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FUZZY PARALLEL SYSTEM RELIABILITY ANALYSIS BASED ON LEVEL (λ, ρ) INTERVAL-VALUED FUZZY NUMBERS

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Abstract. *In this paper, we examine the reliability of a parallel system. We use the level (λ, ρ) interval-valued fuzzy numbers to find the fuzzy reliability of parallel systems and obtain the estimated reliability of the systems in the fuzzy sense by employing the signed distance method.*

Keywords: Fuzzy reliability, Interval-valued fuzzy number, Statistical data, Confidence interval, Signed distance

1. **Introduction.** In this article, we consider the reliability of a parallel system. We know that in a factory production process if we want to consider the reliability of the production process experiments are necessary. It is difficult to obtain significant results with this reliability problem if we only consider a model without using experiments. Conventional optimization methods assume that all parameters and goals of a model are precisely known. However, in many practical problems incomplete and unreliable information exists. Therefore, we use the fuzzy concept to treat this parallel system reliability problem.

Because the population reliability R_j of the subsystem P_j $(j = 1, 2, \ldots, n)$ is unknown, we can obtain reliable statistical data R_{jq}, $q = 1, 2, \ldots, n_j$ from the subsystem P_j in the parallel system. If we use the average value $\bar{R}_j$ as the point estimate R_j from past statistical data, we will not know the probability of the error $\bar{R}_j - R_j$. Moreover, the system reliability may fluctuate around the point estimate $\bar{R}_j$ during a time interval. It follows that using the point estimate $\bar{R}_j$ to estimate the population reliability R_j is not suitable for real cases. Therefore, it is more desirable to use the statistical confidence interval. We use the statistical confidence interval instead of the point estimate. We transfer the statistical confidence interval into the level (λ, ρ) *i-v* fuzzy number. We consider the fuzzy reliable system through these level (λ, ρ) *i-v* fuzzy numbers. We fuzzify the reliability of parallel systems. Through defuzzifying the fuzzy parallel system reliability using the signed distance method we obtain a fuzzy reliability estimate in the fuzzy sense.

There are two fundamental hypotheses in conventional reliability theory, namely the probability assumption and the binary-state assumption. (1^0) The probability assumption: The system behavior is fully characterized in the context of the probability measure. (2^0) The binary-state assumption: At any given time, the system has only two states. One is the functioning state and the other is the failed state. In earlier papers [1-4], the authors modified (2^0) to (2^*) as follows. (2^*) The fuzzy state assumption: at any given time, the system has only two states. One is the fuzzy success state and the other is the fuzzy failure state. In [5], the authors used the α-cut of level 1 fuzzy numbers to obtain the interval and find the fuzzy reliability of a serial system and the fuzzy reliability of a parallel system. In [6], they used fuzzy numbers to find the fuzzy reliability of a serial system and the fuzzy reliability of a parallel system. Yao et al. [10] used triangular fuzzy numbers and statistical data to find the fuzzy reliabilities of both systems. Singer [11] did not use the statistical method. He used the L-R type fuzzy number to consider the fuzzy reliability problem.

In this paper, we consider the reliability of a parallel system. We use the statistical confidence interval concept to fuzzify the parallel system using a level (λ, ρ) *i-v* fuzzy number and defuzzify the parallel system using the signed distance method. Section 2 presents some properties of fuzzy sets. Section 3.1 uses a statistical point estimate to examine the reliability of the parallel system. In Section 3.2, we use the statistical confidence interval concept converted to level (λ, ρ) *i-v* fuzzy number. We give a numerical example in Section 4 and we offer some discussions in Section 5 and conclusions in Section 6.

