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BUILDING THE THREE-GROUP CAUSAL PATH IN THE VAR MODEL

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Abstract. *To exhibit the leading/lagging relationship among groups in a VAR process, we construct a three-group causal path (an extended Granger causality) as well as an identification procedure for the pathway which includes the independent, the intermediate and the dependent groups. In addition, we impose the unidirectional restriction on the pathway. Consequently, our method can organize more detailed and practical causal structure in a dynamic system than the conventional methods. The property concerning the impulse response function is derived when the three-group causality occurs in the VAR model. Finally, we show that these techniques can be easily implemented in the U.S. economic model consisting of the stock return, the inflation rate and the industrial production growth rate.*

Keywords: Vector autoregressive process (VAR), Granger causality, Three-group causal path, Impulse response analysis

1. **Introduction.** For many years, the following question has been discussed in the fields of both academy and business: What kind of causalities can be used to make meaningful predictions concerning the economic system? Answers to this question have been provided with various economic theories and statistical methods. Several studies dealt with this question, relating it to the measure of the causality via spectral decomposition in a VECM model [1], to the causal connectivity analysis in neural mechanisms [2] or to frequency-modified causality of the network structure in a non-linear system [3]. Recently, the identification of the causality has applications in bidirectional function learning method [4], the dynamic economic models [5], the knowledge acquisition [6] and many others.

However, Granger's [7] causality has become a fairly popular technique to measure dynamic relationship between groups of variables in the time series process. It has been widely used in economics [8,9] since the 1960s, and its applications in neuroscience [10-12] have been in favor in the last years.

The concept of Granger causality is based on two-group framework in which all variables are partitioned into two groups x and z. We say that x is Granger-causal for z if x is useful in forecasting z. In this setting, the information set for forecasting z contains two groups x and z. However, in order to extend the idea of the two-group framework, Lütkepohl [13] and Dufour and Renault [14] considered a higher dimensional system in which all variables are partitioned into three groups, say x, z and w. In the three-group setting, the question that whether x is Granger-causal for z is reconsidered while the information set adds the auxiliary group w besides the original groups x and z.

Although Lütkepohl [13] and Dufour and Renault [14] suggested a three-group framework involving x, z and w, their concept still focused on the causality between the two groups x and z. The remaining group w is only used to expand the information set but is excluded from the causality we concerned. As a matter of fact, the framework they had proposed should be viewed as a two-group causal structure.

The idea of our research is different from those of Lütkepohl [13] and Dufour and Renault [14]. We intend to construct a complete causal link by employing all of the three groups x, z and w in a real dynamic system. That is, the third group w is used not only to expand the information set but also to join the causality link. Correspondingly, the truly three-group causality, which will present more detailed causal relationship, is actually motivated by the 'causal pathway' of the *Structural Equation Model* (SEM).

In addition, we provide an identification procedure to detect whether there exists a three-group causality in a VAR model. We also investigate the impulse response function between the two variables y_{jt} and y_{kt} deriving from different groups under the three-group pathway. Moreover, it's demonstrated that there is no reaction of y_{jt} to an impulse in y_{kt} if the group of y_{jt} leads the group of y_{kt}.

The remainder of the paper is organized as follows. Section 2 briefly introduces the concept of Granger causality and the leading relationship between groups. Section 3 discusses in detail the construction of the three-group causal path in a VAR model. A procedure which can judge the existence of the three-group causality is discussed in Section 4. The impulse response analysis for the three-group causality is investigated in Section 5. Section 6 shows an empirical application of the three-group causality. The final section illustrates the conclusion and issue about this research.

2. **Granger Causality and Leading Relationship.** Granger causality is a statistical concept of causality that is based on whether one process is useful in forecasting another process. Let $\Omega_z = (z_t, z_{t-1}, \ldots, z_1)$ and $\Omega_{xz} = (x_t, z_t, x_{t-1}, z_{t-1}, \ldots, x_1, z_1)$ denote the information sets, where x_t and z_t refer to the groups of variables, and let $z_t(h\,|\Omega)$ be the optimal predictor of z_{t+h} based on the available information set Ω. If $z_t(h\,|\Omega_{xz})$ has smaller prediction error than $z_t(h\,|\Omega_z)$ for any h, then x is said to be *Granger-causal* for z. In other word, Granger causality is used for examining whether eliminating $\{x_t\}$ from the information set Ω_{xz} will enlarge the prediction error or not.

Granger causality is particularly easy to be dealt with in the p^{th}-order VAR model written by

$$Y_t = v + \sum_{i=1}^{p} A_i Y_{t-i} + \varepsilon_t, \quad \varepsilon_t \sim WN(\mathbf{0}, \Sigma_\varepsilon), \quad t = 1, \ldots, T, \tag{1}$$

where Y_t is a $(K \times 1)$ random vector and A_i is a $(K \times K)$ coefficient matrix for $i = 1, 2, \ldots, p$. To partition Y_t into two groups z_t and x_t, the $VAR(p)$ model can also be represented in the following form

$$\begin{bmatrix} z_t \\ x_t \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \sum_{i=1}^{p} \begin{bmatrix} A_{11,i} & A_{12,i} \\ A_{21,i} & A_{22,i} \end{bmatrix} \begin{bmatrix} z_{t-i} \\ x_{t-i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}, \tag{2}$$

where z_t and x_t are $M \times 1$ and $(K-M) \times 1$ random vectors, respectively. Granger causality can be determined only through the coefficient submatrices in Model (2) because of the following property:

$$z_t(h\,|\Omega_{xz}) = z_t(h\,|\Omega_z) \Leftrightarrow A_{12,i} = 0 \quad \text{for } i = 1, \ldots, p.$$

It implies that x_t is Granger causal for z_t, if there exists at least one non-zero matrix among the set of matrices $A_{12,i}$, $i = 1, \ldots, p$.

Usually, we have a two-way (bidirectional) causality between x_t and z_t in Granger sense, i.e., x_t is Granger causal for z_t and z_t is Granger causal for x_t, too. This phenomenon is called a feedback in which the roles of cause and effect can exchange with each other. In a $VAR(p)$ model, feedback indicates that the following conditions are held simultaneously:

(a) there exists at least one nonzero matrix among $A_{12,i}$, $i = 1, \ldots, p$, and
(b) there exists at least one nonzero matrix among $A_{21,i}$, $i = 1, \ldots, p$.

