International Journal of Innovative Computing, Information and Control
Volume **22**, Number **1**, February **2026**

ICIC International ©2026 ISSN 1349-4198
pp. **209–225**

SYNTHESIS CONTROL TUNING ANALYSIS FOR CASCADE SYSTEM STRUCTURE

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Received April 2025; revised August 2025

Abstract. *Cascade system structure is a special closed loop system due to its dual feedback paths, meanwhile being a dynamic network system with two unknown plants and two unknown controllers, i.e., inner controller and outer controller. This paper concerns on synthesis control tuning analysis about these dual controllers, existing in cascade system. More specifically, firstly nonparametric method is used to design these dual unknown controllers, while combining system identification and power spectrum theory to derive their nonparametric forms and their equivalences. Secondly, inner loop stability analysis is considered through small gain theorem and loop shaping theory, thus resulting in some improved stability conditions. Thirdly, from an applied point of view, we adjust our proposed control tuning to its improved adaptive switching mechanism, more suited to practical engineering. Finally, one platform is established and simulation example confirms our proposed theoretical results. Generally, this paper is to design those two unknown controllers, existing in one cascade system structure, i.e., inner controller and outer controller, respectively, while also covering our new subjects about stability analysis and adaptive switching mechanism through combining system identification, power spectrum and adaptive control, etc.*
Keywords: Cascade system, Nonparametric method, Control tuning, Stability analysis, Adaptive switching mechanism

1. **Introduction.** Control theory includes lots of different aspects, for example, control strategy, stability analysis, robustness, adaptation, disturbance rejection, tracking, filtering, estimation and optimization. Each control strategy has its own goal and property, corresponding to its advantage and shortcoming, so our mission is to choose one appropriate control strategy from our control purpose, while considering some necessary external of internal factors. To the best of our knowledge, two control structures exist during all control strategies, i.e., open loop system and closed loop system, deemed as control system or plant. Due to divergence or bad tracking property, open loop system is more replaced by closed loop system, which introduces one feedback path to achieve dual goals, i.e.,

DOI: 10.24507/ijicic.22.01.209

perfect tracking and disturbance rejection. As two feedback path and feedforward path exist in the same closed loop system structure, the consequent two controllers are needed to design from the expected control mission, i.e., feedforward controller and feedback controller, thus leading to more computational complexity and other hardware or software requirement. These additional conditions are supported by some advanced devices during our new big data era.

Consider interesting problem of closed loop controller design, there are lots of different control strategies for our references, such as classical proportional integral derivation (PID) control, adaptive control, robust control, model predictive control, learning control, and nonlinear control. Before to design controller one prior mathematical model for plant is needed, thus depending on physical modeling or system identification, which corresponds to model based control. The other data driven control is similar to the main essence of system identification, i.e., designing controller only through the measured input-output data directly without any priori knowledge about the unknown plant, so the first modeling process for the unknown plant is avoided. Based on the existing contributions about closed loop system identification and closed loop controller design, more complex systems appear in academia and engineering, such as network system, large scale system and our considered cascade system.

Cascade system structure was firstly studied in our book [1], where system identification, direct data driven control and optimization theory were cooperated together to propose the named direct data driven control strategy, for cascade system. Specifically, the total number of observations, used to extract the intrinsic principle of cascade system, is defined as the sample size [2]. In case that the number of observations exceeds this sample size, then the input is persistent excitation, meaning the identification model satisfies the desired accuracy. From the knowledge of system identification theory, the situation with observed disturbance or noise in output brings the coming of robust system identification for cascade system [3]. When applying the probabilistic or statistical inference in both system identification and learning control, [4] measures the asymptotic accuracy to testify whether the final identification model or controller is efficient. During recent years, risk sensitive theory and reinforcement learning are all introduced into system theory and advanced control theory, see [5] and [6], where risk decision and limitations of policies are mentioned during the iterative identification process and control process. Furthermore, one second step model structure choice is related with graph theory in [7], i.e., the chosen model is constructed as one complex network system, being formulated as graph theory. As we know that some identification problem and control problem can be transferred into their corresponding constrain optimization problem [8], i.e., parameter estimation, then some existing optimization algorithms are applied directly, for example, convex optimization [9], scenarion optimization [10], and robust stochastic optimization [11]. Other than system identification and controller design in control theory, additional subjects are also very important, such as model or controller validation [12], experiment design [13], and statistical performance analysis [14]. All above mentioned subjects are also suited for cascade system as cascade system is a special closed loop system, meaning two feedback paths exist simultaneously to generate inner controller and outer controller. Our previous paper [15] introduces an adaptive idea into direct data driven control, and the detailed mathematical derivations are shown to give the nonparametric controller and parameterized controller adaptively. Moreover, to guarantee both perfect tracking and asymptotic unbias controller, Lyapunov function is established to derive the accurate adaptive controller.

As a consequence, cascade system structure has two closed loop systems, called as inner loop and outer loop, and then two controllers exist within above two closed loop systems, such as inner controller and outer controller. In this paper, our control purpose is to design these two inner controller and outer controller, while satisfying some desired or expected closed loop performances. Similarly, cascade system structure is also one special case of dynamic network system, which includes some unknown plants or systems simultaneously, and then cascade system identification applies the observations to identifying all of these unknown plants through the statistical strategy, for example, prediction error method, maximum likelihood method, and Bayesian method, while achieving the desired asymptotic accuracy. As cascade system structure is suited for aircraft system, such as mathematical models for aircraft, motor and tunable device area, it is necessary to consider cascade system structure from different views, such as cascade system identification, and cascade controller design.

This paper considers the problem of designing these two unknown controllers, existing in one cascade system structure, i.e., inner controller and outer controllers without any parametrized forms. We combine prediction error method and power spectrum theory together to derive these two controllers in detail, and then their equivalence is also proved through our own mathematical derivations for inner controller and outer controller. Furthermore, it is well known that stability is an important factor in control theory, so for completeness, stability about our considered cascade system structure with two controllers and two plants is also analyzed by virtue of small gain theorem and loop shaping theory. In addition, some necessary and sufficient conditions are given to satisfy the stability property. Above contributions about cascade system lie around how to design the two unknown controllers, while satisfying the stability property, we regard them as control tuning analysis. Tuning means two controllers are generated iteratively with each other. Later we construct one adaptive switching mechanism for controller tuning analysis for cascade system structure, i.e., establishing multi-controllers to suit for different plants at different time instants. It tells us that the whole control interval is divided into many subintervals, then during each subinterval, and the different controller is implemented from the idea of adaptive learning strategy, which means the switching mechanism will switch among these subintervals with time increases.

Two forms exist for every plant or controller, i.e., parametric form and nonparametric form. Specifically, parametric form means a group of unknown parameters exist in one rational expression, and then the controller design is turned to identify these unknown parameters. Because for parametric form, a premise condition is that the priori parametric form is known, but it is unrealistic due to nothing about plant or controller in reality, during this paper, we use nonparametric method without anything about controller in priori, and extract useful information from closed loop input-output data, corresponding to nonparametric controller.