2. **Preliminaries.**

In order to consider the fuzzy system reliability analysis based on level 1-α fuzzy numbers and level (λ, ρ) interval-valued fuzzy numbers, we provide several definitions as follows [7,9,16]:

Definition 2.1. *A fuzzy set $\tilde{A}$ defined on R is called the level λ triangular fuzzy number if its membership function is*

$$\mu_{\tilde{A}}(x) = \begin{cases} \lambda\frac{(x-a)}{b-a}, & a \le x \le b \\ \lambda\frac{(c-x)}{c-b}, & b \le x \le c \\ 0, & \textit{otherwise}, \end{cases}$$

where $a < b < c$, $0 < \lambda \le 1$, then $\tilde{A}$ is called the level λ fuzzy number and denoted by $\tilde{A} = (a, b, c; \lambda)$. When $\lambda = 1$ is called a triangular fuzzy number and denoted by $\tilde{A} = (a, b, c)$.

Definition 2.2. *[7]. An interval-valued fuzzy set (i-v fuzzy set for short) $\tilde{A}$ on R is given by*

$$\tilde{A} \triangleq \{(x, [\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x)])\}, \quad \forall x \in R,$$

where $\mu_{\tilde{A}^L}(x) \le \mu_{\tilde{A}^U}(x)$ and $\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x) \in [0, 1]$.

We denote it by $\bar{\mu}_{\tilde{A}}(x) = [\mu_{\tilde{A}^L}(x), \mu_{\tilde{A}^U}(x)]$, $\forall x \in R$, or

$$\tilde{A} = [\tilde{A}^L, \tilde{A}^U] \tag{1}$$

Let $\tilde{A}^L = (a, b, c; \lambda)$ and $\tilde{A}^U = (e, b, h; \rho)$, where $0 < \lambda \le \rho \le 1$ and $e < a < b < c < h$. Then, the *i-v* fuzzy set is written as

$$\tilde{A} = [(a, b, c; \lambda), (e, b, h; \rho)] \tag{2}$$

We view $\tilde{A}$ as a level (λ, ρ) *i-v* fuzzy number. The family of all level (λ, ρ) fuzzy numbers will then be denoted by

$$F_{IV}(\lambda, \rho) = \{[(a, b, c; \lambda), (e, b, h; \rho)] | e < a < b < c < h\},$$

where $0 < \lambda \le \rho \le 1,\ a, b, c, e, h \in R$.

We let $\tilde{A} = [(a, b, c; \lambda), (e, b, h; \rho)] = [\tilde{A}^L, \tilde{A}^U] \in F_{IV}(\lambda, \rho)$ and then have the α-cut of $\tilde{A}^L$ and $\tilde{A}^U$, $0 \le \alpha \le 1$ at the left and right endpoint as follows:

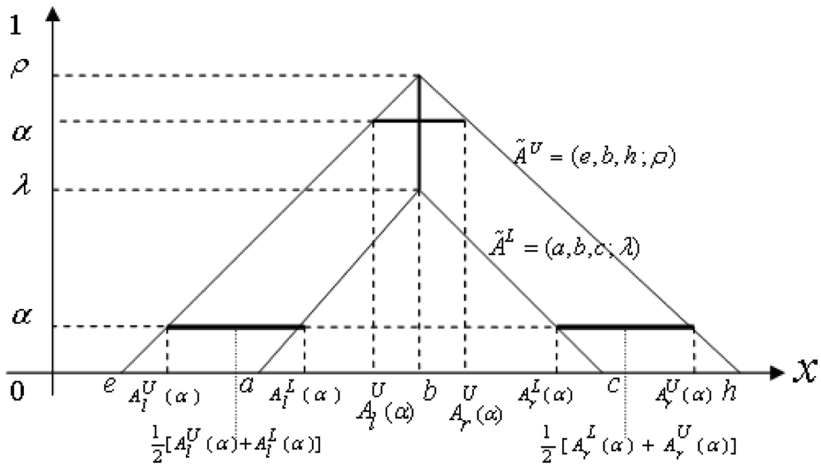

FIGURE 1. α-cut of level (λ, ρ) *i-v* fuzzy number $\tilde{A}$

If $0 \le \alpha < \lambda$ then

$$\begin{aligned} \tilde{A}_l^L(\alpha) &= a + (b-a)\frac{\alpha}{\lambda}, \quad \tilde{A}_r^L(\alpha) = c - (c-b)\frac{\alpha}{\lambda}, \\ \tilde{A}_l^U(\alpha) &= e + (b-e)\frac{\alpha}{\rho}, \quad \tilde{A}_r^U(\alpha) = h - (h-b)\frac{\alpha}{\rho}, \end{aligned} \tag{3}$$

