a_1\cdots a_m\} = h_{(\{a_1\cdots a_m\})}(x).$$

so we have

$$\overline{C_\beta^H}(U) = U \Rightarrow \overline{C_\beta^H}\left(\{a_1\cdots a_m\}\right) = \{a_1\cdots a_m\}.$$

From $h_{\underline{C_\beta^H}(\phi)}(x) = \overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y)\underline{\vee}\{0\cdots 0\}\right\} = \overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y)\right\} = h_\phi(x) = \{0\cdots 0\}$, we can get

$$h_{\underline{C_\beta^H}(\{a_1\cdots a_m\})}(x) = \overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y)\,\underline{\vee}\,h_{(\{a_1\cdots a_m\})}(y)\right\}$$
$$= \overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^\beta\right)^c}(y)\,\underline{\vee}\,\{a_1\cdots a_m\}\right\}$$
$$= \{a_1\cdots a_m\}\,\underline{\vee}\,\overline{\wedge}_{y\in U}\left\{h_{\left(\tilde{H}_x^\beta\right)}(y)\right\} = \{a_1\cdots a_m\} = h_{\{a_1\cdots a_m\}}(x),$$

so we have

$$\underline{C_\beta^H}(\phi) = \phi \Rightarrow \underline{C_\beta^H}\left(\{a_1\cdots a_m\}\right) = \{a_1\cdots a_m\}.$$

3.2. **Axiomatic characterization of HF $\boldsymbol{\beta}$-covering-based approximation operators.** Following we will consider the axiomatic characterization of approximation operators of CHFRS. That is, for a pair of abstract operators $E, F$: $HF(U) \to HF(U)$, under what conditions they should meet, we can always find an HF $\beta$-covering on $U$ such that $E$, $F$ are the HF $\beta$-covering rough approximation operators.

**Theorem 3.3.** *Let $E$: $HF(U) \to HF(U)$ be an HF set-theoretic operator, $\beta \in (0, 1]$, and then for $\boldsymbol{A}, \boldsymbol{B} \in HF(U)$, there exists an HF $\beta$-covering on $U$ satisfying $\overline{C_\beta^H}(\boldsymbol{A}) = E(\boldsymbol{A})$ iff $E$ meets the following axioms.*

*(E1) For any $x \in U$, $\min\left(h_{E(U)}(x)\right) \geq \beta$;*

*(E2) $E\left(\{a_1\cdots a_m\} \cap \boldsymbol{A}\right) = \{a_1\cdots a_m\} \cap E(\boldsymbol{A})$;*

*(E3) $E(\boldsymbol{A}\cup\boldsymbol{B}) = E(\boldsymbol{A}) \cup E(\boldsymbol{B})$.*

**Proof:** "$\Rightarrow$" From Theorem 3.1(8) and Theorem 3.2, we can achieve the conclusion directly.

"$\Leftarrow$" $\forall x, y \in U$, let $U = \{x_1, \ldots, x_m\}$, and the HF set-theoretic operator $E$ satisfies conditions (E1), (E2) and (E3). Now we define a family of HF sets $\tilde{C}_i^H = \left\{h_{E(\{y\})}(x_i) : i = 1, \ldots, m\right\}$, and $\tilde{C}^H = \left\{\tilde{C}_i^H : i = 1, \ldots, m\right\}$. Then from (E1), we know that $\tilde{C}^H$ is an HF $\beta$-covering on $U$.

For any $x \in U$ and $\boldsymbol{A} \in HF(U)$,

$$h_{\bigcup_{y\in U}(\{y\}\cap\{h_{\boldsymbol{A}}(y)\})}(x) = \underline{\vee}_{y\in U}\left\{h_{\{y\}}(x)\,\overline{\wedge}\,h_{\{h_{\boldsymbol{A}}(y)\}}(x)\right\}$$
$$= \left\{h_{\{x\}}(x)\,\overline{\wedge}\,h_{\{h_{\boldsymbol{A}}(x)\}}(x)\right\}\,\underline{\vee}\left(\underline{\vee}_{y\neq x}\left\{h_{\{y\}}(x)\,\overline{\wedge}\,h_{\{h_{\boldsymbol{A}}(y)\}}(x)\right\}\right)$$
$$= \left\{\{1\}\,\overline{\wedge}\,h_{\{h_{\boldsymbol{A}}(x)\}}(x)\right\}\,\underline{\vee}\left(\underline{\vee}_{y\neq x}\left\{\{0\}\,\overline{\wedge}\,h_{\{h_{\boldsymbol{A}}(y)\}}(x)\right\}\right)$$
$$= h_{\{h_{\boldsymbol{A}}(x)\}}(x) = h_{\boldsymbol{A}}(x).$$

So we have $\boldsymbol{A} = \bigcup_{y\in U}\left(\{y\}\bigcap\{h_{\boldsymbol{A}}(y)\}\right)$.

According to (E2) and (E3), define $h_{\tilde{H}_x^\beta}(y) = h_{E(\{y\})}(x)$,

$$h_{\overline{C_\beta^H}(\boldsymbol{A})}(x) = \underline{\vee}_{y\in U}\left\{h_{\tilde{H}_x^\beta}(y)\,\overline{\wedge}\,h_{\boldsymbol{A}}(y)\right\}$$

$$= \underline{\vee}_{y\in U}\left\{h_{E(\{y\})}(x)\,\overline{\wedge}\,h_{\boldsymbol{A}}(y)\right\} = \underline{\vee}_{y\in U}\left\{h_{E(\{y\})}(x)\,\overline{\wedge}\,h_{\{h_{\boldsymbol{A}}(y)\}}(x)\right\}$$
$$= \underline{\vee}_{y\in U}\left\{h_{E(\{y\}\cap\{h_{\boldsymbol{A}}(y)\})}(x)\right\} = h_{E\left(\bigcup_{y\in U}(\{y\}\cap\{h_{\boldsymbol{A}}(y)\})\right)}(x) = h_{E(\boldsymbol{A})}(x).$$

Therefore, $\overline{C_\beta^H}(\boldsymbol{A}) = E(\boldsymbol{A})$.

**Theorem 3.4.** *Let $F$: $HF(U) \to HF(U)$ be an HF set-theoretic operator, $\beta \in (0, 1]$, and then for any $\boldsymbol{A}, \boldsymbol{B} \in HF(U)$, there exists an HF $\beta$-covering on $U$ satisfying $\underline{C_\beta^H}(\boldsymbol{A}) = F(\boldsymbol{A})$ iff $F$ meets the following axioms.*

*(F1) For any $x \in U$, $\max\left(h_{F(\phi)}(x)\right) \le 1 - \beta$;*
*(F2) $F\left(\{a_1 \cdots a_m\} \bigcup \boldsymbol{A}\right) = \{a_1 \cdots a_m\} \bigcup F(\boldsymbol{A})$;*
*(F3) $F(\boldsymbol{A} \cap \boldsymbol{B}) = F(\boldsymbol{A}) \cap F(\boldsymbol{B})$.*

According to Theorem 3.1(1) and Theorem 3.3, we know that $\overline{C_\beta^H}(\boldsymbol{A})$ and $\underline{C_\beta^H}(\boldsymbol{A})$ are dual. Therefore, for the proof of Theorem 3.4, for any $\boldsymbol{A} \in HF(U)$, we only set that $\boldsymbol{A} = \bigcap_{y\in U}\left(\{U - \{y\}\} \bigcup \{h_{\boldsymbol{A}}(y)\}\right)$, $h_{\tilde{H}_x^\beta}(y) = h_{F(U-\{y\})}(x)$. Similar to the proof of Theorem 3.3, we obtain the above conclusion.

## 4. MCGDM Method Based on VIKOR and CHFRS Model (VIKOR-CHFRS).

As for decision scenarios like research proposals evaluation and postgraduate admission, they share common features such as multi-criteria assessment, and minimum threshold requirements. Consequently, the solution process exhibits structural similarity across contexts, primarily differing in criteria specification and threshold configuration. Furthermore, in real-world applications, such as e-commerce decision-making, textual reviews play a crucial role. By leveraging natural language processing (NLP) technology, these unstructured text data can be transformed into quantifiable hesitant fuzzy sets. Subsequently, a decision-making model can be established using our proposed method. This approach provides a generalized framework within the context of hesitant fuzzy $\beta$-covering for this type of problems, with research proposal evaluation serving as the demonstrative application.