Generally, in summary, new contributions of this paper are formulated as follows.

1) Combinations with prediction error method and power spectrum theory are proposed to design the detailed nonparametric forms for these two controllers in cascade system.

2) Some detailed stability analysis is given for cascade system, and small gain theorem and loop shaping theory are proposed to yield the stability condition.

3) Due to the varying time or environment, multi-controllers with switching mechanism are constructed, and then each controller is suited for each time subinterval.

This paper is organized as follows. In Section 2, after reviewing cascade system structure, we turn to the controller design problem, i.e., designing inner controller and outer controller. Section 3 shows our main control tuning analysis in detail. Specifically, prediction error method is proposed to design these two unknown controllers for cascade system, while no any parametrical form is assumed, i.e., nonparametric controller. Moreover, other power spectrum forms are combined together to prove the equivalent relations.

In Section 4, stability analysis is studied by virtue of small gain theorem and loop shaping theory in order to derive closed loop stability condition. Section 5 proposes adaptive switching mechanism to better improve the control performance through adaptive learning strategy. Section 6 establishes one platform and gives some simulation examples to testify our proposed theoretical results. Finally, Section 7 formulates the main conclusion and points out our future work.

2. **Cascade System Structure.** Consider the following cascade system, plotted in Figure 1, which includes two closed loop paths. The reason why this cascade system is studied here will be exemplified in later platform, where one aircraft velocity loop is always modeled as our considered cascade system structure.

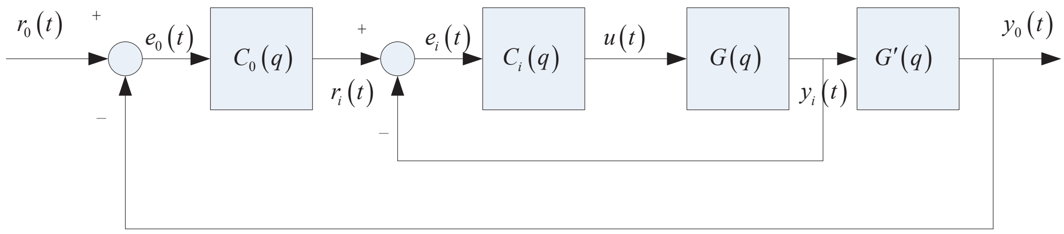


FIGURE 1. Cascade system with two closed loops

In Figure 1, two closed loops exist, i.e., inner loop and outer loop. $G(q)$ and $G'(q)$ are two unknown plants for inner loop and outer loop, respectively; similarly two controllers $C_i(q)$ and $C_0(q)$ correspond to two unknown controllers for inner loop and outer loop, respectively, too. $r_0(t)$ and $r_i(t)$ are two input signals for these two closed loops; similarly $e_0(t)$ and $e_i(t)$ are two error signals. $y_0(t)$ is the closed loop output for this cascade control system, and variable q is the shift operator.

In our previous paper, the special cases of two parametrized controllers, $\{C_i(q, \theta_i), C_0(q, \theta_0)\}$ are considered, i.e., those two unknown controllers $\{C_i(q, \theta_i), C_0(q, \theta_0)\}$ are parametrized by two unknown parameter vectors θ_i and θ_0, respectively. Then the problem of designing the two controllers is changed to identify those unknown parameter vectors $\{\theta_i, \theta_0\}$. However, in this new paper, we do not assume these parametrized controllers in priori, as this parametrized form is an ideal case. In practice, more non-parametrized forms are needed, and it is our mission to propose non-parametric way to design these unknown controllers $\{C_i(q), C_0(q)\}$. However, the following controller design process for these two unknown controllers $\{C_i(q), C_0(q)\}$ is dependent of other two unknown plants $\{G(q), G^{'}(q)\}$, so firstly the explicit forms for those two unknown plants $\{G(q), G^{'}(q)\}$ are identified within this cascade system. Furthermore, prediction error method is also applied to achieving this identification goal through our previous work.

3. **Synthesis Control Tuning Analysis.** In order to design these two unknown controllers $\{C_i(q), C_0(q)\}$ without any parametrized forms, existing in that cascade control system with inner loop and outer loop, we use our own mathematical derivations to do them. Due to the fact that two unknown controllers, i.e., inner controller $C_i(q)$ and outer controller $C_0(q)$, exist, we separate two parts to design them, respectively.

3.1. **Inner controller.** Before to design that inner controller $C_i(q)$ firstly, the cascade control system with two external noises $\{n_i(t), n_0(t)\}$ is reformulated in Figure 2.

In Figure 2, two additive noises $\{n_i(t), n_0(t)\}$ are added. They may be white noises or color noises.

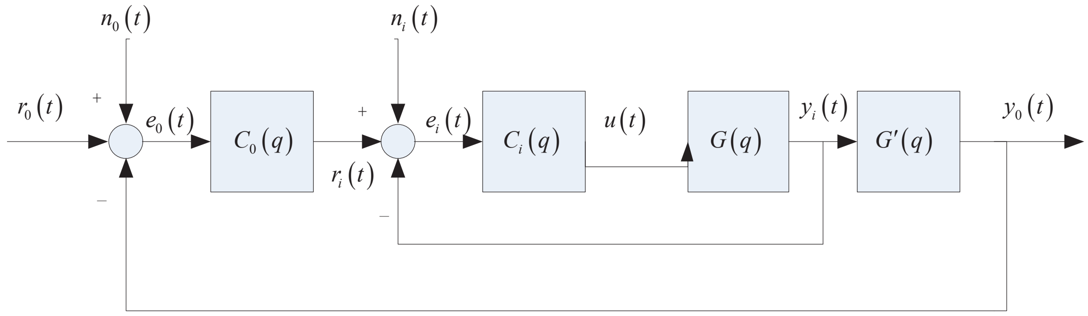


Figure 2. Cascade system with two noises

Consider that inner loop with additive noises, we have the following input-output relation:

$$y_i(t) = G(q)C_i(q)r_i(t) - G(q)C_i(q)y_i(t) + G(q)C_i(q)n_i(t) \tag{1}$$

i.e.,

$$y_i(t) = \frac{G(q)C_i(q)}{1 + G(q)C_i(q)} r_i(t) + \frac{G(q)C_i(q)}{1 + G(q)C_i(q)} n_i(t) \tag{2}$$

To apply the non-parametric prediction error method, the one step ahead prediction $\hat{y}_i(t)$ is defined as follows:

$$\hat{y}_i(t) = r_i(t) - \frac{1}{G(q)C_i(q)} y_i(t) \tag{3}$$

The prediction error is given as

$$y_i(t) - \hat{y}_i(t) = y_i(t) - r_i(t) + \frac{1}{G(q)C_i(q)} y_i(t) = \frac{1 + G(q)C_i(q)}{G(q)C_i(q)} y_i(t) - r_i(t) \tag{4}$$

As $y_i(t)$ is the observed output, $r_i(t)$ is the input signal for inner loop. Combine them together to be the input-output data sequence $\{r_i(t), y_i(t)\}_{t=1}^{N}$ and construct a quadratic cost function to minimize with respect to the inner controller $C_i(q)$, i.e.,

$$\frac{1}{N} \sum_{t=1}^{N} [y_i(t) - \hat{y}_i(t)]^2 = \frac{1}{N} \sum_{t=1}^{N} \left[\frac{1 + G(q)C_i(q)}{G(q)C_i(q)} y_i(t) - r_i(t) \right]^2 \tag{5}$$

where N is the total number of observed data.