and if $\lambda \le \alpha \le \rho$ then

$$\tilde{A}_l^U(\alpha) = e + (b-e)\frac{\alpha}{\rho}, \quad \tilde{A}_r^U(\alpha) = h - (h-b)\frac{\alpha}{\rho}. \tag{4}$$

Using the decomposition theory with Figure 1, we can represent $\tilde{A}$ as:

$$\tilde{A} = \bigcup_{0 \le \alpha < \lambda} \left(\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha); \alpha \right] \cup \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha); \alpha \right] \right) \cup \bigcup_{\lambda \le \alpha \le \rho} \left[\tilde{A}_l^U(\alpha), \tilde{A}_r^U(\alpha); \alpha \right] \tag{5}$$

Using the similar method in [9], we discuss the signed distance and ranking of level (λ, ρ) *i-v* fuzzy number defined on $F_{IV}(\lambda, \rho)$. First of all, we consider the definition of the signed distance on R.

Definition 2.3. *[9]. Let $a, 0 \in R$. Define $d_0(a, 0) = a$, $d_0(a, 0)$ is called the signed distance from a to 0.*

Using Definition 2.3, the signed distances from intervals $\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)\right]$ and $\left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)\right]$ to 0 can be defined by $d_0\left(\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)\right], 0\right) = \frac{1}{2}\left[\tilde{A}_l^U(\alpha) + \tilde{A}_l^L(\alpha)\right]$ and $d_0\left(\left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)\right], 0\right) = \frac{1}{2}\left[\tilde{A}_r^L(\alpha) + \tilde{A}_r^U(\alpha)\right]$, respectively. Since $\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)\right] \cap \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)\right] = \emptyset$, from (3) and (4), the signed distance from $\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)\right] \cup \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)\right]$ to 0 can be defined by

$$\begin{aligned} & d_0\left(\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)\right] \cup \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)\right], 0\right) \\ &= \frac{1}{4}\left[a + c + e + h + (2b - a - c)\frac{\alpha}{\lambda} + (2b - h - e)\frac{\alpha}{\rho}\right]. \end{aligned}$$

For each $\alpha \in [0, \lambda)$, $\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha)\right] \leftrightarrow \left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha); \alpha\right]$, $\left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha)\right] \leftrightarrow \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha); \alpha\right]$ and $0 \leftrightarrow \tilde{0}$ are one to one mappings. Therefore, we define the signed distance from $\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha); \alpha\right] \cup \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha); \alpha\right]$ to $\tilde{0}$ by

$$\begin{aligned} &d\left(\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha); \alpha\right] \cup \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha); \alpha\right], \tilde{0}\right) \\ =&\frac{1}{4}\left[a + c + e + h + (2b - a - c)\frac{\alpha}{\lambda} + (2b - h - e)\frac{\alpha}{\rho}\right]. \end{aligned}$$

Using integration we find the average value.

$$\begin{aligned} &\frac{1}{\lambda}\int_0^\lambda d\left(\left[\tilde{A}_l^U(\alpha), \tilde{A}_l^L(\alpha); \alpha\right] \cup \left[\tilde{A}_r^L(\alpha), \tilde{A}_r^U(\alpha); \alpha\right], \tilde{0}\right) d\alpha \\ =&\frac{1}{8}\left[2b + a + c + 2e + 2h + (2b - h - e)\frac{\lambda}{\rho}\right]. \end{aligned} \tag{6}$$

Similarly, when $\lambda \le \alpha \le \rho$, one can define

$$d\left(\left[\tilde{A}_l^U(\alpha), \tilde{A}_r^U(\alpha); \alpha\right], \tilde{0}\right) = \frac{1}{2}\left[e + h + (2b - h - e)\frac{\alpha}{\rho}\right]$$

and

$$\frac{1}{\rho - \lambda}\int_\lambda^\rho d\left(\left[\tilde{A}_l^U(\alpha), \tilde{A}_r^U(\alpha); \alpha\right]\right) d\alpha = \frac{1}{4}\left[2b + e + h + (2b - h - e)\frac{\lambda}{\rho}\right] \tag{7}$$

By (6), (7), we give the following definition.