### 4.1. Decision background and problem description.

It is well known that science and technology plays a crucial part in promoting social development and economic progress. Therefore, many nations make all their efforts to invest significant financial funds to the scientific projects aiming to support research activities. In this context, the university of China also establishes their own research funding to help some potential teachers to carry out their scientific researches. Every teacher who has the prepared copy can apply to the foundation support. However, only a small number of teachers with excellent research proposals will receive further financial support. Meanwhile, how to fairly select some excellent research proposals is a critical issue, which becomes more difficult for the head of the scientific research management department of university. As we know, evaluation based on expert opinions can effectively ensure the fairness of proposals competition, the smooth implementation of the research plans and programs, and can further avoid precious financial funds to be wasted. Hence, they usually organize some sophisticated domain experts to evaluate the collected research proposals based on some score criteria. The most focused criteria includes 6 aspects, i.e., academic idea innovativeness ($c_1$), theory and application practicality ($c_2$), proposal setting scientificity ($c_3$), content reasonability ($c_4$), approach feasibility ($c_5$), and research basis sufficiency ($c_6$). Obviously, the review and selection process is an MCGDM problem.

Consequently, many methods for MCGDM problems have been achieved from various aspects. However, the current researches often have the tendency to the interdisciplinary features, which makes the evaluation process become more complex and not easy to operation. Along with the limited knowledge of experts, the evaluation value often has some uncertainties and imprecision. Therefore, the deterministic and classical MCGDM methods are no longer working in dealing with the above proposal review problems. First, HFSs can be used to depict the imprecision and incompleteness effectively, thereby leading to producing more realistic decision results. It is more reasonable to express the opinions on the involved criteria of the proposals to be evaluated in HFS. Second, RS has been successfully used in abundant decision-making fields and also plays a critical role in data preprocessing and information integration. So it is natural to combine HFS and RS with some traditional decision methods for uncertain MCGDM problems. This paper will make an attempt to construct a VIKOR decision method by using HFS and RS, which is suitable for proposal review problems and can contribute to the reasonability of evaluation result.

In practical review processes, a minimum threshold $\beta$ is often predefined to streamline expert evaluation efforts while ensuring the selection of high-quality research proposals. Only proposals surpassing all thresholds proceed to comprehensive evaluation and ranking. For instance, in Natural Science Foundation reviews, proposals scoring below 4.0 (on a 5-point scale) in innovativeness are eliminated during preliminary screening before in-depth assessment. However, thresholds are not fixed, which can be adjusted flexibly according to varying situations. The usual methods may include historical data benchmarking, elimination rate control method and so on. For the above example, the innovativeness threshold was raised from 4.0 to 4.2 due to higher overall quality based on the submission trends or the funds budget. Therefore, based on the given threshold value $\beta$, each evaluation data from an expert forms an HF $\beta$-covering. In the example, assuming the evaluation value is bigger than the threshold value $\beta$ for all the considered proposals. Meanwhile, VIKOR method based on CHFRS model is used to establish an MCGDM method.

Let $U = \{x_1, \ldots, x_n\}$ be the set of the collected research proposals, $C = \{c_1, \ldots, c_m\}$ is the evaluation criteria set, $w = \{w_1, \ldots, w_m\}$ is the criteria weight with $w_j > 0$, and $\sum_{j=1}^{m} w_j = 1$, $d = \{d_1, \ldots, d_t\}$ is the set of experts who are invited to take part in the assessment process, $\omega = \{\omega_1, \ldots, \omega_t\}$ is the weight of expert with $\omega_k > 0$, and $\sum_{k=1}^{t} \omega_k = 1$. Here, $c_j^k(x_i)$ denotes the HF information of research proposal $x_i$ on the criterion $c_j$ from the $k$th decision-maker, namely $c_j^k(x_i) = \left\{h^1_{c_j^k(x_i)}, \ldots, h^l_{c_j^k(x_i)}\right\}$ and $h^s_{c_j^k(x_i)}$, $s = 1, \ldots, l$ is the possible membership degree of $c_j^k(x_i)$. For simplicity, we assume the length of HFE is the same; otherwise, we can add the maximum element. Then we can get $t$ decision matrix $D(k) = \left\{c_j^k(x_i)\right\}_{n\times m}$, $k = 1, \ldots, t$.

$$D(k) = \left\{\begin{array}{ccc} c_1^k(x_1) = \left\{h^1_{c_1^k(x_1)}, \ldots, h^l_{c_1^k(x_1)}\right\} & \cdots & c_m^k(x_1) = \left\{h^1_{c_m^k(x_1)}, \ldots, h^l_{c_m^k(x_1)}\right\} \\ c_1^k(x_2) = \left\{h^1_{c_1^k(x_2)}, \ldots, h^l_{c_1^k(x_2)}\right\} & \cdots & c_m^k(x_2) = \left\{h^1_{c_m^k(x_2)}, \ldots, h^l_{c_m^k(x_2)}\right\} \\ \vdots & \vdots & \vdots \\ c_1^k(x_n) = \left\{h^1_{c_1^k(x_n)}, \ldots, h^l_{c_1^k(x_n)}\right\} & \cdots & c_m^k(x_n) = \left\{h^1_{c_m^k(x_n)}, \ldots, h^l_{c_m^k(x_n)}\right\} \end{array}\right\}.$$

4.2. **Determining the experts' weights.** For an MCGDM problem, each expert usually has different education backgrounds and specialties so that they have different opinions concerning the problems under consideration. At the same time, each member's influence exerted on a group should be taken into account for making the final decision during the process of group preference aggregation. Due to the varying social status, expertise and practical experience, each expert will have a different importance in information aggregation. Therefore, the role of an expert on the group can be reflected through the weight. In this section, we do not address the consensus-reaching process but instead propose a consistency-based method for determining expert weights. As we know, the essential objective of MCGDM is to obtain a satisfactory solution so that attaining a broad agreement among the experts on the problem. In the following section, we firstly give a concept of consensus measure of each expert, that is, the preferences agreement degree with the most experts.

The consistency index can be mathematically derived by measuring the difference between each expert's preferences and the group's collective preferences. To operationalize this concept, following we introduce a distance metric for hesitant fuzzy sets, which effectively quantifies the divergence degree among expert opinions.

**Definition 4.1.** [18] *Let* $U = \{x_1, x_2, \ldots, x_n\}$ *be a nonempty finite set and* $\boldsymbol{A}, \boldsymbol{B} \in HF(U)$*, and the hesitant fuzzy normalized distance can be defined as*

$$D(\boldsymbol{A}, \boldsymbol{B}) = \left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{l}\sum_{k=1}^{l}\left|h_{\boldsymbol{A}}^{\sigma(k)}(x_i) - h_{\boldsymbol{B}}^{\sigma(k)}(x_i)\right|\right)\right].$$

The expert weights are then derived based on this consistency measure. Guided by "the majority rule" principle of GDM, an expert with a higher consensus measure should be assigned a higher weight, vice versa. The specific steps for determining expert weights are outlined below.

First, we give the consensus measure of expert $d_k$ on each research proposal $x_i$ as follows:

$$CM_k(x_i) = \sum_{r=1, r\neq k}^{t}\sum_{j=1}^{m} D\left(c_j^r(x_i), c_j^k(x_i)\right).$$

$c_j^k(x_i)$ denotes the evaluation value of research proposal $x_i$ on the criterion $c_j$ from the $k$th decision-maker, and it is a hesitant fuzzy set. $D(\boldsymbol{A}, \boldsymbol{B})$ can be obtained by Definition 4.1.

Second, the consensus measure of expert $d_k$ on all the research proposals is

$$CM_k = \sum_{i=1}^{n} CM_k(x_i) = \sum_{r=1, r\neq k}^{t}\sum_{i=1}^{n}\sum_{j=1}^{m} D\left(c_j^r(x_i), c_j^k(x_i)\right).$$

Then we can get the consensus measure of each expert $CM_k$, $k = 1, 2, \ldots, t$.

Through the above obtained consensus measure, we can determine the weight of each expert. Suppose $\omega = \{\omega_1, \ldots, \omega_t\}$ is the weight of expert, and then we have

$$\omega_k = \frac{CM_k}{\sum_{r=1}^{t} CM_r}.$$

Obviously, $0 < \omega_k < 1$ and $\sum_{k=1}^{t}\omega_k = 1$. In the following, we will apply the obtained weight vector to the MCDGM for group aggregation.

4.3. **VIKOR method based on CHFRS model (VIKOR-CHFRS).** According to the problem description, we will give the steps of VIKOR-CHFRS method with HF evaluation information by means of CHFRS model.