In empirical studies Grange causality has been frequently used to describe the leading/lagging relationship between variables for a period of time. Hence, a feedback system implies that z_t leads x_t and x_t leads z_t, too, while this will induce contradiction in chronology with the Granger's claiming.

Since a two-way causality will make more conflicting results than that of a one-way (unidirectional) causality in some applications, our research will focus on the investigation of the latter. We define the leading relationships between the two groups z_t and x_t based on the one-way case. For simplicity, we form the two conditions to be:

$$\begin{aligned} &C_1 : A_{12,i} = \mathbf{0}, \quad \text{for } i = 1, \ldots, p, \text{ and} \\ &C_2 : A_{21,i} = \mathbf{0}, \quad \text{for } i = 1, \ldots, p \end{aligned} \tag{3}$$

and further evaluate the conditions for their true or false. Note that if C_1 is false, it indicates that there exists at least a nonzero matrix among $A_{12,i}$, $i = 1, \ldots, p$. Similarly, if C_2 is false, it indicates that there exists at least a nonzero matrix among $A_{21,i}$, $i = 1, \ldots, p$.

Definition 2.1. (**Leading Relationship**) *The two groups z_t and x_t are defined in Model (2). With the true or the false for C_1 and C_2 defined in (3), the variety of the leading relationship between z_t and x_t are defined in the following table:*

TABLE 1. Leading relationship between z_t and x_t

case	*condition*		*conclusion*	*symbol*
	C_1	C_2		
1	F	F	*failing to judge leading relationship*	$z \rightleftarrows x$
2	T	T	*z is irrelevant to x*	$z \parallel x$
3	T	F	*z leads x*	$z \to x$
4	F	T	*x leads z*	$x \to z$

In this definition, we use the term '*lead*' instead of '*Granger-causes*' to represent a more strict causal link. Case (1) in Table 1 just represents a feedback system which is treated as a controversy case. So, we say that we can not judge the leading relationship between z_t and x_t. In Case (2), z_t and x_t do not affect each other. We accordingly hold that z is irrelevant to x. In Case (3), z_t is a Granger-causal for x_t but not vice versa. For the one-way causality, we say that z leads x.

Example 2.1. *Let's consider a VAR(2) model*

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \\ y_{3t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} y_{1t-2} \\ y_{2t-2} \\ y_{3t-2} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \tag{4}$$

We want to detect the leading relationship between (y_{1t}, y_{2t}) and y_{3t}, where (y_{1t}, y_{2t}) refers to the one group containing the variables y_{1t} and y_{2t}. As in (3), we set the conditions:

$$C_1 : A_{12,1} = A_{12,2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and}$$
$$C_2 : A_{21,1} = A_{21,2} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

To judge true/false of C_1 and C_2, the actual values of related coefficient matrices are derived from Model (4):

$$A_{12,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{12,2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_{21,1} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad A_{21,2} = \begin{bmatrix} 0 & 2 \end{bmatrix}.$$

It is easily found that C_1 is true while C_2 is false. These are corresponding to Case (3) in Table 1. Thus, we claim that the group (y_{1t}, y_{2t}) leads y_{3t} in the VAR model, denoted by $(y_1, y_2) \to y_3$.

3. **Three-Group Causal Path.** In order to analyze a complete causality system in a higher dimensional situation, we propose a three-group causal structure by the use of SEM. SEM is originally a statistical technique to show causal dependency between variables, which can be represented as a path diagram. For example, the path diagram $x \to w \to z$ indicates a specific causality with the independent variable x, the intermediate variable w and the dependent variable z.

Hence, we make a three-group causal path with three types of groups: (a) the independent group, (b) the intermediate group and (c) the dependent group. Giving a simple example to illustrate the application of the three-group causality, assume that there is a 3th-order VAR model with three variables, investment (y_1), income (y_2) and consumption (y_3). Our goal is to identify which one is independent, which one is intermediate and which one is dependent among them. If the causal path is presented with the following form:

$$\begin{array}{ccc} \textit{income } y_2 \to & \textit{investment } y_1 \to & \textit{consumption } y_3, \\ \textit{(independent)} & \textit{(intermediate)} & \textit{(dependent)} \end{array}$$

we call it a Three-Group Causal Path with the independent group y_2, the intermediate group y_1 and the dependent group y_3.

Let us illustrate the background of our picture for processing three-group causality problem. As in the representation (1), let Y_t be divided into three groups y_{1t}, y_{2t} and y_{3t}. Then we have

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \sum_{i=1}^{p} \begin{bmatrix} A_{11,i} & A_{12,i} & A_{13,i} \\ A_{21,i} & A_{22,i} & A_{23,i} \\ A_{31,i} & A_{32,i} & A_{33,i} \end{bmatrix} \begin{bmatrix} y_{1t-i} \\ y_{2t-i} \\ y_{3t-i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}, \tag{5}$$

where it is assumed that y_{1t}, y_{2t} and y_{3t} are independent, intermediate and dependent groups with $M \times 1$, $L \times 1$ and $(K - M - L) \times 1$ random vectors, respectively, and could be diagramed as

$$y_1 \to y_2 \to y_3. \tag{6}$$

In terms of intuitive thinking for the three-group case, it is reasonable to suppose that the pathway (6) satisfies the following conditions for the leading relationship as in Definition 2.1:

(a) The independent group y_{1t} leads the group (y_{2t}, y_{3t}) which contains both intermediate and dependent ones. It can be diagramed as $y_{1t} \to (y_{2t}, y_{3t})$.

(b) The group $(y_{1t},\ y_{2t})$ containing both independent and intermediate ones leads the dependent group y_{3t}. It is diagramed as $(y_{1t}, y_{2t}) \to y_{3t}$.

In the following paragraphs, we are going to explain the mathematical aspects of the both conditions separately.

Condition (a): $y_{1t} \to (y_{2t}, y_{3t})$.