Definition 2.4. *Let* $\tilde{A} = [(a, b, c; \lambda), (e, b, h; \rho)] \in F_{IV}(\lambda, \rho)$. *The signed distance from* $\tilde{A}$ *to* $\tilde{0}$ *is defined as follows:*
(1^0) *when* $0 < \lambda < \rho \le 1$,

$$d\left(\tilde{A}, \tilde{0}\right) = \frac{1}{8}\left[6b + a + c + 4e + 4h + \frac{3\lambda}{\rho}(2b - h - e)\right] \tag{8}$$

(2^0) *when* $0 < \lambda = \rho \le 1$,

$$d(\tilde{A}, \tilde{0}) = \frac{1}{8}[4b + a + c + e + h] \tag{9}$$

Definition 2.5. *For* $\tilde{A}, \tilde{B} \in F_{IV}(\lambda, \rho)$*, their ordering is defined by*

$$\begin{aligned} \tilde{A} \prec \tilde{B} &\Leftrightarrow d(\tilde{A}, \tilde{0}) < d(\tilde{B}, \tilde{0}); \\ \tilde{A} \approx \tilde{B} &\Leftrightarrow d(\tilde{A}, \tilde{0}) = d(\tilde{B}, \tilde{0}). \end{aligned}$$

By the linear order "$<, =$" on R and Definition 2.5, we obtain the following property.
Property 1. $\tilde{A}, \tilde{B}, C \in F_{IV}(\lambda, \rho)$
(1^0) One and only one of $\tilde{A} \prec \tilde{B}$, $\tilde{A} \approx \tilde{B}$, $\tilde{B} \prec \tilde{A}$ will occur.
(2^0) The "$\prec, \approx$" on $F_{IV}(\lambda, \rho)$ satisfies the following 3 ordering relations.

$$\begin{aligned} &(a) \quad \tilde{A} \precsim \tilde{A}; \\ &(b) \quad \tilde{A} \precsim \tilde{B},\ \tilde{B} \precsim \tilde{A} \Rightarrow \tilde{A} \approx \tilde{B}; \\ &(c) \quad \tilde{A} \precsim \tilde{B},\ \tilde{B} \precsim \tilde{C} \Rightarrow \tilde{A} \precsim \tilde{C}. \end{aligned}$$

Due to Property 1, we know that "$\prec, \approx$" is the linear order on $F_{IV}(\lambda, \rho)$.

Consider a level (λ, ρ) *i-v* fuzzy number $\tilde{A}$ in (2). Let $a = b$, $c = b$, and $\lambda = 0$. (see Figure 1). Therefore, the level (λ, ρ) *i-v* fuzzy number $\tilde{A}$ can be reduced to the level ρ fuzzy number $\tilde{A}^U$, i.e., the level ρ fuzzy number is a special case of the level (λ, ρ) *i-v* fuzzy number. The family of all level ρ fuzzy numbers is denoted by

$$F_N(\rho) = \{(e, b, h; \rho) | e < b < h,\ e, b, h \in R\}.$$

By Definition 2.5, we have

Definition 2.6. *Let $\tilde{C} = (a, b, c; \rho) \in F_N(\rho)$. The signed distance from $\tilde{C}$ to $\tilde{0}$ is*

$$d^*\left(\tilde{C}, \tilde{0}\right) = \frac{1}{2\rho}\int_0^\rho \left[\tilde{C}_l(\alpha) + \tilde{C}_r(\alpha)\right] d\alpha = \frac{1}{4}(2b + a + c).$$