Firstly, determine the best and the worst HF ideal research proposal $x_{HF}^{+}$, $x_{HF}^{-}$, respectively. The above involved 6 criteria are benefit criteria. Therefore, the bigger the HF evaluation value on a criterion is, the better the proposal is. For the $k$th decision maker, we can get the following

$$x_{HF}^{k,+} = \left\{c_1^k\left(x_{HF,1}^{k,+}\right), c_2^k\left(x_{HF,2}^{k,+}\right), \ldots, c_m^k\left(x_{HF,m}^{k,+}\right)\right\},$$

$$x_{HF}^{k,-} = \left\{c_1^k\left(x_{HF,1}^{k,-}\right), c_2^k\left(x_{HF,2}^{k,-}\right), \ldots, c_m^k\left(x_{HF,m}^{k,-}\right)\right\},\ k = 1, 2, \ldots, t;$$

where, $c_j^k\left(x_{HF,j}^{k,+}\right) = \underline{\vee}_{1\le i\le n}\left(c_j^k(x_i)\right)$, $c_j^k\left(x_{HF,j}^{k,-}\right) = \overline{\wedge}_{1\le i\le n}\left(c_j^k(x_i)\right)$, $i = 1, \ldots, n$, $j = 1, \ldots, m$.

Secondly, compute the HF group utility

$$s_{HF}^k\left(x_i^k\right) = \oplus_{j=1}^m\left\{w_j\left\{\left(x_{HF}^{k,+} \ominus x_{ij}^k\right) \oslash \left(x_{HF}^{k,+} - x_{HF}^{k,-}\right)\right\}\right\};$$

the HF individual regret

$$r_{HF}^k\left(x_i^k\right) = \underline{\vee}_{1\le j\le m}\left\{w_j\left\{\left(x_{HF}^{k,+} \ominus x_{ij}^k\right) \oslash \left(x_{HF}^{k,+} - x_{HF}^{k,-}\right)\right\}\right\}.$$

Thirdly, compute the HF compromise value

$$\begin{aligned} q_{HF}^k\left(x_i^k\right) = {} & \varphi\left\{s_{HF}^k\left(x_i^k\right) \ominus s_{HF}^{k,-}\left(x_i^k\right)\right\} \oslash \left\{s_{HF}^{k,+}\left(x_i^k\right) \ominus s_{HF}^{k,-}\left(x_i^k\right)\right\} \\ & \oplus(1-\varphi)\left\{r_{HF}^k\left(x_i^k\right) \ominus r_{HF}^{k,-}\left(x_i^k\right)\right\} \oslash \left\{r_{HF}^{k,+}\left(x_i^k\right) \ominus r_{HF}^{k,-}\left(x_i^k\right)\right\}, \end{aligned}$$

where, $s_{HF}^{k,+}\left(x_i^k\right) = \underline{\vee}_{1\le i\le n} s_{HF}^k\left(x_i^k\right)$, $s_{HF}^{k,-}\left(x_i^k\right) = \overline{\wedge}_{1\le i\le n} s_{HF}^k\left(x_i^k\right)$, $r_{HF}^{k,+}\left(x_i^k\right) = \underline{\vee}_{1\le i\le n} r_{HF}^k\left(x_i^k\right)$, $r_{HF}^{k,-}\left(x_i^k\right) = \overline{\wedge}_{1\le i\le n} r_{HF}^k\left(x_i^k\right)$. $\varphi \in [0, 1]$ is the coefficient of the decision-making mechanism of the "majority criterion" strategy, reflecting the importance of the criterion or the preference of the decision maker. If $\varphi > 0.5$, it means a risk-seeing tendency, favoring the majority opinion; if $\varphi = 0.5$, it represents a risk-neutral approach, balancing the interests of both the majority and minority groups; if $\varphi < 0.5$, it suggests a risk-aversion stance, prioritizing minority objections. Typically, we take $\varphi = 0.5$.

Fourthly, three HFSs can be obtained as follows:

$$\begin{aligned} s_{HF}^k &= \left\{\frac{s_{HF}^k\left(x_1^k\right)}{x_1}, \frac{s_{HF}^k\left(x_2^k\right)}{x_2}, \ldots, \frac{s_{HF}^k\left(x_n^k\right)}{x_n}\right\}, \\ r_{HF}^k &= \left\{\frac{r_{HF}^k\left(x_1^k\right)}{x_1}, \frac{r_{HF}^k\left(x_2^k\right)}{x_2}, \ldots, \frac{r_{HF}^k\left(x_n^k\right)}{x_n}\right\}, \\ q_{HF}^k &= \left\{\frac{q_{HF}^k\left(x_1^k\right)}{x_1}, \frac{q_{HF}^k\left(x_2^k\right)}{x_2}, \ldots, \frac{q_{HF}^k\left(x_n^k\right)}{x_n}\right\}. \end{aligned}$$

$s_{HF}^k$, $r_{HF}^k$, $q_{HF}^k$ are called group utility HFS, individual regret HFS, comprise HFS, respectively.

Fifthly, given a $\beta \in (0, 1]$, determine the $\tilde{H}_x^\beta$, compute the lower and upper approximations of $s_{HF}^k$, $r_{HF}^k$, $q_{HF}^k$, respectively, denoted as $\underline{C_\beta^H}\left(s_{HF}^k\right)$, $\overline{C_\beta^H}\left(s_{HF}^k\right)$, $\underline{C_\beta^H}\left(r_{HF}^k\right)$, $\overline{C_\beta^H}\left(r_{HF}^k\right)$, $\underline{C_\beta^H}\left(q_{HF}^k\right)$, $\overline{C_\beta^H}\left(q_{HF}^k\right)$.

Sixthly, compute the expert's weight $\omega_k$, $k = 1, \ldots, t$; then synthesize the lower and upper approximations using the weight average operator,

$$\begin{aligned} \underline{C_\beta^H}\left(s_{HF}\right) &= \oplus_{k=1}^t\left\{\omega_k \underline{C_\beta^H}\left(s_{HF}^k\right)\right\}, & \overline{C_\beta^H}\left(s_{HF}\right) &= \oplus_{k=1}^t\left\{\omega_k \overline{C_\beta^H}\left(s_{HF}^k\right)\right\}, \\ \underline{C_\beta^H}\left(r_{HF}\right) &= \oplus_{k=1}^t\left\{\omega_k \underline{C_\beta^H}\left(r_{HF}^k\right)\right\}, & \overline{C_\beta^H}\left(r_{HF}\right) &= \oplus_{k=1}^t\left\{\omega_k \overline{C_\beta^H}\left(r_{HF}^k\right)\right\}, \\ \underline{C_\beta^H}\left(q_{HF}\right) &= \oplus_{k=1}^t\left\{\omega_k \underline{C_\beta^H}\left(q_{HF}^k\right)\right\}, & \overline{C_\beta^H}\left(q_{HF}\right) &= \oplus_{k=1}^t\left\{\omega_k \overline{C_\beta^H}\left(q_{HF}^k\right)\right\}. \end{aligned}$$

Then, compute the score value of $SC\left(\underline{C_\beta^H}(s_{HF})\right)$, $SC\left(\overline{C_\beta^H}(s_{HF})\right)$, $SC\left(\underline{C_\beta^H}(r_{HF})\right)$, $SC\left(\overline{C_\beta^H}(r_{HF})\right)$, $SC\left(\underline{C_\beta^H}(q_{HF})\right)$, $SC\left(\overline{C_\beta^H}(q_{HF})\right)$. Since the above obtained values lie within $[0,1]$, we will use a fuzzy logical operator $\vartheta(a,b) = a+b-ab$ to get three fuzzy sets

$$s^*_{HF} = \vartheta\left(SC\left(\underline{C_\beta^H}(s_{HF})\right), SC\left(\overline{C_\beta^H}(s_{HF})\right)\right),$$
$$r^*_{HF} = \vartheta\left(SC\left(\underline{C_\beta^H}(r_{HF})\right), SC\left(\overline{C_\beta^H}(r_{HF})\right)\right),$$
$$q^*_{HF} = \vartheta\left(SC\left(\underline{C_\beta^H}(q_{HF})\right), SC\left(\overline{C_\beta^H}(q_{HF})\right)\right).$$

Next, rank all the research proposals according to the above obtained values in increasing order.

Finally, determine the compromise value of the proposal. Rank the alternatives $x^1, x^2, \ldots, x^n$ by $q^*_{HF}$ in growing order. If $x^1$ meets the next two conditions, then $x^1$ is a compromise solution.

($C_1$) Acceptable advantage: $q^*_{HF}(x^2) - q^*_{HF}(x^1) \geq \frac{1}{n-1}$.

($C_2$) Acceptable decision stability: $x^1$ is also the best solution by $s^*_{HF}$, $r^*_{HF}$.

If one of the above conditions cannot be met, then a set of compromise solutions can be gotten as follows:

(1) If $C_2$ cannot be satisfied, then we can take $x^1$, $x^2$.

(2) If $C_1$ cannot be satisfied, then we can take $x^1, x^2, \ldots, x^m$, and $m$ is the maximal value satisfying the condition $q^*_{HF}(x^m) - q^*_{HF}(x^1) < \frac{1}{n-1}$.