Then regarding that unknown inner controller $C_i(q)$ as one decision variable for the above quadratic cost function, the optimal inner controller is solved through the following minimization problem, i.e.,

$$\arg\min_{C_i(q)} \frac{1}{N} \sum_{t=1}^{N} \left[\frac{1 + G(q)C_i(q)}{G(q)C_i(q)} y_i(t) - r_i(t) \right]^2 \tag{6}$$

After differentiating with respect to inner controller, the optimal inner controller $\hat{C}_i(q)$ must satisfy

$$\frac{1}{G(q)C_i(q)} = \frac{r_i(t)}{y_i(t)} - 1 = \frac{r_i(t) - y_i(t)}{y_i(t)}; \quad G(q)C_i(q) = \frac{y_i(t)}{r_i(t) - y_i(t)} \tag{7}$$

i.e.,

$$\hat{C}_i(q) = G^{-1}(q) \frac{y_i(t)}{r_i(t) - y_i(t)} \tag{8}$$

As above mathematical derivations are independent of any parametric forms, we call that optimal inner controller $\hat{C}_i(q)$ as one non-parametric controller. Observing Equation (8) again, the optimal inner controller $\hat{C}_i(q)$ is related with $y_i(t)$, $r_i(t)$ and $G(q)$. Due to the fact that input-output data sequence $\{r_i(t), y_i(t)\}$ are measured in priori, the optimal inner controller depends on the plant $G(q)$, i.e., model-based control.

3.2. **Outer controller.** Similarly, outer controller $\hat{C}_0(q)$ can also be defined through above non-parametric prediction error method. More specifically, from Figure 2, we have

$$\begin{aligned} y_0(t) = & \frac{G'(q)G(q)\hat{C}_i(q)}{1+G(q)\hat{C}_i(q)}C_0(q)r_0(t) - \frac{G'(q)G(q)\hat{C}_i(q)}{1+G(q)\hat{C}_i(q)}C_0(q)y_0(t) \\ & + \frac{G'(q)G(q)}{1+G(q)\hat{C}_i(q)}\left(\hat{C}_i(q)C_0(q)n_0(t)+n_i(t)\right) \end{aligned} \tag{9}$$

i.e.,

$$\begin{aligned} & \frac{1+G(q)\hat{C}_i(q)+G'(q)G(q)\hat{C}_i(q)}{1+G(q)\hat{C}_i(q)}y_0(t) \\ = & \frac{G'(q)G(q)\hat{C}_i(q)}{1+G(q)\hat{C}_i(q)}r_0(t) + \frac{G'(q)G(q)}{1+G(q)\hat{C}_i(q)}\left(\hat{C}_i(q)C_0(q)n_0(t)+n_i(t)\right) \end{aligned} \tag{10}$$

It holds that

$$\begin{aligned} y_0(t) = & \frac{G'(q)G(q)\hat{C}_i(q)C_0(q)}{1+G(q)\hat{C}_i(q)+G'(q)G(q)\hat{C}_i(q)C_0(q)}r_0(t) \\ & + \frac{G'(q)G(q)}{1+G(q)\hat{C}_i(q)+G'(q)G(q)\hat{C}_i(q)C_0(q)}r_0(t)\left(\hat{C}_i(q)C_0(q)n_0(t)+n_i(t)\right) \end{aligned} \tag{11}$$

Then the one step ahead prediction is defined as follows:

$$\begin{aligned} \hat{y}_0(t) = & \frac{1+G(q)\hat{C}_i(q)+G'(q)G(q)\hat{C}_i(q)C_0(q)}{G'(q)G(q)\hat{C}_i(q)C_0(q)}\frac{G'(q)G(q)\hat{C}_i(q)C_0(q)}{1+G(q)\hat{C}_i(q)+G'(q)G(q)\hat{C}_i(q)C_0(q)}r_0(t) \\ & - \left[1-\frac{1+G(q)\hat{C}_i(q)+G'(q)G(q)\hat{C}_i(q)C_0(q)}{G'(q)G(q)\hat{C}_i(q)C_0(q)}\right]y_0(t) \\ = & r_0(t) - \frac{1+G(q)\hat{C}_i(q)}{G'(q)G(q)\hat{C}_i(q)C_0(q)}y_0(t) \end{aligned} \tag{12}$$

Then the optimal outer controller $\hat{C}_0(q)$ is obtained through solving the following minimization problem, i.e.,

$$\hat{C}_0(q) = \frac{1}{N}\sum_{t=1}^{N}\left[1+\frac{G(q)\hat{C}_i(q)}{G'(q)G(q)\hat{C}_i(q)C_0(q)}y_0(t)-r_0(t)\right]^2 \tag{13}$$

Similarly, the ideal case is

$$1+\frac{G(q)\hat{C}_i(q)}{G'(q)G(q)\hat{C}_i(q)C_0(q)} = \frac{r_0(t)}{y_0(t)} \tag{14}$$

i.e.,

$$\frac{G(q)\hat{C}_i(q)}{G'(q)G(q)\hat{C}_i(q)C_0(q)} = \frac{r_0(t)}{y_0(t)} - 1 = \frac{r_0(t)-y_0(t)}{y_0(t)} \tag{15}$$

Then the optimal outer controller $\hat{C}_0(q)$ is

$$\hat{C}_0(q) = \frac{1 + G(q)\hat{C}_i(q)}{G'(q)G(q)\hat{C}_i(q)} \frac{y_0(t)}{r_0(t) - y_0(t)} \tag{16}$$

Combining the optimal inner controller $\hat{C}_i(q)$ and optimal outer controller $\hat{C}_0(q)$, they are all dependent of two models or plants $\{G(q), G'(q)\}$, i.e., model-based controller design.

3.3. **Extending.** Whatever the optimal inner controller (8) and optimal outer controller (16), only input-output data at time instant t are used. However, it is not enough, as we have lots of data from time instant t to N. These two optimal controllers are needed to design based on these input-output data sequence $\{r_i(t), y_i(t), r_0(t), y_0(t)\}_{t=1}^N$, not only on the simple data $\{r_i(t), y_i(t), r_0(t), y_0(t)\}$ at time instant t.