Same as Definition 2.5, we also have

Definition 2.7. *For $\tilde{B}, C \in F_N(\rho)$, their ordering is defined by*

$$\begin{aligned}\tilde{B} \prec \tilde{C} \Leftrightarrow d^*(\tilde{B}, \tilde{0}) < d^*(\tilde{C}, \tilde{0});\\ \tilde{B} \approx \tilde{C} \Leftrightarrow d^*(\tilde{B}, \tilde{0}) = d^*(\tilde{C}, \tilde{0}).\end{aligned}$$

Definition 2.8. *Let $\tilde{A} = \left[\tilde{A}^L, \tilde{A}^U\right]$, $\tilde{B} = \left[\tilde{B}^L, \tilde{B}^U\right] \in F_{IV}(\lambda, \rho)$, and let $k \in R$.*
We define the following operations:

$$\begin{aligned}
&\tilde{A} \oplus \tilde{B} = \left[\tilde{A}^L \oplus \tilde{B}^L, \tilde{A}^U \oplus \tilde{B}^U\right],\\
&k\tilde{A} = \left[k\tilde{A}^L, k\tilde{A}^U\right],\\
&\tilde{A}^L = (a_{L1}, a_{L2}, a_{L3}; \lambda), \quad \tilde{A}^U = (a_{U1}, a_{U2}, a_{U3}; \rho),\\
&\tilde{B}^L = (b_{L1}, b_{L2}, b_{L3}; \lambda), \quad \tilde{B}^U = (b_{U1}, b_{U2}, b_{U3}; \rho),\\
&\tilde{A}^L \oplus \tilde{B}^L = (a_{L1} + b_{L1}, a_{L2} + b_{L2}, a_{L3} + b_{L3}; \lambda),\\
&\tilde{A}^U \oplus \tilde{B}^U = (a_{U1} + b_{U1}, a_{U2} + b_{U2}, a_{U3} + b_{U3}; \rho),\\
&k\tilde{A}^L = \begin{cases} (ka_{L1}, ka_{L2}, ka_{L3}; \lambda), & \text{if } k > 0\\ (ka_{L3}, ka_{L2}, ka_{L1}; \lambda), & \text{if } k < 0.\end{cases}
\end{aligned}$$

3. **Fuzzy Parallel System Reliability Analysis.** In this section, we use statistical data to consider the general parallel system. Figure 2 represents the subsystems of a parallel system, which are $P_1, P_2, \ldots, P_n$ with reliabilities $R_1, R_2, \ldots, R_n$ (if known), respectively. The reliability of the parallel system is then

$$1 - \prod_{j=1}^{n}(1 - R_j). \tag{10}$$

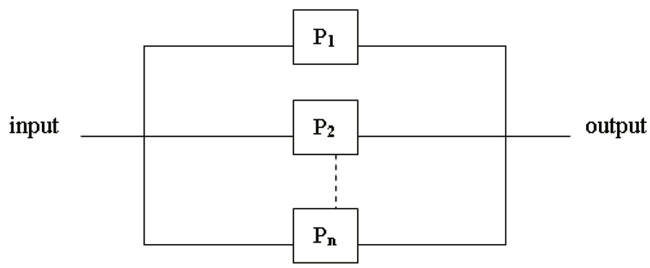

FIGURE 2. Configuration of a parallel system

3.1. **Reliability using point estimates.** If R_j, $j = 1, 2, \ldots, n$ are unknown and for each subsystem P_j, $j = 1, 2, \cdots, n$, we may obtain n records of reliability of P_j. Let R_{jq}, $q = 1, 2, \ldots, n_j$ be the statistical data. Then, the average value is $\bar{R}_j = \frac{1}{n_j}\sum_{q=1}^{n_j} R_{jq} \in [0, 1]$ and it can be viewed as the reliability of P_j.