**Remark 4.1.** *By combining the VIKOR method and CHFRS model, the proposed MCGDM method possesses many advantages: (1) the obtained compromise solutions consider the maximal group utility and minimal individual regret simultaneously, thereby making the decision results more reliable; (2) the CHFRS model is used to obtain three pairs of lower and upper approximations value, which can depict the ranking information comprehensively by the aid of the score function and fuzzy logical operator. All these help our method adapt to various uncertain situations.*

**Remark 4.2.** *The lower approximation (objects definitely belonging to a certain class) and upper approximation (objects possibly belonging to a certain class) in CHFRS model can correspond to the "most conservative" and "most aggressive" compromise solutions in VIKOR, respectively, generating multiple compromise solutions rather than a single solution. And the compromise solutions also change with the value of $\varphi$, which can be further demonstrated in Table 13.*

### 4.4. **Algorithm for VIKOR-CHFRS.**

**Remark 4.3.** *According to the above problem description, there involves $n$ alternatives, $m$ criteria and $t$ decision-makers. Suppose the maximum length of each hesitant fuzzy information of research proposal is $l$. The overall complexity of Algorithm 1 is the synthesis of each step. The complexity of Step 1, Step 2 and Step 3 is $O(lmnt)$, respectively. Step 4 computes the lower and upper approximation with the complexity of $O(lmn^2t)$. Step 5 computes the consensus measure of each expert with the complexity of $O(lmnt)$ and the weights with $O(t)$. The complexity of Step 6, and Step 7 is $O(lmnt)$, respectively. Step 8 computes the overall value with the complexity $O(1)$; Step 9 ranks all the alternatives with the complexity of $O(n^2)$, Step 4 judges if the required condition is satisfied with $O(1)$. Therefore, the complexity of Algorithm 1 is $O(lmn^2t)$.*

**Algorithm 1** Implementation of VIKOR-CHFRS

**Input:** $t$ HF decision matrix $D(k)$, $k = 1, \ldots, t$, preference value $\phi$, $\beta$, and criteria weights $w = \{w_1, \ldots, w_m\}$.

**Output:** Rank all the research proposals.

1: Decide the best and the worst HF ideal research proposal $x^+_{HF}$, $x^-_{HF}$, respectively;

2: Compute the HF group utility $s^k_{HF}$, the HF individual regret $r^k_{HF}$ and the HF compromise value $q^k_{HF}$, respectively;

3: Determine the $\tilde{H}^\beta_x$;

4: Compute the lower and upper approximations $\underline{C^H_\beta}\left(s^k_{HF}\right)$, $\overline{C^H_\beta}\left(s^k_{HF}\right)$, $\underline{C^H_\beta}\left(r^k_{HF}\right)$, $\overline{C^H_\beta}\left(r^k_{HF}\right)$, $\underline{C^H_\beta}\left(q^k_{HF}\right)$, $\overline{C^H_\beta}\left(q^k_{HF}\right)$, respectively;

5: Compute the expert's weight $\omega = \{\omega_1, \ldots, \omega_t\}$;

6: Synthesize the lower and upper approximations using the weight average operator to obtain $\underline{C^H_\beta}(s_{HF})$, $\overline{C^H_\beta}(s_{HF})$, $\underline{C^H_\beta}(r_{HF})$, $\overline{C^H_\beta}(r_{HF})$, $\underline{C^H_\beta}(q_{HF})$, $\overline{C^H_\beta}(q_{HF})$;

7: Compute the score value of $SC\left(\underline{C^H_\beta}(s_{HF})\right)$, $SC\left(\overline{C^H_\beta}(s_{HF})\right)$, $SC\left(\underline{C^H_\beta}(r_{HF})\right)$, $SC\left(\overline{C^H_\beta}(r_{HF})\right)$, $SC\left(\underline{C^H_\beta}(q_{HF})\right)$, $SC\left(\overline{C^H_\beta}(q_{HF})\right)$;

8: Compute three overall fuzzy value $s^*_{HF}$, $r^*_{HF}$, $q^*_{HF}$;

9: Rank all the research proposals according to $s^*_{HF}$, $r^*_{HF}$, $q^*_{HF}$ in increasing order;

10: Obtain the optimal proposal based on $\mathrm{C}_1$ and $\mathrm{C}_2$.

## 5. An Illustrative Example.

### 5.1. The evaluation data description of research proposals.

The university always makes a budget for the research funding, which is used to support the excellent research plans. On the one hand, it can help the university to raise the scientific research level; on the other hand, it can help the teachers to further conduct their researches. Therefore, the selection of some excellent proposals is very vital to the managers of the research management departments.

Following will give some data on the collected research proposals of a university in May 2024. There are 10 research proposals $U = \{x_1, \ldots, x_{10}\}$ applying for funding support. To select the outstanding proposals, three experts $d = \{d_1, d_2, d_3\}$ with rich experiences in project review are invited to assess the quality of the research proposals from the aforesaid 6 aspects. In this paper, suppose the subjective weight of three experts is the same. And the weight of each criterion is given by the management departments as $w = \{0.35, 0.25, 0.2, 0.1, 0.05, 0.05\}$. The evaluation values are presented by HFEs, indicating the possible membership degree of a proposal to a criterion. The following Tables 3-5 show the HF evaluation information of the 10 proposals.

Here, we give $\beta = 0.6$ and suppose that a proposal should be necessarily good on at least one criterion, i.e., for any proposal $x_i$, there is a criterion $c_j$ such that $\min\left(c^k_j(x_i)\right) \geq \beta$ (for $\min\left(c^1_5(x_1)\right) = 0.7 \geq \beta = 0.6$); otherwise, the proposal will be eliminated in the first evaluation. Then, $C = \{c_1, \ldots, c_6\}$ constructs an HF $\beta$-covering of $U$. Therefore, the proposal selection problem is an MCGDM in an HFCAS. Next, we will employ VIKOR-CHFRS to settle the proposal selection problem.

### 5.2. Data analysis based on VIKOR-CHFRS method.

Firstly, for $d_1$, the best and the worst HF ideal research proposals are decided as

$$x^{1+}_{HF} = \{\{0.8, 0.9, 0.9\}, \{0.7, 0.9, 0.9\}, \{0.7, 0.8, 0.9\}, \{0.6, 0.7, 0.9\}, \{0.7, 0.8, 0.9\}, \{0.6, 0.8, 0.9\}\},$$
$$x^{1-}_{HF} = \{\{0.1, 0.3, 0.6\}, \{0.1, 0.2, 0.3\}, \{0.1, 0.3, 0.5\}, \{0.1, 0.3, 0.6\}, \{0.1, 0.2, 0.4\}, \{0.1, 0.2, 0.4\}\}.$$

TABLE 3. HF evaluation information of 10 proposals by $d_1$

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | $\{.1,.4,.7\}$ | $\{.1,.3,.6\}$ | $\{.3,.4,.7\}$ | $\{.6,.7,.8\}$ | $\{.7,.8,.9\}$ | $\{.2,.3,.5\}$ |
| $x_2$ | $\{.3,.7,.9\}$ | $\{.1,.4,.9\}$ | $\{.4,.6,.9\}$ | $\{.3,.5,.9\}$ | $\{.1,.2,.5\}$ | $\{.6,.7,.8\}$ |
| $x_3$ | $\{.1,.3,.6\}$ | $\{.1,.5,.6\}$ | $\{.7,.8,.9\}$ | $\{.4,.7,.8\}$ | $\{.1,.2,.7\}$ | $\{.2,.4,.5\}$ |
| $x_4$ | $\{.5,.6,.7\}$ | $\{.1,.3,.4\}$ | $\{.1,.4,.5\}$ | $\{.1,.3,.6\}$ | $\{.7,.8,.9\}$ | $\{.2,.3,.5\}$ |
| $x_5$ | $\{.4,.5,.6\}$ | $\{.7,.8,.8\}$ | $\{.3,.5,.9\}$ | $\{.3,.7,.8\}$ | $\{.1,.2,.4\}$ | $\{.1,.2,.4\}$ |
| $x_6$ | $\{.2,.4,.7\}$ | $\{.1,.2,.3\}$ | $\{.1,.3,.5\}$ | $\{.6,.7,.7\}$ | $\{.7,.8,.9\}$ | $\{.1,.6,.7\}$ |
| $x_7$ | $\{.4,.6,.8\}$ | $\{.5,.6,.7\}$ | $\{.1,.5,.9\}$ | $\{.3,.4,.7\}$ | $\{.2,.4,.5\}$ | $\{.6,.8,.9\}$ |
| $x_8$ | $\{.8,.9,.9\}$ | $\{.2,.4,.7\}$ | $\{.7,.8,.9\}$ | $\{.5,.6,.7\}$ | $\{.6,.8,.8\}$ | $\{.5,.7,.7\}$ |
| $x_9$ | $\{.1,.3,.8\}$ | $\{.6,.9,.9\}$ | $\{.2,.3,.6\}$ | $\{.5,.7,.8\}$ | $\{.7,.8,.9\}$ | $\{.4,.7,.9\}$ |
| $x_{10}$ | $\{.2,.5,.6\}$ | $\{.2,.3,.8\}$ | $\{.4,.6,.6\}$ | $\{.1,.5,.8\}$ | $\{.6,.7,.8\}$ | $\{.1,.6,.9\}$ |