Consider the controller design problem based on all input-output data sequence $\{r_i(t), y_i(t), r_0(t), y_0(t)\}_{t=1}^N$, firstly we rewrite that quadratic cost function (6) again.

$$J_1(C_i(q)) = \frac{1}{N}\sum_{t=1}^{N}\left[\left(1 + \frac{1}{G(q)C_i(q)}\right) y_i(t) - r_i(t)\right]^2 \tag{17}$$

Taking the partial derivative with respect to that inner controller $C_i(q)$, it holds that

$$\frac{\partial J_1(C_i(q))}{\partial C_i(q)} = \frac{2}{N}\sum_{t=1}^{N}\left[\frac{1 + G(q)C_i(q)}{G(q)C_i(q)} y_i(t) - r_i(t)\right] \frac{G(q)}{[G(q)C_i(q)]^2} y_i(t) \tag{18}$$

By differentiating with respect to inner controller and by setting the derivative equal to zero, i.e.,

$$\begin{gathered}\sum_{t=1}^{N}\left[\frac{1 + G(q)C_i(q)}{G(q)C_i(q)} y_i(t) - r_i(t)\right] \frac{G(q)}{[G(q)C_i(q)]^2} y_i(t) = 0; \\ \frac{G(q)[1 + G(q)C_i(q)]}{[G(q)C_i(q)]^3}\sum_{t=1}^{N} y_i^T(t)y_i(t) = \frac{G(q)}{[G(q)C_i(q)]^3}\sum_{t=1}^{N} y_i^T(t)r_i(t)\end{gathered} \tag{19}$$

i.e.,

$$\frac{1}{G(q)C_i(q)} = \left[\sum_{t=1}^{N} y_i^T(t)y_i(t)\right]^{-1}\left[\sum_{t=1}^{N} y_i^T(t)r_i(t)\right] - 1 \tag{20}$$

then we have

$$\hat{C}_i(q) = G^{-1}(q)\frac{\phi_{y_i}(w)}{\phi_{y_i r_i}(w) - \phi_{y_i}(w)} \tag{21}$$

where $\phi_{y_i}(w)$ and $\phi_{y_i r_i}(w)$ are power spectrum and cross spectrum, i.e.,

$$\phi_{y_i}(w) = E\left[\sum_{t=1}^{N} y_i^T(t)y_i(t)\right]; \quad \phi_{y_i r_i}(w) = E\left[\sum_{t=1}^{N} y_i^T(t)r_i(t)\right] \tag{22}$$

The optimal inner controller (21) is obtained from input-output data sequence $\{r_i(t), y_i(t)\}_{t=1}^N$ through using some power spectrum forms $\phi_{y_i}(w)$ and $\phi_{y_i r_i}(w)$.

Observing those two optimal inner controllers in Equations (8) and (21), Equation (21) holds in case of one operation on Equation (8). The detained operation is described by multiplying $y_i(t)$ on both sides of Equation (8), and then the expectation operation is taken to get Equation (22), showing in the latter.

$$\hat{C}_i(q) = G^{-1}(q)\frac{\phi_{y_i}(w)}{\phi_{y_i r_i}(w) - \phi_{y_i}(w)} \tag{23}$$

The above Equation (23) shows the equivalence property for those two different inner controller forms. Similarly, the other spectrum form for that optimal outer controller is

$$\hat{C}_0(q) = \frac{1 + G(q)\hat{C}_i(q)}{G'(q)G(q)\hat{C}_i(q)} \frac{\phi_{y_0 r_0}(w)}{\phi_{r_0}(w) - \phi_{y_0 r_0}(w)} \tag{24}$$

where power spectrum $\phi_{y_0 r_0}(w)$ and $\phi_{r_0}(w)$ are defined just as above.

Specifically, when the number of data sequences is limited, the rough optimal inner controller and outer controller can be chosen from Equations (8) and (16).

4. **Synthesis Stability Analysis.** Section 3 is our main contribution about optimal controller tuning analysis for cascade system. Here during this short part, we state and discuss the small gain theorem or loop shaping, which can be thought as a stability equivalence result.

Rewrite the inner closed loop system from cascade control system again in Figure 3.

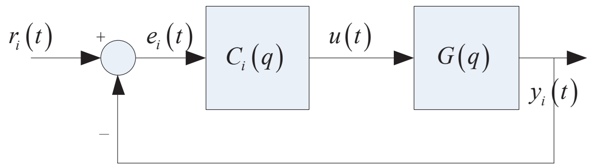


FIGURE 3. Inner closed loop system

Our general problem under investigation is that: Given some assumptions on $G(q)$ and $C_i(q)$, it is shown that if $r_i(t)$ belongs to some class, then $y_i(t)$ and $u(t)$ also belong to the same class. The equations of that closed loop system structure in Figure 3 are

$$u(t) = C_i(q)e_i(t) = C_i(q)(r_i(t) - y_i(t));\ 0 = y_i(t) - G(q)u(t) \tag{25}$$

where $\{C_i(q), G(q)\}$ are operators which act on their inputs $u(t)$ and $y_i(t)$ respectively to produce the outputs $y_i(t)$ and $C_i(q)y_i(t)$.

From control theory, we have the following two obvious propositions.

Proposition 4.1. *If $G(q)$ is a linear map, and if $(I+G(q)C_i(q))^{-1}$ and $(I+C_i(q)G(q))^{-1}$ are well defined maps, then*

$$G(q)(I + C_i(q)G(q))^{-1} = (I + G(q)C_i(q))^{-1}G(q) \tag{26}$$

The proof of Equation (26) is very easy, i.e.,

$$\begin{aligned} G(q)(I + C_i(q)G(q))^{-1} &= [(I + G(q)C_i(q))G(q)^{-1}]^{-1} = [G(q)^{-1} + C_i(q)]^{-1} \\ &= [G(q)^{-1}(I + G(q)C_i(q))]^{-1} = (I + G(q)C_i(q))^{-1}G(q) \end{aligned} \tag{27}$$

Proposition 4.2. *Suppose that $(I + C_i(q)G(q))^{-1}$ is a well defined map, and then for the closed loop system structure in Figure 3, we have*

$$y_i(t) = G(q)u(t) = G(q)C_i(q)[r_i(t) - y_i(t)] \tag{28}$$

i.e.,

$$y_i(t) = \frac{G(q)C_i(q)}{1 + G(q)C_i(q)} r_i(t) \tag{29}$$

The small gain theorem is a very generated theorem, which gives sufficient conditions under which a bounded input produces a bounded output. Our stability analysis is listed as the following two theorems.