Let R_j (unknown), $j = 1, 2, \ldots, n$ be the population reliability of the subsystem P_j. The reliability of the parallel system is then

$$1 - \prod_{j=1}^{n} (1 - \bar{R}_j). \tag{11}$$

3.2. **Reliability based on level $(\boldsymbol{\lambda}, \boldsymbol{\rho})$ *i-v* fuzzy numbers.** When we measure the reliability R_j of the subsystem P_j ($j = 1, 2, \ldots, n$) for a parallel system a general error will occur. It would be more realistic to say that the reliability will be around R_j. It must be notified that saying "around R_j" is to use the fuzzy language. It appears that using a fuzzy set to express its meaning is better. Due to the unknown probability of error in point estimation $\bar{R}_j$ and R_j is unknown, we use the confidence interval of R_j instead.

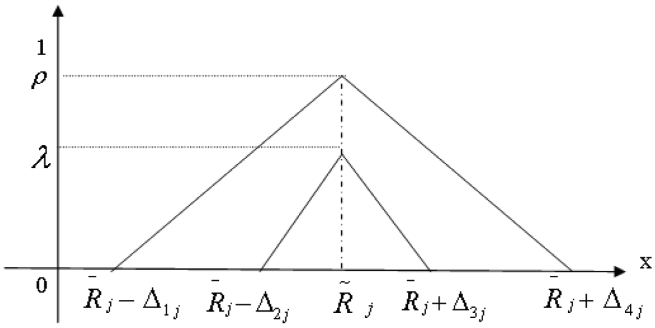

FIGURE 3. Level (λ, ρ) interval-valued fuzzy number $\tilde{R}_j$

The $(1 - \alpha) \times 100\%$ confidence interval of R_j is

$$\left[\bar{R}_j - \Delta_{1j}, \bar{R}_j + \Delta_{4j}\right], \quad j = 1, 2, \ldots, n, \tag{12}$$

where $\Delta_{1j} = t_{n_j-1}(\alpha_1)\frac{s_j}{\sqrt{n_j}}$, $\Delta_{4j} = t_{n_j-1}(\alpha_2)\frac{s_j}{\sqrt{n_j}}$, $0 < \alpha_j < 1$, $j = 1, 2$, $\alpha_1 + \alpha_2 = \alpha$, $0 < \alpha < 1$, $0 < \Delta_{1j}$, $\Delta_{4j} < \bar{R}_j$ and $s_j^2 = \frac{1}{n_j-1}\sum_{q=1}^{n_j}\left(R_{jq} - \bar{R}_j\right)^2$, $j = 1, 2, \ldots, n$.

Let T be the random variable of t distribution with $n_j - 1$ degrees of freedom. $t_{n_j-1}(\alpha_k)$ then satisfies $p(T \geq t_{n_j-1}(\alpha_k)) = \alpha_k$, $k = 1, 2$. Since the $(1-\alpha)\times 100\%$ confidence interval in (12) is an interval, the decision makers choose a point within the interval to estimate R_j. We find that if the chosen point is R_j, then there is no deference between the chosen point and the point to estimate $\bar{R}_j$ and the error is definitely 0. We obtain the maximum confidence level. Let it be $\rho = 1 - \alpha$. According to the same reason as above, we have level ρ fuzzy number (13) corresponding to (12) as follows:

$$\tilde{R}_j^U = (\bar{R}_j - \Delta_{1j}, \bar{R}_j, \bar{R}_j + \Delta_{4j}; \rho), \quad j = 1, 2, \ldots, n, \tag{13}$$

where

$$0 < \bar{R}_j - \Delta_{1j} < 1, \quad \alpha_1 + \alpha_2 = \alpha, \quad \rho = 1 - \alpha, \quad j = 1, 2, \ldots, n. \tag{14}$$

Similarly, let $0 < \alpha < \beta < 1$, $0 < \alpha_j < \beta_j < 1$, $j = 1, 2$ and $\alpha_1 + \alpha_2 = \alpha$, $\beta_1 + \beta_2 = \beta$. We have $(1 - \beta) \times 100\%$ confidence interval

$$\left[\bar{R}_j - \Delta_{2j}, \bar{R}_j + \Delta_{3j}\right], \quad j = 1, 2, \ldots, n, \tag{15}$$

where $\Delta_{2j} = t_{n_j-1}(\beta_1)\frac{s_j}{\sqrt{n_j}}$ and $\Delta_{3j} = t_{n_j-1}(\beta_2)\frac{s_j}{\sqrt{n_j}}$.