TABLE 4. HF evaluation information of 10 proposals by $d_2$

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | $\{.1,.3,.3\}$ | $\{.2,.3,.6\}$ | $\{.3,.4,.7\}$ | $\{.6,.7,.8\}$ | $\{.7,.8,.9\}$ | $\{.3,.5,.5\}$ |
| $x_2$ | $\{.3,.7,.9\}$ | $\{.1,.3,.4\}$ | $\{.4,.6,.9\}$ | $\{.3,.5,.9\}$ | $\{.1,.2,.5\}$ | $\{.7,.8,.9\}$ |
| $x_3$ | $\{.2,.3,.5\}$ | $\{.1,.5,.6\}$ | $\{.7,.8,.9\}$ | $\{.4,.7,.8\}$ | $\{.1,.2,.7\}$ | $\{.2,.4,.5\}$ |
| $x_4$ | $\{.5,.6,.7\}$ | $\{.1,.3,.4\}$ | $\{.1,.4,.5\}$ | $\{.1,.3,.6\}$ | $\{.7,.8,.9\}$ | $\{.2,.3,.5\}$ |
| $x_5$ | $\{.6,.7,.9\}$ | $\{.7,.8,.8\}$ | $\{.3,.5,.9\}$ | $\{.6,.7,.9\}$ | $\{.3,.5,.6\}$ | $\{.1,.2,.4\}$ |
| $x_6$ | $\{.3,.5,.8\}$ | $\{.1,.2,.3\}$ | $\{.3,.4,.5\}$ | $\{.7,.7,.7\}$ | $\{.7,.8,.9\}$ | $\{.4,.6,.7\}$ |
| $x_7$ | $\{.4,.6,.8\}$ | $\{.6,.7,.8\}$ | $\{.5,.6,.7\}$ | $\{.5,.6,.7\}$ | $\{.2,.4,.5\}$ | $\{.6,.8,.9\}$ |
| $x_8$ | $\{.8,.9,.9\}$ | $\{.1,.4,.7\}$ | $\{.7,.8,.9\}$ | $\{.5,.6,.7\}$ | $\{.5,.6,.8\}$ | $\{.5,.7,.7\}$ |
| $x_9$ | $\{.2,.3,.7\}$ | $\{.6,.9,.9\}$ | $\{.2,.3,.6\}$ | $\{.5,.7,.8\}$ | $\{.7,.8,.9\}$ | $\{.4,.7,.9\}$ |
| $x_{10}$ | $\{.3,.5,.6\}$ | $\{.4,.7,.8\}$ | $\{.4,.6,.8\}$ | $\{.1,.5,.8\}$ | $\{.6,.7,.8\}$ | $\{.3,.6,.8\}$ |

TABLE 5. HF evaluation information of 10 proposals by $d_3$

| | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
|---|---|---|---|---|---|---|
| $x_1$ | $\{.4,.6,.7\}$ | $\{.1,.3,.6\}$ | $\{.6,.7,.8\}$ | $\{.3,.4,.7\}$ | $\{.7,.8,.9\}$ | $\{.3,.5,.5\}$ |
| $x_2$ | $\{.6,.7,.9\}$ | $\{.1,.4,.9\}$ | $\{.4,.6,.9\}$ | $\{.2,.5,.6\}$ | $\{.1,.2,.5\}$ | $\{.6,.7,.8\}$ |
| $x_3$ | $\{.7,.8,.9\}$ | $\{.1,.5,.6\}$ | $\{.1,.3,.6\}$ | $\{.4,.7,.8\}$ | $\{.2,.5,.6\}$ | $\{.2,.4,.4\}$ |
| $x_4$ | $\{.5,.6,.7\}$ | $\{.1,.3,.4\}$ | $\{.3,.4,.5\}$ | $\{.1,.3,.6\}$ | $\{.7,.8,.9\}$ | $\{.2,.3,.5\}$ |
| $x_5$ | $\{.4,.5,.6\}$ | $\{.7,.8,.8\}$ | $\{.3,.5,.9\}$ | $\{.3,.7,.8\}$ | $\{.1,.2,.4\}$ | $\{.1,.2,.4\}$ |
| $x_6$ | $\{.2,.4,.4\}$ | $\{.6,.7,.7\}$ | $\{.1,.3,.5\}$ | $\{.1,.2,.3\}$ | $\{.7,.8,.9\}$ | $\{.1,.6,.7\}$ |
| $x_7$ | $\{.4,.6,.8\}$ | $\{.6,.7,.9\}$ | $\{.4,.5,.7\}$ | $\{.3,.4,.7\}$ | $\{.2,.4,.4\}$ | $\{.6,.8,.9\}$ |
| $x_8$ | $\{.8,.9,.9\}$ | $\{.5,.6,.7\}$ | $\{.7,.8,.9\}$ | $\{.5,.6,.7\}$ | $\{.6,.8,.8\}$ | $\{.6,.7,.8\}$ |
| $x_9$ | $\{.5,.6,.8\}$ | $\{.3,.3,.9\}$ | $\{.2,.3,.6\}$ | $\{.7,.8,.9\}$ | $\{.5,.7,.8\}$ | $\{.1,.5,.8\}$ |
| $x_{10}$ | $\{.2,.5,.6\}$ | $\{.2,.6,.8\}$ | $\{.5,.6,.9\}$ | $\{.3,.5,.7\}$ | $\{.6,.7,.8\}$ | $\{.4,.6,.9\}$ |

Secondly, we compute the HF group utility $s^1_{HF}\left(x^1_i\right)$, the HF individual regret $r^1_{HF}\left(x^1_i\right)$, the HF compromise value $q^1_{HF}\left(x^1_i\right)$ for every research proposal which are indicated in Table 6.

Thirdly, for $\beta = 0.6$, we can get the $\tilde{H}^{0.6}_{x_i}$ of $x_i$ $(i = 1, 2, \ldots, 10)$. To save space, we only list $\tilde{H}^{0.6}_{x_1}$.

$$\begin{aligned}\tilde{H}^{0.6}_{x_1} =&\{\{0.6, 0.7, 0.8\}, \{0.1, 0.2, 0.5\}, \{0.1, 0.2, 0.7\}, \{0.1, 0.3, 0.6\}, \{0.1, 0.2, 0.4\},\\ &\{0.6, 0.7, 0.7\}, \{0.2, 0.4, 0.5\}, \{0.5, 0.6, 0.7\}, \{0.5, 0.7, 0.8\}, \{0.1, 0.5, 0.8\}\}.\end{aligned}$$

Table 6. The values of $s_{HF}^1(x_i^1)$, $r_{HF}^1(x_i^1)$, $q_{HF}^1(x_i^1)$

| | $s_{HF}^1(x_i^1)$ | $r_{HF}^1(x_i^1)$ | $q_{HF}^1(x_i^1)$ |
|---|---|---|---|
| $x_1$ | $\{.4983, .6793, .8217\}$ | $\{.2333, .2917, .3500\}$ | $\{.7190, .8024, .8511\}$ |
| $x_2$ | $\{.0400, .5133, .6843\}$ | $\{.0400, .2083, .2500\}$ | $\{.0, .3322, .4050\}$ |
| $x_3$ | $\{.5283, .6317, .6943\}$ | $\{.3500, .3500, .3500\}$ | $\{.8681, .8741, .8777\}$ |
| $x_4$ | $\{.6583, .7293, .8250\}$ | $\{.2083, .2143, .2500\}$ | $\{.5971, .6124, .7617\}$ |
| $x_5$ | $\{.3000, .5140, .6850\}$ | $\{.2000, .2333, .3500\}$ | $\{.4389, .5102, .7389\}$ |
| $x_6$ | $\{.7033, .7617, .8317\}$ | $\{.2500, .2917, .3000\}$ | $\{.8295, .8334, .8387\}$ |
| $x_7$ | $\{.2933, .4850, .6238\}$ | $\{.1167, .1750, .2000\}$ | $\{.2264, .2383, .3879\}$ |
| $x_8$ | $\{.1183, .2302, .2733\}$ | $\{.0833, .1786, .2083\}$ | $\{.0120, .0237, .1289\}$ |
| $x_9$ | $\{.2667, .5467, .6274\}$ | $\{.1500, .3500, .3500\}$ | $\{.4955, .6336, .7098\}$ |
| $x_{10}$ | $\{.4167, .6867, .8360\}$ | $\{.2333, .3000, .3500\}$ | $\{.6661, .8223, .8527\}$ |

Thus, by using the CHFRS model, we can compute the lower approximation $\underline{C_\beta^H}(s_{HF}^1)$, $\underline{C_\beta^H}(r_{HF}^1)$, $\underline{C_\beta^H}(q_{HF}^1)$, and the upper approximation $\overline{C_\beta^H}(s_{HF}^1)$, $\overline{C_\beta^H}(r_{HF}^1)$, $\overline{C_\beta^H}(q_{HF}^1)$, respectively, which are listed as Tables 7 and 8.