Theorem 4.1. *Considering that closed loop system in Figure 3, define*

$$r_i(t) = y_i(t) + C_i^{-1}(q)u(t); \quad 0 = y_i(t) - G(q)u(t)$$

Suppose that there are constants β_1, β_2, $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, such that

$$\|G(q)u(t)\| \leq \gamma_1\|u(t)\| + \beta_1; \quad \|C_i(q)y_i(t)\| \leq \gamma_2\|y_i(t)\| + \beta_2 \tag{30}$$

If $\gamma_1\gamma_2 < 1$, then

$$\begin{aligned}\|u(t)\| &\leq (1-\gamma_1\gamma_2)^{-1}(\|C_i(q)r_i(t)\| + \beta_2 + \gamma_2\beta_1);\\ \|y_i(t)\| &\leq (1-\gamma_1\gamma_2)^{-1}(\gamma_1\|C_i(q)r_i(t)\| + \beta_1 + \gamma_1\beta_2)\end{aligned} \tag{31}$$

Proof: From $r_i(t) = y_i(t) + C_i^{-1}(q)u(t)$, we have $u(t) = C_i(q)(r_i(t) - y(t))$. Taking norm on both sides of above equation, i.e.,

$$\|u(t)\| \leq \|C_i(q)r_i(t)\| + \|C_i(q)y_i(t)\| \leq \|C_i(q)r_i(t)\| + \gamma_2\|y_i(t)\| + \beta_2$$

Similar result is obtained.

$$\|y_i(t)\| \leq \|G(q)u(t)\| \leq \gamma_1\|u(t)\| + \beta_1$$

Hence using the fact that $\gamma_2 \geq 0$, it means that

$$\begin{aligned}\|u(t)\| &\leq \|C_i(q)r_i(t)\| + \gamma_1\gamma_2\|u(t)\| + \gamma_2\beta_1 + \beta_2\\ [1-\gamma_1\gamma_2]\|u(t)\| &\leq \|C_i(q)r_i(t)\| + \gamma_2\beta_1 + \beta_2\end{aligned}$$

Since $\gamma_1\gamma_2 < 1$,

$$\|u(t)\| \leq (1-\gamma_1\gamma_2)^{-1}(\|C_i(q)r_i(t)\| + \beta_2 + \gamma_2\beta_1)$$

Similarly, we get

$$\begin{aligned}\|y_i(t)\| &\leq \gamma_1\|u(t)\| + \beta_1 \leq \gamma_1\|C_i(q)r_i(t)\| + \gamma_1\gamma_2\|y_i(t)\| + \beta_1 + \gamma_1\beta_2\\ [1-\gamma_1\gamma_2]\|y_i(t)\| &\leq \gamma_1\|C_i(q)r_i(t)\| + \beta_1 + \gamma_1\beta_2\end{aligned}$$

Since $\gamma_1\gamma_2 < 1$,

$$\|y_i(t)\| \leq (1-\gamma_1\gamma_2)^{-1}\left(\gamma_1\|C_i(q)r_i(t)\| + \beta_1 + \gamma_1\beta_2\right)$$

The proof is complete.

Theorem 4.2. *Define*

$$r_i(t) = y_i(t) + C_i^{-1}(q)u(t); \quad y_i(t) = G(q)u(t)$$

Suppose that there exist γ_{21}, γ_1, β_{21} and β_1, with $\gamma_{21} \geq 0$, $\gamma_1 \geq 0$, such that

$$\|C_i(q)G(q)u(t)\| \leq \gamma_{21}\|u(t)\| + \beta_2; \quad \|G(q)u(t)\| \leq \gamma_1\|u(t)\| + \beta_1 \tag{32}$$

If $\gamma_{21} < 1$, then

$$\|u(t)\| \leq \frac{1}{1-\gamma_{21}}(\|C_i(q)r_i(t)\| + \beta_{21}); \quad \|y_i(t)\| \leq \frac{\gamma_1}{1-\gamma_{21}}(\|C_i(q)r_i(t)\| + \beta_{21}) + \beta_1 \tag{33}$$

Proof: Due to

$$u(t) = C_i(q)r_i(t) - C_i(q)y_i(t)$$

we have

$$\begin{aligned}\|u(t)\| &= \|C_i(q)r_i(t) - C_i(q)y_i(t)\|\\ &\leq \|C_i(q)r_i(t)\| + \|C_i(q)G(q)u(t)\|\end{aligned}$$

$$\leq \|C_i(q)r_i(t)\| + \gamma_{21}\|u(t)\| + \beta_{21}$$

i.e.,

$$(1-\gamma_{21})\|u(t)\| \leq \|C_i(q)r_i(t)\| + \beta_{21}$$

Then

$$\|u(t)\| \leq \frac{1}{1-\gamma_{21}}(\|C_i(q)r_i(t)\| + \beta_{21})$$

and from $y_i(t) = G(q)u(t)$, we get

$$\|y_i(t)\| = \|G(q)u(t)\| \leq \gamma_1\|u(t)\| + \beta_1 \leq \frac{\gamma_1}{1-\gamma_{21}}(\|C_i(q)r_i(t)\| + \beta_{21}) + \beta_1$$

The proof is complete.

Remark 4.1. *Once the loop gain is smaller than 1, the closed loop system has finite gain. γ_1, γ_2 are called the gain of $G(q)$ and $C_i(q)$. We are interested in the smallest γ_1, γ_2 that work in analyzing stability. If γ_1 and γ_2 corresponding to the gain of $G(q)$ and $C_i(q)$ have a produce smaller than 1, then a solution exists. In the sense, any bounded input gain produces a bounded output gain. The above stability analysis holds for that inner closed loop system in cascade control system, and finally the similar stability analysis can be done for the outer closed loop system.*

Above Theorems 4.1 and 4.2 analyze closed loop stability in detail through small gain theorem, showing when plants $G(q)$ and $C_i(q)$ satisfy inequality condition (30), and then input-output $\{u(t), y_i(t)\}$ belong to their corresponding intervals, i.e., two upper bounds, benefiting for guaranteed accuracy.

5. **Synthesis Adaptive Switching Mechanism.** From above two Sections 3 and 4, more specifically, Section 3 proposes to design those two unknown controllers $\{C_i(q), C_0(q)\}$ in their nonparametric forms, and then Section 4 extracts the inner loop to study its stability analysis, applied to the case of outer loop similarly. During Section 5, we also extract that inner loop as our considered case and analyze it deeply, meaning the obtained result can also be suited for the outer loop.

From our previous contribution about adaptive data driven control, safe data driven control and application into the practical engineering, we find a fact that the unknown plant in inner loop $G(q)$ will change with time or environment varying, so the corresponding inner controller $C_i(q)$ must be also changed greatly.

From a practical point of view, there are n designed candidate controllers $\{C_i(q), i = 1, 2, \ldots, n\}$, and then an active controller must be selected and inserted into the closed loop control loop based on the mentioned performance assessment, plotted in Figure 4.