The leading relationship is originally defined in the two-group framework. Hence, when considering Condition (a), we take the coefficient matrices A_i in (5) as the following 2×2 partitioning form

$$A_i = \begin{array}{|c|cc|} \hline * & A_{12,i} & A_{13,i} \\ \hline A_{21,i} & * & * \\ A_{31,i} & * & * \\ \hline \end{array}$$

where the symbol '*' means that the submatrix of coefficients located in the corresponding position need not to be used in the analysis process. Then, from Case (3) in Table 1 we obtain

$$\begin{cases} \begin{bmatrix} A_{12,i} & A_{13,i} \end{bmatrix} = \mathbf{0} \quad \text{for } i = 1, 2, \ldots, p, \quad \text{and} \\ \text{there exists at least one non-zero matrix among } \begin{bmatrix} A_{21,i} \\ A_{31,i} \end{bmatrix}, \quad i = 1, \ldots, p. \end{cases} \tag{7}$$

Condition (b): $(y_{1t}, y_{2t}) \to y_{3t}$.

We partition the coefficient matrix A_i as

$$A_i = \begin{array}{|cc|c|} \hline * & * & A_{13,i} \\ * & * & A_{23,i} \\ \hline A_{31,i} & A_{32,i} & * \\ \hline \end{array}$$

and from Case (3) in Table 1, we have

$$\begin{cases} \begin{bmatrix} A_{13,i} \\ A_{23,i} \end{bmatrix} = \mathbf{0} \quad \text{for } i = 1, 2, \ldots, p, \quad \text{and} \\ \text{there exists at least one non-zero matrix among } \begin{bmatrix} A_{31,i} & A_{32,i} \end{bmatrix}, \quad i = 1, \ldots, p. \end{cases} \tag{8}$$

We believe that (7) and (8) will hold simultaneously if the three-group causal path (6) exists. Thus, the intersection of (7) and (8)

$$\begin{cases} A_{12,i}, A_{13,i} \text{ and } A_{23,i} \text{ are zero matrices for } i = 1, 2, \ldots, p, \\ \text{there exists at least one non-zero matrix among } \begin{bmatrix} A_{21,i} \\ A_{31,i} \end{bmatrix}, \quad i = 1, \ldots, p, \quad \text{and} \\ \text{there exists at least one non-zero matrix among } \begin{bmatrix} A_{31,i} & A_{32,i} \end{bmatrix}, \quad i = 1, \ldots, p; \end{cases} \tag{9}$$

should be treated as the background for defining the causal path (6).

In order to further analyze Rule (9), we design a 3×3 grid to convey the conditions related to the submatrix $A_{jk,i}$. That is, if the $(j,k)^{th}$ entry of the grid is equal to 0, it means $A_{jk,i} = \mathbf{0}$ for $i = 1, \ldots, p$. Alternatively, if the $(j,k)^{th}$ entry is identical to 'N', then it means there exists at least one non-zero matrix among $A_{jk,i}$, $i = 1, \ldots, p$. For example, if a grid is given below

	0	
	0	
N		

it means that the three conditions hold simultaneously: (a) $A_{12,i} = \mathbf{0}$ for $i = 1, 2, \ldots, p$, (b) $A_{22,i} = \mathbf{0}$ for $i = 1, 2, \ldots, p$ and (c) there exists at least one non-zero matrix among $A_{31,i}$, $i = 1, \ldots, p$.

In the following, we present the way to employ the grid tool to demonstrate Rule (9). It can be found that there are five types of grids satisfying Rule (9) in total, which are shown in the figure below.

(1)

	0	**0**
N		**0**
N	N	

(2)

	0	**0**
N		**0**
0	N	

(3)

	0	**0**
0		**0**
N	**0**	

(4)

	0	**0**
N		**0**
N	**0**	

(5)

	0	**0**
0		**0**
N	N	

FIGURE 1. Five types of grids satisfying Rule (9)

Notice that Case (4) in Figure 1 presents two relationships (a) $y_1 \to (y_2, y_3)$ and (b) $(y_1, y_3) \to y_2$ simultaneously, which can be checked according to Table 1. That invites a pathway

$$y_1 \to y_3 \to y_2. \tag{10}$$

Similarly, another pathway

$$y_2 \to y_1 \to y_3. \tag{11}$$

can also be suggested from Case (5) in Figure 1. We treat (10) and (11) as distinct causal paths from (6). As discussed in Section 2, we do not allow the irrational causality in which a variety of pathways occur at one time, such as feedback in the two-group case. Rule (9), therefore, seems to reveal the problem with contradiction in chronology.

In order to avoid these conflicting problems, we need to give the three-group causal path more refined descriptions than Rule (9). Specifically, we must exclude Cases (4) and (5), which generate extra conflicting pathways, when making the formal definition of the three-group causal path $y_1 \to y_2 \to y_3$.

Definition 3.1. (**Three-Group Causal Path**) *Suppose that Y_t follows a $VAR(p)$ process as in (5) and that $A_{12,i}$, $A_{13,i}$ and $A_{23,i}$ are zero submatrices for $i = 1, 2, \ldots, p$. For the following two conditions:*

(a) There exists at least one non-zero submatrix among $A_{21,i}$, $i = 1, 2, \ldots, p$, and there exists at least one non-zero submatrix among $A_{32,i}$, $i = 1, \ldots, p$.

(b) $A_{21,i} = \mathbf{0}$ and $A_{32,i} = \mathbf{0}$ for $i = 1, \ldots, p$, and there exists at least one non-zero submatrix among $A_{31,i}$, $i = 1, \ldots, p$.

If the condition (a) or (b) holds, then this system is called a three-Group Causality with the pathway $y_1 \to y_2 \to y_3$, where y_1 is an independent group, y_2 is an intermediate group and y_3 is a dependent group.

In Definition 3.1, Condition (a) implies that Case (1) or Case (2) in Figure 1 remains, while Condition (b) implies that Case (3) holds. Hence, Cases (4) and (5) have been excluded from the definition of the three-group causality.