Table 7. The lower approximation of $s_{HF}^1$, $r_{HF}^1$, $q_{HF}^1$

| | $h_{\underline{C_\beta^H}(s_{HF}^1)}$ | $h_{\underline{C_\beta^H}(r_{HF}^1)}$ | $h_{\underline{C_\beta^H}(q_{HF}^1)}$ |
|---|---|---|---|
| $x_1$ | $\{.2667, .4000, .5000\}$ | $\{.2000, .3000, .4000\}$ | $\{.3000, .4000, .5000\}$ |
| $x_2$ | $\{.2000, .3000, .5000\}$ | $\{.1167, .2000, .4000\}$ | $\{.2000, .2383, .4000\}$ |
| $x_3$ | $\{.1000, .2302, .3000\}$ | $\{.1000, .2000, .3000\}$ | $\{.1000, .2000, .3000\}$ |
| $x_4$ | $\{.2000, .2302, .4000\}$ | $\{.1500, .2000, .3000\}$ | $\{.2000, .2000, .4000\}$ |
| $x_5$ | $\{.1000, .4850, .6238\}$ | $\{.1000, .2333, .3500\}$ | $\{.1000, .4000, .5000\}$ |
| $x_6$ | $\{.2667, .4000, .5000\}$ | $\{.2000, .3000, .4000\}$ | $\{.3000, .4000, .5000\}$ |
| $x_7$ | $\{.2000, .3000, .5000\}$ | $\{.1167, .2000, .4000\}$ | $\{.2000, .2383, .4000\}$ |
| $x_8$ | $\{.2000, .2302, .4000\}$ | $\{.2000, .2000, .4000\}$ | $\{.2000, .2000, .4000\}$ |
| $x_9$ | $\{.2667, .5467, .6274\}$ | $\{.1500, .3500, .4000\}$ | $\{.3000, .6000, .7098\}$ |
| $x_{10}$ | $\{.2000, .2302, .4000\}$ | $\{.1500, .2000, .3000\}$ | $\{.2000, .2000, .4000\}$ |

Table 8. The upper approximation of $s_{HF}^1$, $r_{HF}^1$, $q_{HF}^1$

| | $h_{\overline{C_\beta^H}(s_{HF}^1)}$ | $h_{\overline{C_\beta^H}(r_{HF}^1)}$ | $h_{\overline{C_\beta^H}(q_{HF}^1)}$ |
|---|---|---|---|
| $x_1$ | $\{.6000, .7000, .8000\}$ | $\{.2500, .3500, .3500\}$ | $\{.6000, .7000, .8000\}$ |
| $x_2$ | $\{.2933, .6000, .8360\}$ | $\{.2000, .3500, .3500\}$ | $\{.4000, .6336, .8527\}$ |
| $x_3$ | $\{.5283, .6317, .7000\}$ | $\{.3500, .3500, .3500\}$ | $\{.7000, .8000, .8777\}$ |
| $x_4$ | $\{.7000, .7617, .8317\}$ | $\{.2500, .3500, .3500\}$ | $\{.7000, .8000, .8511\}$ |
| $x_5$ | $\{.3000, .5467, .8000\}$ | $\{.2000, .3500, .3500\}$ | $\{.4955, .6336, .8000\}$ |
| $x_6$ | $\{.6000, .7000, .8000\}$ | $\{.2500, .3500, .3500\}$ | $\{.6000, .7000, .8000\}$ |
| $x_7$ | $\{.2933, .6000, .8360\}$ | $\{.2000, .3500, .3500\}$ | $\{.4000, .6336, .8527\}$ |
| $x_8$ | $\{.2000, .5000, .7000\}$ | $\{.2000, .3000, .3500\}$ | $\{.2000, .5000, .7000\}$ |
| $x_9$ | $\{.2667, .5467, .8000\}$ | $\{.2000, .3500, .3500\}$ | $\{.4955, .6336, .8000\}$ |
| $x_{10}$ | $\{.7000, .7617, .8317\}$ | $\{.2500, .3500, .3500\}$ | $\{.7000, .8000, .8511\}$ |

Similarly, for $d_2$, $d_3$, the lower and upper approximation can be obtained, respectively. Here, we omit the results.

Fourthly, compute the expert's weight as $\omega_1 = 0.2737$, $\omega_2 = 0.3211$, $\omega_3 = 0.4052$; then synthesize the three lower and upper approximations as Tables 9 and 10.

TABLE 9. Synthesizing the lower approximation

| | $h_{\underline{C_\beta^H}(s_{HF})}$ | $h_{\underline{C_\beta^H}(r_{HF})}$ | $h_{\underline{C_\beta^H}(q_{HF})}$ |
|---|---|---|---|
| $x_1$ | $\{.2397, .3190, .4595\}$ | $\{.2000, .2595, .3679\}$ | $\{.2595, .3190, .4595\}$ |
| $x_2$ | $\{.2107, .3000, .4530\}$ | $\{.1451, .2405, .3679\}$ | $\{.1814, .2749, .4000\}$ |
| $x_3$ | $\{.1000, .2121, .2595\}$ | $\{.1000, .1621, .2595\}$ | $\{.1000, .1595, .2595\}$ |
| $x_4$ | $\{.2000, .2725, .4321\}$ | $\{.1703, .2046, .3000\}$ | $\{.2000, .2642, .4321\}$ |
| $x_5$ | $\{.1321, .4037, .5018\}$ | $\{.1321, .2736, .3661\}$ | $\{.1321, .3679, .4679\}$ |
| $x_6$ | $\{.2741, .4000, .5000\}$ | $\{.2000, .3203, .3679\}$ | $\{.3000, .4000, .5000\}$ |
| $x_7$ | $\{.2107, .3635, .4935\}$ | $\{.1274, .2726, .4000\}$ | $\{.2000, .3236, .4405\}$ |
| $x_8$ | $\{.1679, .2729, .3679\}$ | $\{.1679, .2432, .3679\}$ | $\{.1679, .2405, .3679\}$ |
| $x_9$ | $\{.2741, .4632, .5679\}$ | $\{.1500, .2732, .3595\}$ | $\{.3000, .4869, .5895\}$ |
| $x_{10}$ | $\{.2000, .2725, .4321\}$ | $\{.1703, .2046, .3000\}$ | $\{.2000, .2642, .4321\}$ |

TABLE 10. Synthesizing the upper approximation

| | $h_{\overline{C_\beta^H}(s_{HF})}$ | $h_{\overline{C_\beta^H}(r_{HF})}$ | $h_{\overline{C_\beta^H}(q_{HF})}$ |
|---|---|---|---|
| $x_1$ | $\{.5419, .6676, .7681\}$ | $\{.2943, .3500, .3500\}$ | $\{.6223, .6909, .8000\}$ |
| $x_2$ | $\{.3303, .5615, .7426\}$ | $\{.2362, .3500, .3500\}$ | $\{.3713, .5687, .7438\}$ |
| $x_3$ | $\{.4810, .6028, .7054\}$ | $\{.2934, .3500, .3500\}$ | $\{.6157, .7031, .7764\}$ |
| $x_4$ | $\{.6550, .7478, .8132\}$ | $\{.2943, .3500, .3500\}$ | $\{.7000, .8000, .8866\}$ |
| $x_5$ | $\{.4001, .5847, .7236\}$ | $\{.1974, .3339, .3500\}$ | $\{.4430, .6177, .7679\}$ |
| $x_6$ | $\{.6000, .7000, .7681\}$ | $\{.2753, .3500, .3500\}$ | $\{.6223, .7000, .8000\}$ |
| $x_7$ | $\{.2914, .5679, .7334\}$ | $\{.2000, .3339, .3500\}$ | $\{.3668, .6070, .7843\}$ |
| $x_8$ | $\{.2726, .5035, .6648\}$ | $\{.2201, .3203, .3500\}$ | $\{.2726, .5000, .6418\}$ |
| $x_9$ | $\{.3272, .5459, .7094\}$ | $\{.2324, .3339, .3500\}$ | $\{.4377, .6475, .7699\}$ |
| $x_{10}$ | $\{.6550, .7478, .8132\}$ | $\{.2943, .3500, .3500\}$ | $\{.7000, .8000, .8866\}$ |

Fifthly, we compute the overall score of group utility fuzzy value $s^*_{HF}$, individual regret fuzzy value $r^*_{HF}$ and compromise fuzzy value $q^*_{HF}$ shown as Table 11.