After collecting the measured input-output data sequence $\{u(t), y_i(t)\}_{t=1}^N$, both input-output for inner controller $\{C_i(q), i = 1, 2, \ldots, n\}$ are $\{e_i(t), u(t)\}_{t=1}^N$, where N is the total number of the measured input-output data. Then our mission is to select one appropriate controller $C_i(q)$ from the candidate controller set $\{C_i(q), i = 1, 2, \ldots, n\}$ to guarantee the error $u(t) = C_i(q)e_i(t)$ be sufficiently small, and the corresponding index i is the selected ith controller.

For simplicity, determine one switching logic or switching signal σ_i to satisfy

$$\sigma_i = \arg\min_i \sum_{t=1}^{N}[u(t) - C_i(q)e_i(t)]^2 = \arg\min_i \sum_{t=1}^{N}[u(t) - C_i(q)(r_i(t) - y_i(t))]^2$$
$$i = 1, 2, \ldots, n \tag{34}$$

where σ_i is the two-value switching signal, i.e., 0 or 1.

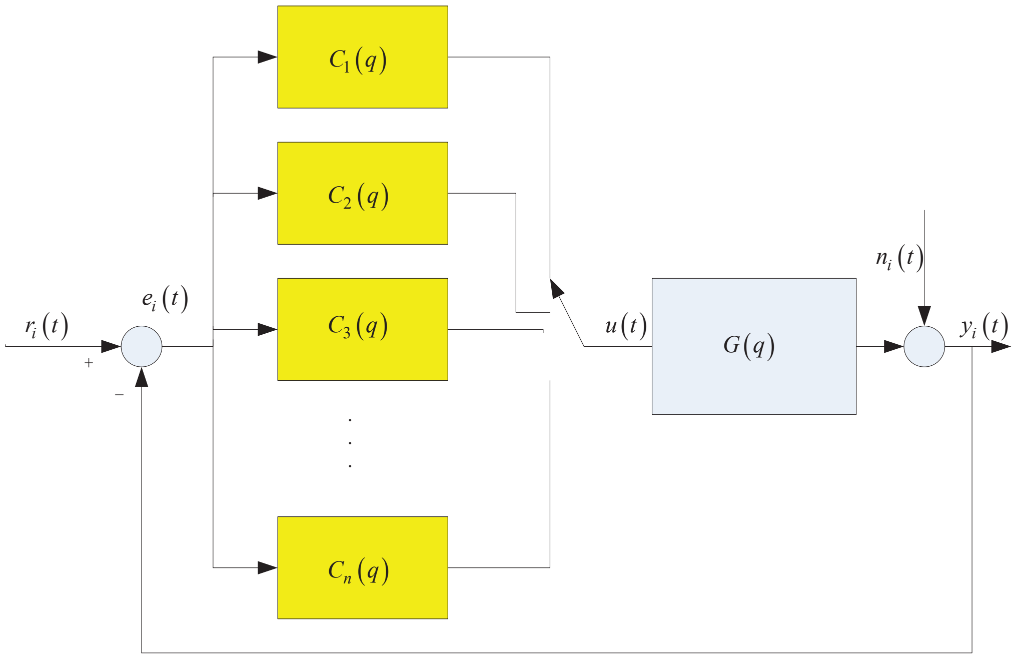


FIGURE 4. Switching mechanism

Equation (34) is similar to the called integer programming problem, i.e.,

$$\min_{\sigma_i} \sum_{i=1}^{n} \sum_{t=1}^{N} [u(t) - \sigma_i C_i(q)(r_i(t) - y_i(t))]^2; \ \sigma_i \in \{0, 1\} \tag{35}$$

Equation (35) tells us to find that ith controller from index set $i = 1, 2, \ldots, n$, which corresponds to the smallest error from the following n errors, i.e.,

$$\sum_{t=1}^{N} [u(t) - C_1(q)(r_i(t) - y_i(t))]^2, \ \sum_{t=1}^{N} [u(t) - C_2(q)(r_i(t) - y_i(t))]^2, \ \ldots$$
$$\sum_{t=1}^{N} [u(t) - C_n(q)(r_i(t) - y_i(t))]^2 \tag{36}$$

Introduction of switching signal σ_i into control tuning analysis for cascade system structure is plotted in Figure 5, where the process of determining switches is solved by one integer programming problem, and those n candidate controllers are prior designed through our proposed control tuning analysis, mentioned in above Section 3.

As here this paper only considers the nonparametric forms for dual controllers $\{C_i(q), C_0(q)\}$, all theoretical results can be applied for the special parametric forms. For example, those n candidate inner controllers $\{C_i(q), i = 1, 2, \ldots, n\}$ can be parametrized by parameter vectors θ_i, i.e., $\{C_i(q, \theta_i), i = 1, 2, \ldots, n\}$, and then based on the parametrized controllers, the controller design is transformed to the parameter estimation, i.e., identifying parameter vectors $\{\theta_i, i = 1, 2, \ldots, n\}$ through statistical methods, such as maximum likelihood method, and classical least squares method. For the case of parametrized controllers, adaptive switching mechanism in Figure 5, is replotted in Figure 6, where the switching process concerns on choosing the appropriate parameter vector θ_i from parameter vector set $\{\theta_i, i = 1, 2, \ldots, n\}$.