Note that Definition 3.1 is devoted to recognizing the pathway $y_1 \to y_2 \to y_3$. However, it can still be used to investigate another type of pathway, such as $y_2 \to y_1 \to y_3$, as long as the random vectors and the coefficient matrices in Model (5) are rearranged. For instance, if we want to check the existence of the causal path $y_2 \to y_1 \to y_3$, we need to rearrange the random vectors and coefficient matrices as

$$Y_t^* = \begin{bmatrix} y_{2t} \\ y_{1t} \\ y_{3t} \end{bmatrix}, \quad A_i^* = \begin{bmatrix} A_{22,i} & A_{21,i} & A_{23,i} \\ A_{12,i} & A_{11,i} & A_{13,i} \\ A_{32,i} & A_{31,i} & A_{33,i} \end{bmatrix}, \quad i = 1, \ldots, p. \tag{12}$$

Then the detecting work can be implemented through Definition 3.1, based on the new VAR model organized with Y_t^* and A_i^*.

4. **An Identification Procedure for the Three-Group Causal Path.** When using Definition 3.1 to detect whether a pathway exists or not, the detected pathway must be explicitly specified in advance. However, in practical applications researchers often have no idea about how to specify a causality of interests before performing the detecting work of Definition 3.1.

To overcome such an inconvenience, we present a procedure which can identify the existing causality without specifying a pathway in advance. This procedure involves the following three steps:

Step 1: Detecting the leading relationship for each combination.

At this step, by partitioning y_1, y_2 and y_3 we set up three types of combinations:

$$\text{(a) } y_1 \; vs \; (y_2, y_3), \;\; \text{(b) } (y_1, y_2) \; vs \; y_3 \text{ and } (c) \; y_2 \; vs \; (y_1, y_3).$$

Then the leading relationship for each combination can be decided according to the rule in Definition 2.1. In the following is the example to illustrate the work.

Example 4.1. *Referring to Example 2.1, we find the leading relationship for each combination.*

(a) y_1 *vs* (y_2, y_3)*:*

As in (3) we set the conditions:

$$C_1 : \begin{bmatrix} A_{12,1} & A_{13,1} \end{bmatrix} = \begin{bmatrix} A_{12,2} & A_{13,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad and$$
$$C_2 : \begin{bmatrix} A_{21,1} \\ A_{31,1} \end{bmatrix} = \begin{bmatrix} A_{21,2} \\ A_{31,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which can be rewritten as

$$C_1 : A_{12,i} = A_{13,i} = 0 \quad for\ i = 1, 2, \; and$$
$$C_2 : A_{21,i} = A_{31,i} = 0 \quad for\ i = 1, 2.$$

In terms of the actual values of those submatrices from Model (4), it is easy to find that C_1 *and* C_2 *are both true. Corresponding to Case (2) in Table 1, we get the conclusion that* y_1 *is irrelevant to* (y_2, y_3)*, denoted by* $y_1 \parallel (y_2, y_3)$*.*

(b) (y_1, y_2) *vs* y_3*:*

For this combination, we refer to the conclusion in Example 2.1. The conclusion is that (y_{1t}, y_{2t}) *leads* y_{3t}*, denoted by* $(y_1, y_2) \to y_3$*.*

(c) y_2 *vs* (y_1, y_3)*:*

Here, we make rearrangement for random vectors and coefficient matrices as in (12), and then obtain a new VAR model. Hence, the conditions are set as

$$C_1 : \begin{bmatrix} A_{21,1} & A_{23,1} \end{bmatrix} = \begin{bmatrix} A_{21,2} & A_{23,2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \quad and$$
$$C_2 : \begin{bmatrix} A_{12,1} \\ A_{32,1} \end{bmatrix} = \begin{bmatrix} A_{12,2} \\ A_{32,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which can be rewritten as

$$C_1 : A_{21,i} = A_{23,i} = 0 \quad for\ i = 1, 2, \quad and$$
$$C_2 : A_{12,i} = A_{32,i} = 0 \quad for\ i = 1, 2.$$

With the actual values of coefficient matrices from Model (4), we can find that C_1 *is true while* C_2 *is false. Corresponding to Case (3) in Table 1, the conclusion is that* y_2 *leads* (y_1, y_3)*, denoted by* $y_2 \to (y_1, y_3)$*.*

The results are gathered in Table 2.

Table 2. Leading relationships in Model (4)

combination	*condition*		*conclusion*
(a) y_1 *vs* (y_2, y_3)	C_1: $A_{12,i} = A_{13,i} = 0,\ i = 1, 2$	T	$y_1 \parallel (y_2, y_3)$
	C_2: $A_{21,i} = A_{31,i} = 0,\ i = 1, 2$	T	
(b) (y_1, y_2) *vs* y_3	C_1: $A_{13,i} = A_{23,i} = 0,\ i = 1, 2$	T	$(y_1, y_2) \to y_3$
	C_2: $A_{31,i} = A_{32,i} = 0,\ i = 1, 2$	F	
(c) y_2 *vs* (y_1, y_3)	C_1: $A_{21,i} = A_{23,i} = \mathbf{0},\ i = 1, 2$	T	$y_2 \to (y_1, y_3)$
	C_2: $A_{12,i} = A_{32,i} = \mathbf{0},\ i = 1, 2$	F	

Step 2: Searching a pathway from two combinations

At this step, we are going to look for a pathway with the results of Step 1. We try to select two among the three combinations so that a three-group pathway can be derived by merging conclusions of the two selected combinations. If we could not find such a pathway from any two combinations, it indicates that a three-group causal path does not exist in the VAR system and we shall stop the identifying procedure. Otherwise, we continue to go to Step 3. In the following is the example using to explain the foregoing work.

Example 4.1. (continued) *From Table 2, we decide to select Combination (b) and (c), because their conclusions* $(y_1, y_2) \to y_3$ *and* $y_2 \to (y_1, y_3)$ *can deduce the three-group pathway* $y_2 \to y_1 \to y_3$*. This step can be diagramed as*

$$\begin{array}{l} (b).\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to y_3 \\ (c).\ y_2 \to \begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \end{array} \quad \Rightarrow y_2 \to y_1 \to y_3.$$

Step 3: Confirming the pathway to be valid

After obtaining a pathway from Step 2, further we confirm its validity by checking the conclusion of 'the remaining one combination' – note that the other two combinations have been used to search the pathway among all three combinations at the previous step. If the conclusion of the remaining one combination is 'failing to judge leading relationship' or 'being irrelevant each other', then the pathway found out at Step 2 is definitely confirmed to be a valid three-group causal path. Otherwise, the three-group causal path does not exist. Finally the identification procedure comes to an end.