TABLE 11. The overall score of $s^*_{HF}$, $r^*_{HF}$ and $q^*_{HF}$

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_9$ | $x_{10}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $s^*_{HF}$ | .7749 | .6910 | .6733 | .8175 | .7184 | .8109 | .6979 | .6204 | .7331 | .8175 |
| $r^*_{HF}$ | .5158 | .4848 | .4474 | .4818 | .4755 | .5249 | .4827 | .4794 | .4866 | .4818 |
| $q^*_{HF}$ | .8067 | .6865 | .7506 | .8566 | .7355 | .8245 | .7191 | .6082 | .7935 | .8566 |

Finally, we rank the 10 research proposals by $s^*_{HF}$, $r^*_{HF}$ and $q^*_{HF}$ in increasing order shown as Table 12.

From Table 12, the ranking result of the 10 research proposals is $x_8 \succ x_2 \succ x_7 \succ x_5 \succ x_3 \succ x_9 \succ x_1 \succ x_6 \succ x_4 \approx x_{10}$ by $q^*_{HF}$. Because the conditions $C_1$ and $C_2$ cannot be satisfied, and $q^*_{HF}(x_2) - q^*_{HF}(x_8) = 0.0786 < 1/(10-1) = 0.1111$, if the university wants to

TABLE 12. The rankings of 10 research proposals

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $s^*_{HF}$ | $x_8$ | $x_3$ | $x_2$ | $x_7$ | $x_5$ | $x_9$ | $x_1$ | $x_6$ | $x_4$ | $x_{10}$ |
| $r^*_{HF}$ | $x_3$ | $x_5$ | $x_8$ | $x_4$ | $x_{10}$ | $x_7$ | $x_2$ | $x_9$ | $x_1$ | $x_6$ |
| $q^*_{HF}$ | $x_8$ | $x_2$ | $x_7$ | $x_5$ | $x_3$ | $x_9$ | $x_1$ | $x_6$ | $x_4$ | $x_{10}$ |

select one or two of 10 proposals to support, we can get that the decision-making results are $x_8$, $x_2$.

5.3. **Decision results with different $\varphi$.** In designing VIKOR-CHFRS method, we know that the parameter $\varphi \in [0, 1]$ is a crucial index. In what follows, we will further discuss the effect of $\varphi$ on the ranking and the optimal solution (compromise solution) of research proposals. To save the space, the concrete computation steps are omitted. The ranking results on different $\varphi$ are shown in Table 13 and Figure 1.

TABLE 13. The ranking results for different $\varphi$

| $\varphi$ | Ranking results | Decision-making results |
|---|---|---|
| $\varphi = 0$ | $x_8 \succ x_2 \succ x_7 \succ x_5 \succ x_3 \succ x_1 \succ x_6 \succ x_9 \succ x_4 \approx x_{10}$ | $x_8, x_2$ |
| $\varphi = 0.3$ | $x_8 \succ x_2 \succ x_7 \succ x_5 \succ x_3 \succ x_1 \succ x_9 \succ x_6 \succ x_4 \approx x_{10}$ | $x_8, x_2$ |
| $\varphi = 0.5$ | $x_8 \succ x_2 \succ x_7 \succ x_5 \succ x_3 \succ x_9 \succ x_1 \succ x_6 \succ x_4 \approx x_{10}$ | $x_8, x_2$ |
| $\varphi = 0.6$ | $x_8 \succ x_2 \succ x_7 \succ x_5 \succ x_3 \succ x_9 \succ x_1 \succ x_6 \succ x_4 \approx x_{10}$ | $x_8, x_2$ |
| $\varphi = 0.9$ | $x_8 \succ x_2 \succ x_7 \succ x_5 \succ x_3 \succ x_9 \succ x_1 \succ x_6 \succ x_4 \approx x_{10}$ | $x_8, x_2$ |
| $\varphi = 1$ | $x_8 \succ x_7 \succ x_5 \succ x_2 \succ x_3 \succ x_9 \succ x_1 \succ x_6 \succ x_4 \approx x_{10}$ | $x_8, x_7, x_5, x_2$ |

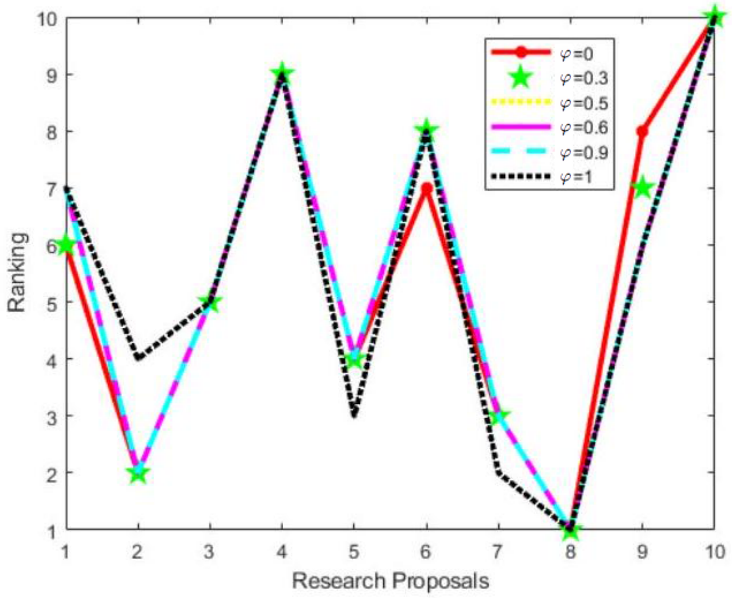


FIGURE 1. The ranking results for different $\varphi$

From Table 13 and Figure 1, we know that there exist some discrepancies on the ranking results based on six $\varphi$, the compromise solutions are all $x_8$, $x_2$ for $\varphi = 0, 0.3, 0.5, 0.6, 0.9$, and the compromise solutions are $x_8$, $x_7$, $x_5$, $x_2$ for $\varphi = 1$. The above analysis shows that $\varphi$ can affect the ranking result to some degree, which indicates our method can embody the subjective consciousness of decision-makers. In real applications, the decision-makers can flexibly choose an appropriate $\varphi$ according to the practical needs. Moreover, Table 13 and Figure 1 tell us that the decision-making result is relatively stable, which verifies that VIKOR-CHFRS method is effective. Therefore, we can take the advantage of VIKOR-CHFRS to solve numerous decision problems under HF environment.

## 6. **Experimental Analysis.**

6.1. **Comparison analysis.** This section will further verify the validity of VIKOR-CHFRS method, and a series of comparative studies with the four existing methods will be conducted on the basis of the examples in Section 5. We will first introduce the four methods.

(1) *The HFA (aggregation) operator method* [20, 22, 23]. The methods mainly developed some HF aggregation operators to integrate the HF information from different fields experts to derive the ranking of all the alternatives. The common used aggregation operators include the HF weighted averaging (HFWA) operator and the HF averaging (HFA) operator [22]. To facilitate the computation, we will use the HF aggregation operators of Definition 2.4 in [23] in this article.

The GDM methods based on the aggregation operator is as follows. Firstly, for every fields expert, aggregate the every research proposal's HF evaluation information on different criteria into an overall HFE using HFWA operator. Secondly, for every research proposal, aggregate the above obtained HFEs from three fields experts into a new overall HFE using HFWA operator. Then, the ranking result of all research proposals can be obtained using the score function of HFEs.

(2) *The HF TOPSIS method* [24, 50, 51]. The methods mainly used the distance between every alternative and the HF positive and negative ideal solutions, by which obtaining the relative closeness coefficient to rank all the alternatives. In this article, firstly, we utilize the TOPSIS method in [48] to determine the individual relative closeness coefficient for every decision-maker. Then, we obtain the group relative closeness coefficient by synthesizing the individual values using weighted average operator to rank all the alternatives.

(3) *The HF VIKOR method* [24, 25]. The method is efficient in dealing with MAGDM with HF conflicting criteria information, which uses the "closeness" measure to the ideal solution to rank all the alternatives. In this article, we firstly use the HF-VIKOR method in [25] to determine HF group utility measure, HF individual regret measure and HF compromise measure from every decision-maker. Then, we synthesize the above three HF measures using HFWA to rank all the alternatives. For the HF Manhattan $l_p$-metric in [25], we set $p = 2$, and the strategy weight of maximizing the group utility $\varphi = 0.5$.