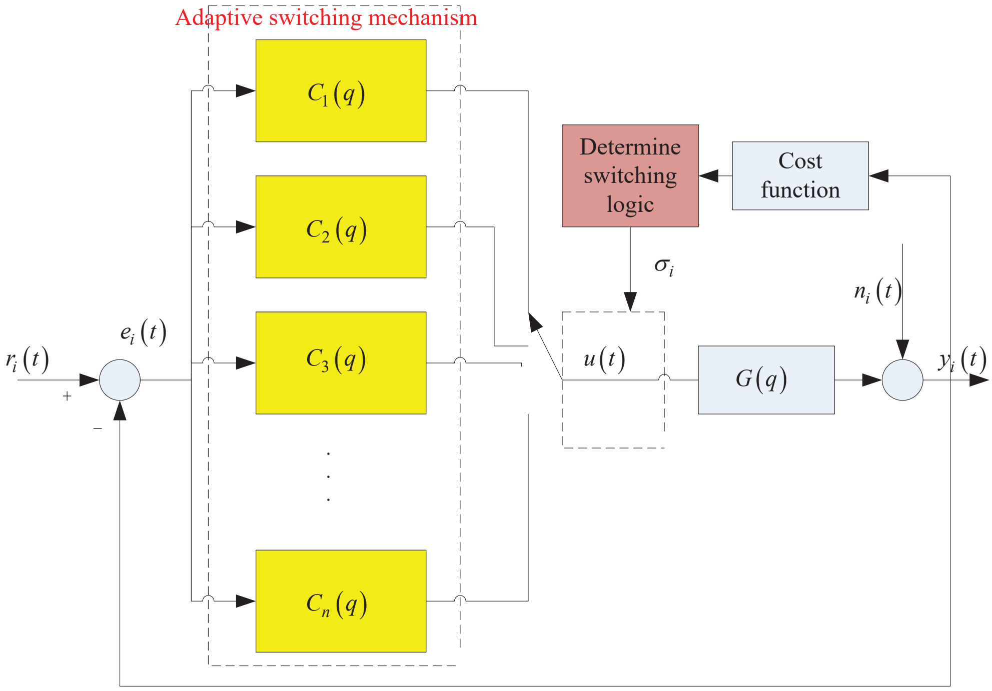


FIGURE 5. Adaptive switching mechanism

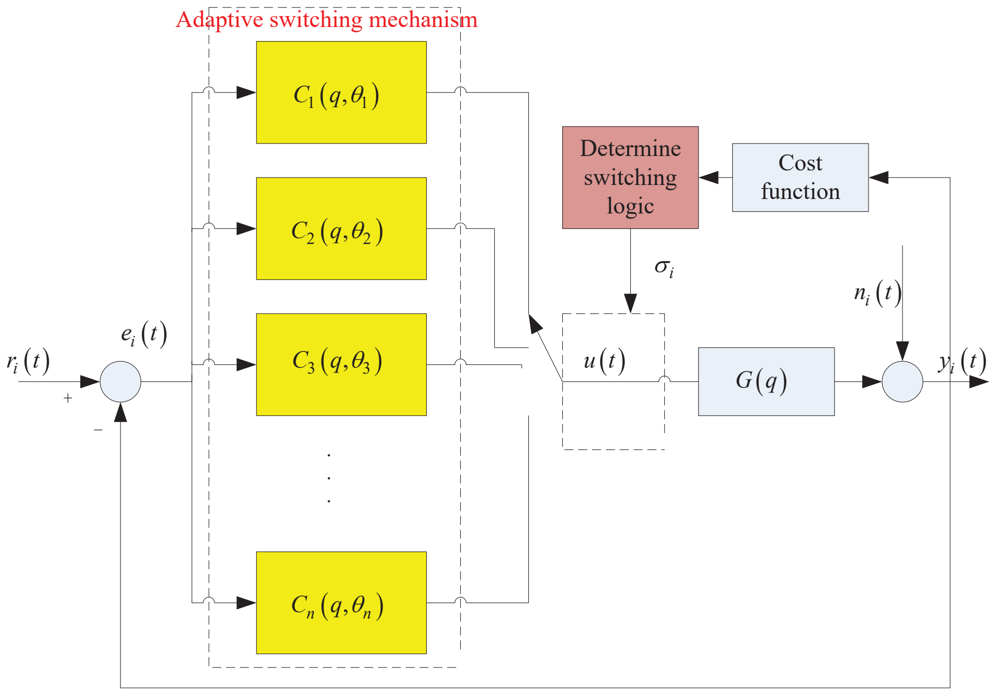


FIGURE 6. Adaptive switching mechanism for the parametrized controllers

6. **Platform and Simulation Example.** To prove our above theoretical results about synthesis control tuning analysis for cascade system structure, we use unmanned helicopter control framework to achieve it. Firstly, one practical real unmanned helicopter is seen in Figure 7, which is a real product for air force in our lab.

FIGURE 7. Real unmanned helicopter

In this simulation example, an unmanned helicopter is used, as it can take off and land vertically. It has strong air control capability, good static flight and low speed flight characteristics. The altitude control of unmanned helicopter is achieved through collective pitch control. The size of the collective pitch determines the size of the main rotor lift. In fact, height control is to compare the real height fed back by the height sensor with the set height, and adjust the size of the collective distance according to the deviation value. In order to increase the control damping, the feedback of the rate of change of the height is introduced to form a cascade control system, shown in Figure 8.

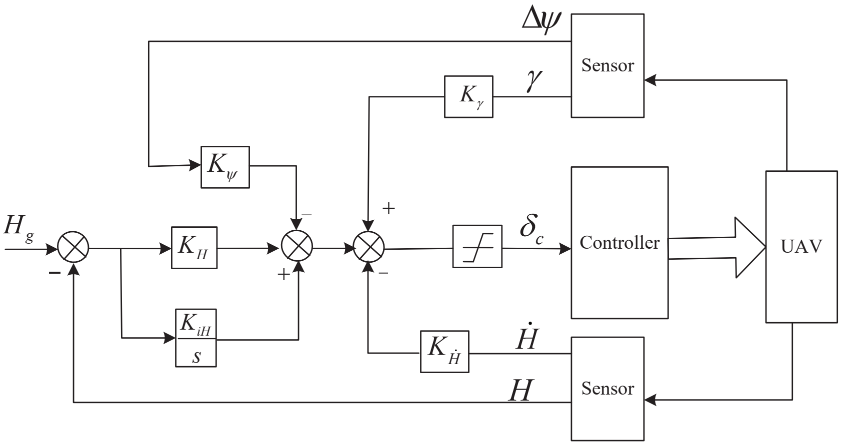


FIGURE 8. Structure of height control for unmanned helicopter

In Figure 8, all physical variables are defined as our previous paper. The altitude control loop adds the effect of the heading channel to the control law, and $\triangle\psi$ is the yaw angle.

Due to the structural characteristics of unmanned helicopter with tail rotor, the lateral force required for left turning is greater than the lateral force required for right turning, so the power consumed by the tail rotor when turning left is relatively large, and the height of unmanned helicopter will be as follows: turn left, go down, turn right, go up. Introduce yaw angle compensation in the altitude loop, properly increase the collective pitch when turning left, and appropriately reduce the collective pitch when turning right, so that the height of unmanned helicopter will not fluctuate too much.

Velocity control for unmanned helicopter refers to the control of the forward flying speed of unmanned helicopter. To make unmanned helicopter fly forward, it is generally necessary to change the longitudinal cyclic pitch unmanned helicopter, and use the pulling force generated by the rotor to pull unmanned helicopter forward to fly. Structure of this velocity control system is seen in Figure 9.

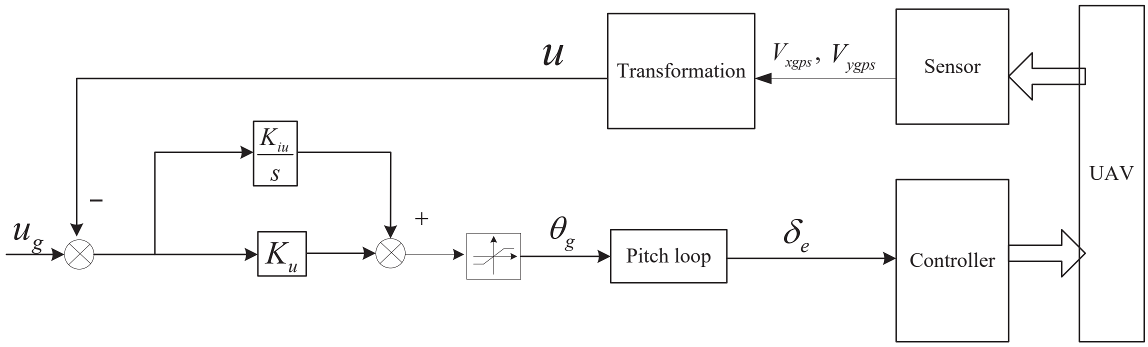


FIGURE 9. Structure of velocity control

In this velocity control loop for unmanned helicopter in Figure 9, u_g is the given forward velocity, corresponding to unmanned helicopter body. The eastward velocity and northward velocity are measured by GPS, and then after comparing them with their given velocities, one velocity error is generated.