Note that at Step 3 the conclusion 'failing to judge leading relationship' just corresponds to Case (1) or (2) in Figure 1, and the conclusion 'being irrelevant with each other' corresponds to Case (3). This two conclusions induce a valid one-way causality. Besides, other conclusions will correspond to Case (4) or (5) which results in a conflicting two-way pathway.

Example 4.1. (continued) *Since Combination (b) and (c) have been used to find out a pathway at Step 2, we need only to check Combination (a) at Step 3. The conclusion of Combination (a) is that* y_1 *and* (y_2, y_3) *is irrelevant from Table 2. So, we verify that the pathway* $y_2 \to y_1 \to y_3$ *is a valid three-group causal path. Finally we obtain a three-group causality with the independent group* y_2*, the intermediate group* y_1 *and the dependent group* y_3 *in Model (4).*

Figure 2 illustrates the identifying procedure for a three-group causality.

We also show the case of absent three-group causality in the following example.

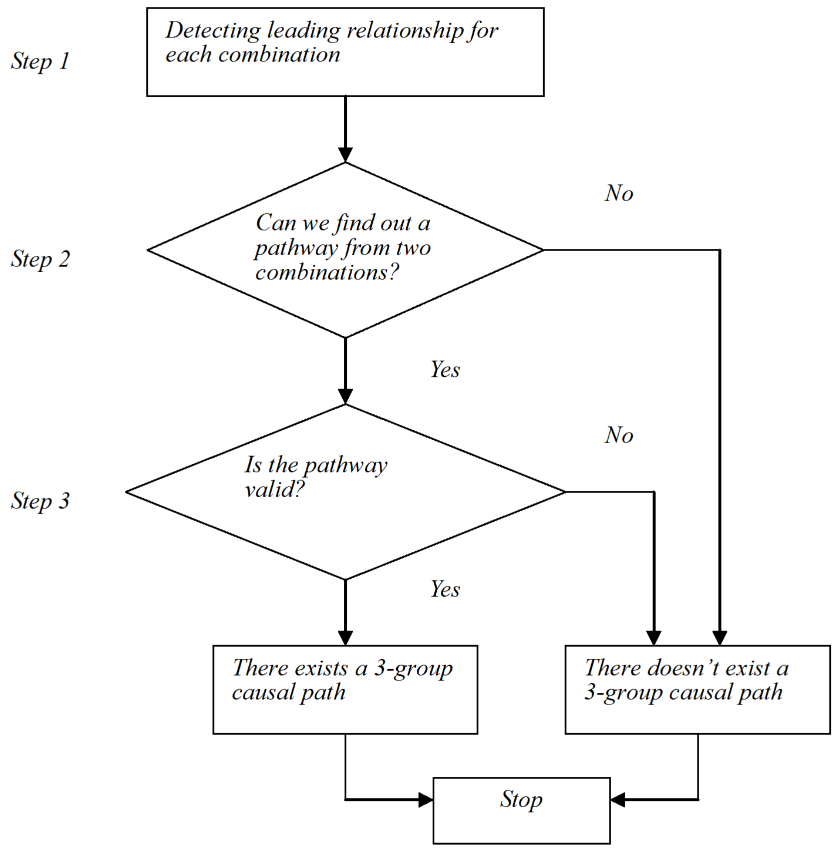

FIGURE 2. Flowchart of the procedure for identifying the three-group causal path

Example 4.2. *Consider a VAR(2) model*

$$\begin{bmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1t-i} \\ y_{2t-i} \\ y_{3t-i} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_{1t-i} \\ y_{2t-i} \\ y_{3t-i} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \\ \varepsilon_{3t} \end{bmatrix}. \tag{13}$$

The procedure for identifying the three-group causal path is as follows.
Step 1: For Model (13), the examination of the leading relationship for each combination is demonstrated at Table 3.

TABLE 3. Leading relationship for each combination

combination	*condition*		*conclusion*
(a) y_1 *vs* (y_2, y_3)	$C_1 : A_{12,i} = A_{13,i} = \mathbf{0},\ i = 1, 2$	T	$y_1 \to (y_2, y_3)$
	$C_2 : A_{21,i} = A_{31,i} = \mathbf{0},\ i = 1, 2$	F	
(b) (y_1, y_2) *vs* y_3	$C_1 : A_{13,i} = A_{23,i} = \mathbf{0},\ i = 1, 2$	F	$y_3 \to (y_1, y_2)$
	$C_2 : A_{31,i} = A_{32,i} = \mathbf{0},\ i = 1, 2$	T	
(c) y_2 *vs* (y_1, y_3)	$C_1 : A_{21,i} = A_{23,i} = \mathbf{0},\ i = 1, 2$	F	$(y_1, y_3) \to y_2$
	$C_2 : A_{12,i} = A_{32,i} = \mathbf{0},\ i = 1, 2$	T	

Step 2: From Table 3, the pathway $y_3 \to y_1 \to y_2$ can be derived by merging the conclusions of Combination (b) and (c). It can be diagramed as

$$\left.\begin{array}{l}(b).\ y_3 \to \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ (c).\ \begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \to y_2 \end{array}\right. \Rightarrow y_3 \to y_1 \to y_2.$$

Step 3: Since the conclusion of Combination (a) is neither 'failing to judge leading relationship' nor 'being irrelevant each other', we finally claim that the three-group causal path does not exist in Model (13).

Note that in Example 4.2, if we select Combination (a) and (c) at Step 2, then a different pathway is derived:

$$\left.\begin{array}{l}(a).\ y_1 \to \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \\ (c).\ \begin{pmatrix} y_1 \\ y_3 \end{pmatrix} \to y_2 \end{array}\right. \Rightarrow y_1 \to y_2 \to y_3.$$

Continue to see Step 3, at present 'the remaining one combination' refers to Combinations (b), and its conclusion is neither '*failing to judge leading relationship*' nor '*being irrelevant each other*'. Hence, it is still claimed that the three-group causal path does not exist. We obtain the same final result although making different selection at Step 2.