(4) *Zhou and Li's method* [39]. This method is an MCDM on HF $\beta$ CRSs (HF $\beta$-covering rough sets). Firstly, we obtain HF $\beta$-neighborhood system, and establish HFRS model. Next, synthesize the HF lower and upper approximations by means of HFA operator of every alternative. Finally, the ranking results are determined by the score function. In this paper, the involved criteria are all benefit type, so we take the maximal value of every alternative on the criteria to form a standard HF set, and set $\beta = \{0.6, 0.7, 0.8\}$. Based on the established HF $\beta$ CRSs, we integrate the score function from three decision-makers.

According to the above stated descriptions and analysis, the results of VIKOR-CHFRS method and the other four methods are achieved in Table 14 and Figure 2.

From Table 14 and Figure 2, we obtain that the results of the VIKOR-CHFRS method are consistent with the existing methods except for Zhou and Li's method. Besides, the three methods (HFA, HF-TOPSIS and HF-VIKOR) are recognized as effective in many decision-making theories and applications. Therefore, the comparison results illustrate

Table 14. The ranking results based on different methods

| Methods | Ranking results |
|---|---|
| HFA | $x_8 \succ x_5 \succ x_7 \succ x_2 \succ x_9 \succ x_{10} \succ x_3 \succ x_1 \succ x_4 \succ x_6$ |
| HF-TOPSIS | $x_8 \succ x_5 \succ x_7 \succ x_9 \succ x_2 \succ x_{10} \succ x_3 \succ x_1 \succ x_4 \succ x_6$ |
| HF-VIKOR | $x_8 \succ x_7 \succ x_2 \succ x_5 \succ x_9 \succ x_4 \succ x_{10} \succ x_3 \succ x_1 \succ x_6$ |
| Zhou and Li's | $x_3 \succ x_{10} \succ x_6 \succ x_4 \succ x_9 \succ x_5 \succ x_2 \succ x_8 \succ x_7 \succ x_1$ |
| VIKOR-CHFRS | $x_8 \succ x_2 \succ x_7 \succ x_5 \succ x_3 \succ x_9 \succ x_1 \succ x_6 \succ x_4 \approx x_{10}$ |

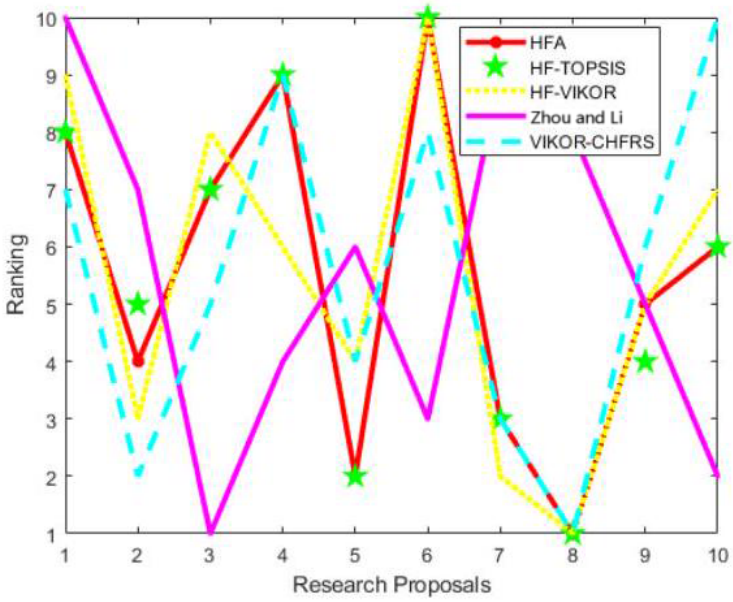


Figure 2. The ranking results based on different methods

that the proposed VIKOR-CHFRS method is sound and effective. Meanwhile, Zhou and Li's decision-making result is largely dependent on the obtained HFS, which will directly lead to the difference with the other methods. This indicates Zhou and Li's method is not suitable for the alternatives selection problems.

Although the obtained ranking results by different methods show some differences, this situation is quite normal in decision-making fields. In real decision-making application, we can select an appropriate method based on the practical requirements and our own decision-making preferences.

6.2. **Correlation analysis.** In this section, we will use the Spearman's correlation coefficient as follows [52, 53] to consider the similarity of the ranking methods.

$$SCC = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n\left(n^2-1\right)}$$

where $d_i$ denotes the ranking position difference of the $i$th alternative based on different ranking methods and $n$ denotes the number of alternatives.

Generally, the greater the value of $SCC$, the higher the similarity of two ranking methods. Specially, it can be recognized that the considered two ranking methods have strong consistency when $SCC \geq 0.8$. The computation results are shown as Table 15.

From Table 15, the correlation coefficients between our method and the classical HFA, HF-TOPSIS and HF-VIKOR are greater than 0.8. These indicate that our method is highly correlated with the classical methods. Moreover, the classical methods have been

TABLE 15. The Spearman correlation coefficient between different methods

| | HFA | HF-TOPSIS | HF-VIKOR | Zhou and Li's | VIKOR-CHFRS |
|---|---|---|---|---|---|
| HFA | 1 | | | | |
| HF-TOPSIS | 0.9879 | 1 | | | |
| HF-VIKOR | 0.8909 | 0.8242 | 1 | | |
| Zhou and Li's | −0.6242 | −0.6 | −0.3939 | 1 | |
| VIKOR-CHFRS | 0.8364 | 0.8182 | 0.8 | −0.4485 | 1 |

proved effective, and these indicate that VIKOR-CHFRS is efficient and feasible. At the same time, Zhou and Li's method has not positive correlation with the classical methods and ours. Thus, we conclude that our method is superior to Zhou and Li's method.

6.3. **Discussions.** Synthesize the previous analysis, we know that the VIKOR-CHFRS method has some merits.

(1) Using the idea of VIKOR method, VIKOR-CHFRS can embody the subjective preference of decision-makers through adjusting parameter $\varphi$ compared with HFA, HF-TOPSIS methods. Moreover, VIKOR-CHFRS inherits the merits of CHFRS which can describe the uncertainty in research proposals evaluation. Therefore, our method is more flexible and practical than VIKOR especially for the decision problems with the evaluation information forming an HF $\beta$-covering.

(2) Considering that the evaluation information on some criteria of every proposal must be above a given minimum value in real selection process, and we reduce the hesitant fuzzy threshold $\beta$ of Zhou and Li's method into a real value. This not only makes the computation easier, but also makes our method more in line with the practical case. Therefore, VIKOR-CHFRS is more general and more applicable.

(3) The expert weights based on the defined consensus measure can effectively reduce the negative influence of inconsistency during group preference aggregation so as to improve the reliability of decision-making.

(4) VIKOR-CHFRS improves the existing methods and it can be suitable for many situations by selecting various HFA operator and fuzzy logical operator.

7. **Conclusion.** In this article, we develop a VIKOR-CHFRS method to MCGDM problems. This method contains two parts: CHFRS construction process, and VIKOR-improving process. During the CHFRS construction process, an HF $\beta$-neighborhood is presented to measure the relationship between objects, and a novel CHFRS is proposed to tackle the hesitancy and vagueness in MCGDM. In the VIKOR-improving process, through combining VIKOR with CHFRS, an algorithm is designed to solve the ranking of alternatives in the case of research proposals evaluation problems. Among them, HFA operator is applied to fusing the HF information on every criterion and every decision-maker together with different weights.

The method's main novelties are presented as follows. (1) VIKOR-CHFRS can address MCGDM problems with HF evaluation information forming an HF $\beta$-covering. (2) The obtained decision-making results are more stable by means of different values of $\varphi$, which indicates VIKOR-CHFRS is viable in MCGDM problem. It can help decision-maker to make right selection according to different decision-making requirements. It also ensures the flexibility and practicability. (3) The results are consistent with other methods, which indicates VIKOR-CHFRS is reasonable and efficient in MCGDM problem. It not only can process the hesitancy, but also the uncertainty and vagueness of decision-making.

Despite the advantages in setting the expert weights, this paper does not consider the consensus reaching process among the HF evaluation information from varied decision-makers. In the future, we will possibly address the adaptive consensus reaching in information aggregation for MCGDM. Also, it deserves study to combine the advantages of CHFRS model with three-way decisions to solve various MCGDM problems.

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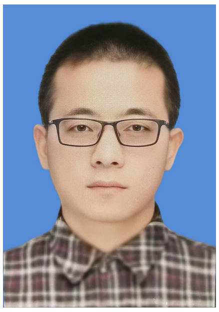

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