During the simulation process for cascade controller design in the velocity loop system structure, the two plants $G(q)$ and $G'(q)$ are chosen as the following transfer functions.

$$G(q) = \frac{14}{q^2 + 5.8q + 14}; \quad G'(q) = \frac{0.9q^2 + 1.8q + 0.9}{q^2 + 1.8q + 0.8}$$

All input-output data sequence $\{r_i(t), y_i(t), r_0(t), y_0(t)\}_{t=1}^{N}$ are measured by some devices, and they are recorded in Figure 10. Two controllers, proposed by our prediction error method, are applied to controlling unmanned helicopter flight trajectory. During the whole simulation process, we apply our proposed control strategy in automatic takeoff for above mentioned unmanned helicopter, i.e., designing two inner controller and outer controller to complete automatic takeoff. The simulation results are plotted in Figure 11,

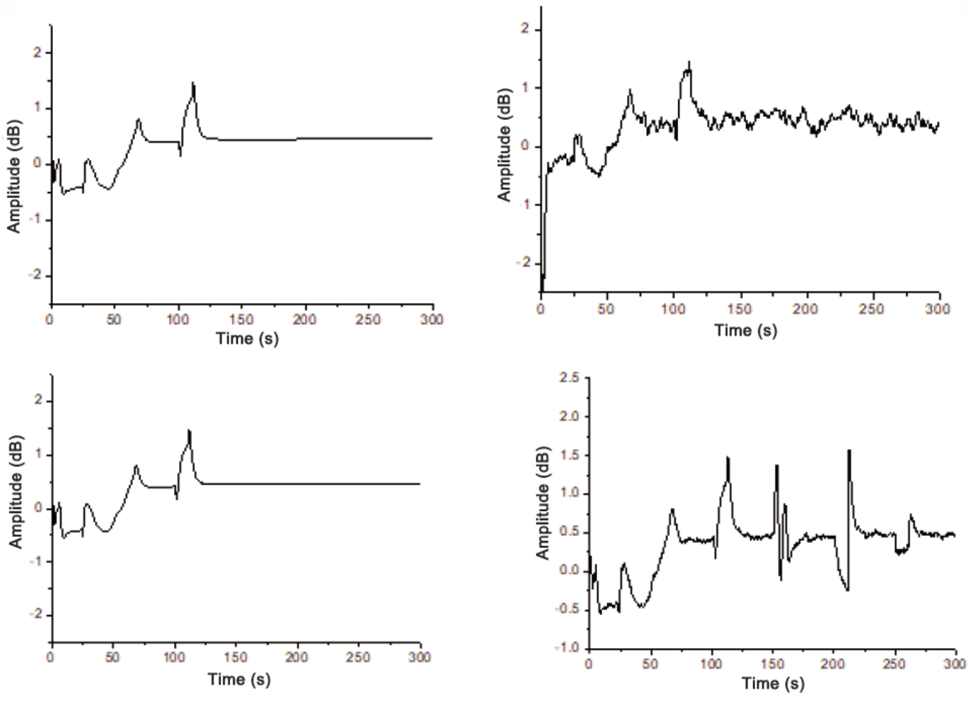
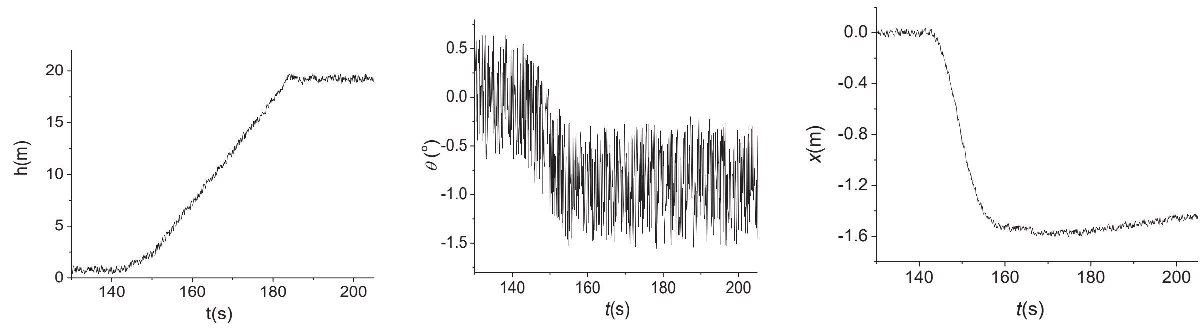


FIGURE 10. Input-output data sequence



(a) Altitude changes in takeoff (b) Pitch angle changes in takeoff (c) Longitudinal slip distance in takeoff

FIGURE 11. Simulation results

where in process of unmanned helicopter takeoff, the height gradually rises from zero to 20 m. From Figure 11(a), the process transition is relatively smooth and no fluctuation, then our controllers also cannot suppress the longitudinal sideslip and prevent unmanned helicopter from rolling over, but also ensure the safety of takeoff process, so the vertical takeoff of unmanned helicopter is realized. Furthermore, Figure 11(b) shows the pitch angle changes from 0° to -10° during takeoff to ensure the balance of the takeoff torque, and the change of the pitch angle is within the constraint range. Figure 11(c) tells us the longitudinal slip during takeoff is within 2 m, i.e., belonging to the permissible range, so that automatic takeoff for unmanned helicopter is completely realized.

7. **Conclusion.** In this paper, cascade system structure is considered about the dual controllers design problem, i.e., designing inner controller and outer controller in their own nonparametric forms. Nonparametric prediction error method is proposed to design these two unknown controllers without any priori parametrized forms, while combining system identification and power spectrum theory to prove the equivalent property. Moreover, closed loop stability is analyzed for cascade system by using small gain theorem and loop shaping theory. As environment always changes with time increases, then adaptive switching mechanism is also established, signifying controller will be appropriately selected from the given controller set through determining the switching logic. Based on the idea of changing controller with plant, next we study adaptive control and robust control and combine game theory or dynamic programming together for more complex closed loop system.

Acknowledgment. This work is supported by Science and Technology Research Project of Jiangxi Provincial Department of Education: GJJ2410304.

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