5. Impulse Response Analysis for Three-Group Causal Path.

We are frequently interested in investigating the relationship between two variables in a system that involves a number of other variables. In a VAR model the impulse response analysis is for quantifying the reaction of one variable to an impulse in another variable with all other variables held constant. To carry out the impulse response analysis for the two variables y_{jt} and y_{kt}, the VAR model in (1) is rewritten in a $MA(\infty)$ representation as

$$Y_t = \mu + \sum_{i=0}^{\infty} \Theta_i w_{t-i}, \quad w_t \sim WN(0, I_k),$$

where

$$\begin{aligned} \Theta_i &= \Phi_i Q \\ \Phi_i &= \begin{cases} I_k & i = 0 \\ \sum_{j=1}^{i} \Phi_{i-j} A_j & i = 1, 2, \ldots, \quad A_j = 0, j > p, \end{cases} \end{aligned} . \tag{14}$$

and the nonsingular lower triangular matrix Q is derived from Choleski decomposition $\Sigma_\varepsilon = QQ'$, and A_j and Σ_ε have been defined in (1). Then the $(j,k)^{th}$ element of the matrix Θ_i just represents the reaction of the variable y_{jt+i} to an impulse in the variable y_{kt}, i.e.,

$$\{\Theta_i\}_{jk} = \partial y_{jt+i} / \partial y_{kt} \quad \text{for } i = 1, 2, \ldots. \tag{15}$$

Especially when $\partial y_{jt+i}/\partial y_{kt} = 0$, it indicates that y_{kt} does not affect y_{jt+i}. It should be kept in mind that the impulse response analysis focuses on the relationship between 'variables', while Granger causality focuses on the relationship between 'groups of variables'.

With the impulse responses analysis, we study the relationship between two variables which belong to different groups in a three-group causality. We provide the following property.

Property 5.1. *Suppose that two variables y_{jt} and y_{kt} belong to different groups in the VAR model with a three-group causal path. If under the pathway the group to which y_{jt} belongs is located in front of the group to which y_{kt} belongs, then there is no reaction of y_{jt+i} to an impulse in y_{kt} for $i = 1, 2, \ldots$; hence, the impulse response function is*

$$\partial y_{jt+i}/\partial y_{kt} = 0, \quad i = 1, 2, \ldots.$$

Proof: Let three groups y_{1t}, y_{2t} and y_{3t} be g-, s- and $(k-g-s)$-dimensional random vectors, respectively. And without loss of generality, we assume that the three-group causal path is with the independent group y_1, the intermediate group y_2 and the dependent group y_3. Then the coefficient matrix A_i in (1) can be represented in the following form:

$$\underset{k\times k}{A_i} = \begin{bmatrix} \underset{g\times g}{{}_iA_{11}} & \underset{g\times s}{\mathbf{0}} & \underset{g\times(k-g-s)}{\mathbf{0}} \\ \underset{s\times g}{{}_iA_{21}} & \underset{s\times s}{{}_iA_{22}} & \underset{s\times(k-g-s)}{\mathbf{0}} \\ \underset{(k-g-s)\times g}{{}_iA_{31}} & \underset{(k-g-s)\times s}{{}_iA_{32}} & \underset{(k-g-s)\times(k-g-s)}{{}_iA_{33}} \end{bmatrix}, \quad i = 1, 2, \ldots, p,$$

and the lower triangular matrix Q in (14) can also be represented in the similar form

$$\underset{k\times k}{Q} = \begin{bmatrix} \underset{g\times g}{Q_{11}} & \underset{g\times s}{\mathbf{0}} & \underset{g\times(k-g-s)}{\mathbf{0}} \\ \underset{s\times g}{Q_{21}} & \underset{s\times s}{Q_{22}} & \underset{s\times(k-g-s)}{\mathbf{0}} \\ \underset{(k-g-s)\times g}{Q_{31}} & \underset{(k-g-s)\times s}{Q_{32}} & \underset{(k-g-s)\times(k-g-s)}{Q_{33}} \end{bmatrix}.$$

When we perform matrix operation on A_i and Q involving addition, subtraction and multiplication, the resulting matrix will has the same form as A_i and Q. Since the matrix Θ_i in (14) is derived by mixing the three operations: addition, subtraction, and multiplication on A_i and Q, therefore, Θ_i must has the following representation form:

$$\underset{k\times k}{\Theta_i} = \begin{bmatrix} \underset{g\times g}{\Theta_{11,i}} & \underset{g\times s}{\mathbf{0}} & \underset{g\times(k-g-s)}{\mathbf{0}} \\ \underset{s\times g}{\Theta_{21,i}} & \underset{s\times s}{\Theta_{22,i}} & \underset{s\times(k-g-s)}{\mathbf{0}} \\ \underset{(k-g-s)\times g}{\Theta_{31,i}} & \underset{(k-g-s)\times s}{\Theta_{32,i}} & \underset{(k-g-s)\times(k-g-s)}{\Theta_{33,i}} \end{bmatrix}, \quad i = 0, 1, 2, \ldots \tag{16}$$

If the group to which y_{jt} belongs is located in front of the group to which y_{kt} belongs in the pathway $y_1 \to y_2 \to y_3$, the matrix element $\{\Theta_i\}_{jk}$ for every $i \geq 1$ must fall within the scope of the three zero matrices in (16). Hence, from (15) we obtain

$$\partial y_{jt+i}/\partial y_{kt} = 0 \quad \text{for } i = 1, 2, \ldots.$$

The property is important in presenting the impact of any variable on others in the three-group system. We give an example to illustrate the application of Property 5.1.

Example 5.1. *Continuing to consider Example 4.1, we have found the three-group causal path $y_2 \to y_1 \to y_3$ in Model (4). Suppose that the two variables y_{jt} and y_{kt} come from the two groups y_2 and y_3, respectively. Based on Property 5.1, we can claim that there is no reaction of y_{jt+i} to an impulse in y_{kt} for $i = 1, 2, \ldots$ since the group y_2 is located in front of the group y_3 in the pathway $y_2 \to y_1 \to y_3$. To give a visual impression, it can be simply diagramed as*

$$\begin{array}{ccccc} y_2 & \to & y_1 & \to & y_3 \\ \Uparrow & & & & \Uparrow \\ y_{jt} & & \not\leftarrow & & y_{kt} \end{array}.$$

Similarly, if we suppose that y_{jt} and y_{kt} come from the two groups y_1 and y_3, respectively, then there is no reaction of y_{jt+i} to an impulse in y_{kt} for $i = 1, 2, \ldots,$ diagramed as

$$\begin{array}{ccccc} y_2 & \rightarrow & y_1 & \rightarrow & y_3 \\ & & \Uparrow & & \Uparrow \\ & & y_{jt} & \not\leftarrow & y_{kt} \end{array}.$$

6. **Empirical Study.** In order to demonstrate the usefulness of the proposed techniques, in this section we investigate the causal relation by taking the stock return, the inflation rate and the industrial production growth rate in a VAR model. We adopt quarterly data from 2001:IV to 2008:III with the three variables: the stock return (MARKET) is the continuously compounded return on S & P 500 index, the inflation rate (INF) refers to the continuously compounded growth rate of U.S. CPI, and the industrial production growth rate (IPG) is the continuously compounded growth rate of U.S. industrial production. The data over this period is displayed in Figure 3.

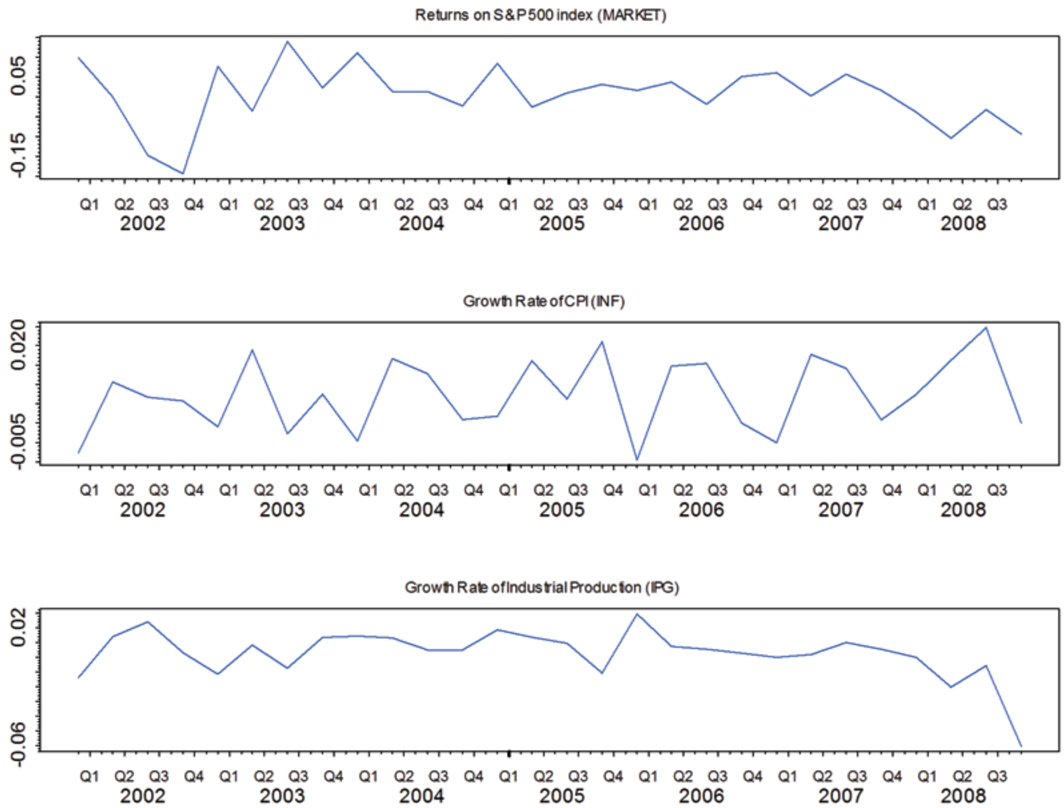

FIGURE 3. Time series plots of MARKET, INF and IPG

A VAR(2) model is fitted with MARKET (y_{1t}), INF (y_{2t}) and IPG (y_{3t}), and the estimated results are presented by the expression

$$Y_t = \begin{bmatrix} \underset{(0.026)}{-0.508} \\ \underset{(0.004)}{0.015} \\ \underset{(0.005)}{0.004} \end{bmatrix} + \begin{bmatrix} \underset{(0.196)}{0.158} & \underset{(1.623)}{3.742} & \underset{(1.206)}{3.582} \\ \underset{(0.026)}{0.009} & \underset{(0.220)}{-0.532} & \underset{(0.163)}{-0.094} \\ \underset{(0.039)}{-0.039} & \underset{(0.323)}{-0.028} & \underset{(0.240)}{-0.041} \end{bmatrix} Y_{t-1} + \begin{bmatrix} \underset{(0.179)}{0.128} & \underset{(1.530)}{0.481} & \underset{(0.832)}{1.406} \\ \underset{(0.024)}{-0.006} & \underset{(0.207)}{-0.403} & \underset{(0.113)}{-0.212} \\ \underset{(0.035)}{0.009} & \underset{(0.305)}{0.241} & \underset{(0.165)}{0.224} \end{bmatrix} Y_{t-2} + \hat{\varepsilon}_t,$$

where the value in the parentheses is the estimated standard deviation of the coefficient.

However, because the coefficients are all unknown in the empirical example, we do not have the actual values but their estimated values. Here the condition C_i ($i = 1, 2$) cannot be correctly evaluated to be true or false. In fact, this problem can be easily overcome by employing the statistical testing technique. First, let's treat C_1 and C_2 as the null hypotheses. And then Wald Test is used to decide whether C_i is accepted or rejected for each $i = 1, 2$ with the significance level α.

With the hypothesis testing, the procedure to identify the three-group causal path is described as follows.

Step 1: The testing result of the leading relationship for each combination is displayed in Table 4.

TABLE 4. Test for the three-group causal path for MARKET/INF/IPG

combination		null hypothesis	p-value	decision	conclusion
(a)	y_{1t} vs (y_{2t}, y_{3t})	$H_0^{(1)}$: $A_{12,i} = A_{13,i} = 0,\ i = 1, 2$	0.003	reject	$(y_2, y_3) \to y_1$
		$H_0^{(2)}$: $A_{21,i} = A_{31,i} = 0,\ i = 1, 2$	0.868	accept	
(b)	(y_{1t}, y_{2t}) vs y_{3t}	$H_0^{(1)}$: $A_{13,i} = A_{23,i} = 0,\ i = 1, 2$	0.002	reject	$y_3 \to (y_1, y_2)$
		$H_0^{(2)}$: $A_{31,i} = A_{32,i} = 0,\ i = 1, 2$	0.824	accept	
(c)	y_{2t} vs (y_{1t}, y_{3t})	$H_0^{(1)}$: $A_{21,i} = A_{23,i} = 0,\ i = 1, 2$	0.325	accept	$y_2 \parallel (y_1, y_3)$
		$H_0^{(2)}$: $A_{12,i} = A_{32,i} = 0,\ i = 1, 2$	0.168	accept	

Table 4 includes six hypotheses testing. In each testing the null $H_0^{(i)}$ will be rejected if the p-value is less than the significant level $\alpha = 0.05$.

Step 2: From Table 4, the conclusions of Combination (a) and (b) imply the pathway $y_3 \to y_2 \to y_1$, diagramed by

$$\left.\begin{array}{l} (a).\ \begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \to y_1 \\ (b).\ y_3 \to \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{array}\right. \Rightarrow y_3 \to y_2 \to y_1.$$

Step 3: Since Combination (c) reveals the irrelevance of relationship between two groups y_{2t} and (y_{1t}, y_{3t}), we conclude that there exists a three-group causal path with the independent variable IPG (y_3), the intermediate variable INF (y_2) and the dependent variable MARKET (y_1), denoted as

$$\text{IPG} \to \text{INF} \to \text{MARKET}. \tag{17}$$

The causal path indicates that the industrial production growth rate and the inflation rate will lead the stock return, but not vice versa.

In order to further realize the dynamic interactions among IPG, INF and MARKET in the three-group causality, we also present the impulse responses analysis for the three variables.

According to Property 5.1, Pathway (17) imposes the restriction that

$$\partial y_{jt+i} / \partial y_{kt} = 0 \quad \text{for } i = 1, 2, \ldots$$

when (y_{kt}, y_{jt}) = (MARKET, INF), (MARKET, IPG) and (INF, IPG). We can see such restriction at the three bottom-left panels of Figure 4. Therefore, the impulse response analysis shows that MARKET does not affect INF and IPG, as well as that INF does not affect IPG.

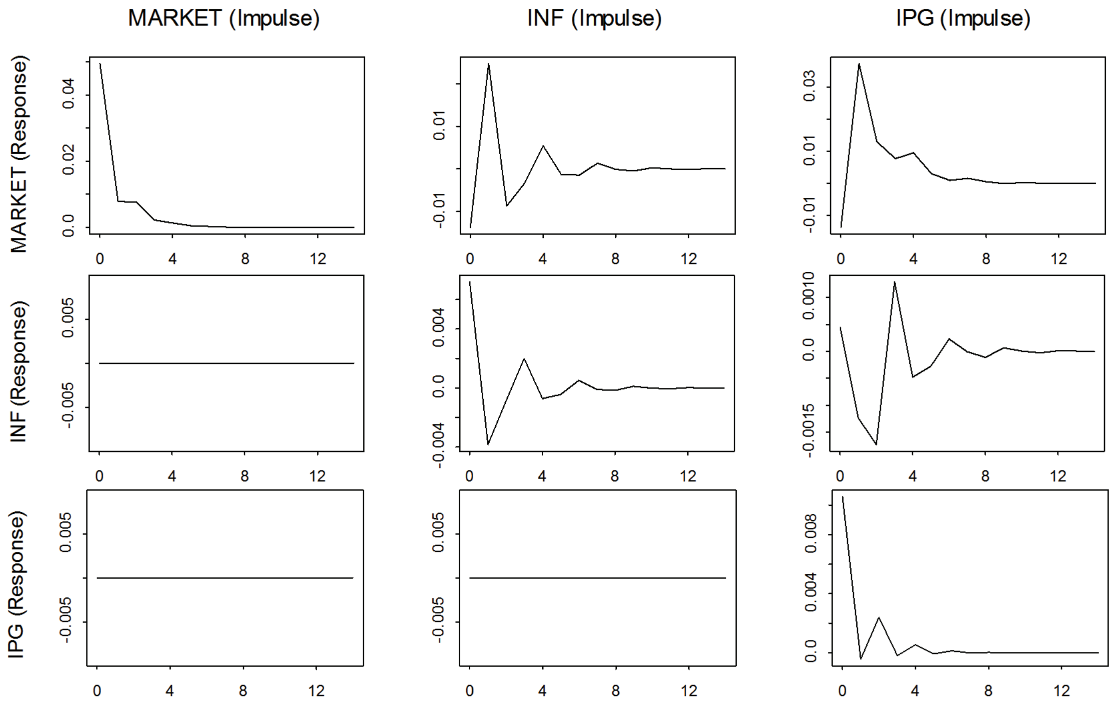

FIGURE 4. Impulse responses of the MARKET/INF/IPG system (impulse $\rightarrow$ response)

7. **Conclusion.** In this paper, we have constructed a three-group causality for capturing the leading/lagging relationship among groups in a VAR process, and the identification procedure for the pathway also has been proposed. Furthermore, we have shown that these techniques can be easily implemented in an U.S. economic model.

This research extends the previous studies with the following points:

(1) Different from Lütkepohl [13] and Dufour and Renault [14], we construct a truly three-group causal link by employing all groups; hence, a more detailed causal relation can be represented in a dynamic system.
(2) SEM is originally used in psychological and educational research. To avoid contradiction in chronology, we impose one-way direction on the causality. This imposition will make SEM popularized to the field of economics instead.

There are several issues of interest which might be studied in the future.

(1) To consider a VECM (Vector Error Correction Model), which is a nonstationary VAR process with cointegration, the construction of a three-group causality in the VECM model will be a appealing topic.
(2) The six hypotheses testing, shown in Section 6, for identifying the causal link can be grouped as one multiple hypothesis testing in the simultaneous sense. How to find an appropriate familywise error rate in the entire multiple skeleton, instead of the significance level α in each individual testing, will be an important issue.

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