In corresponding to confidence interval above, we have the level λ triangular fuzzy number as follows:

$$\tilde{R}_j^L = \left(\bar{R}_j - \Delta_{2j}, \bar{R}_j, \bar{R}_j + \Delta_{3j}; \lambda\right), \quad j = 1, 2, \ldots, n, \tag{16}$$

where

$$0 < \bar{R}_j - \Delta_{2j} < 1, \quad \beta_1 + \beta_2 = \beta, \quad \lambda = 1 - \beta, \quad j = 1, 2, \ldots, n.$$

Since $0 < \lambda < \rho$, from (13) and (16), we have the following level (λ, ρ) *i-v* fuzzy numbers

$$\tilde{R}_j = \left[\tilde{R}_j^L, \tilde{R}_j^U\right] \in F_{IV}(\lambda, \rho), \quad j = 1, 2, \ldots, n. \tag{17}$$

Theorem 3.1. *Using the level (λ, ρ) i-v fuzzy numbers $\tilde{R}_j = [\tilde{R}_j^L, \tilde{R}_j^U]$, $j = 1, 2, \ldots, n$ in (17), we obtain the fuzzy reliability of the parallel system as follows:*

$$\begin{aligned} &\tilde{1} \ominus \left[\left(\tilde{1} \ominus \tilde{R}_1\right) \otimes \left(\tilde{1} \ominus \tilde{R}_2\right) \otimes \cdots \otimes \left(\tilde{1} \ominus \tilde{R}_n\right)\right] \\ &= \left[\left(1 - \prod_{j=1}^{n} (1 - \bar{R}_j + \Delta_{2j}), 1 - \prod_{j=1}^{n} (1 - \bar{R}_j), 1 - \prod_{j=1}^{n} (1 - \bar{R}_j - \Delta_{3j}); \lambda\right),\right. \\ &\quad \left.\left(1 - \prod_{j=1}^{n} (1 - \bar{R}_j + \Delta_{1j}), 1 - \prod_{j=1}^{n} (1 - \bar{R}_j), 1 - \prod_{j=1}^{n} (1 - \bar{R}_j - \Delta_{4j}); \rho\right)\right]. \end{aligned} \tag{18}$$

Note 1. The fuzzy reliability of the serial system is

$$\begin{aligned} \tilde{R}_1 \otimes \tilde{R}_2 \otimes \cdots \otimes \tilde{R}_n = &\left[\left(\prod_{j=1}^{n} (\bar{R}_j - \Delta_{2j}), \prod_{j=1}^{n} \bar{R}_j, \prod_{j=1}^{n} (\bar{R}_j + \Delta_{3j}); \lambda\right)\right. \\ &\left.\left(\prod_{j=1}^{n} (\bar{R}_j - \Delta_{1j}), \prod_{j=1}^{n} \bar{R}_j, \prod_{j=1}^{n} (\bar{R}_j + \Delta_{4j}); \rho\right)\right]. \end{aligned}$$

Theorem 3.2. *Using Definition 2.4, we can defuzzify (18) to get the estimate reliability of the parallel system in the fuzzy sense as follows:*

$$\begin{aligned} &\frac{1}{2} d\left(\tilde{1} \ominus \left[\left(\tilde{1} \ominus \tilde{R}_1\right) \otimes \left(\tilde{1} \ominus \tilde{R}_2\right) \otimes \cdots \otimes \left(\tilde{1} \ominus \tilde{R}_n\right)\right], \tilde{0}\right) \\ &= \frac{1}{16}\left[6b_2 + a_2 + c_2 + 4p_2 + 4r_2 + \frac{3\lambda}{\rho}(2b_2 - p_2 - r_2)\right] \end{aligned} \tag{19}$$

where

$$a_2 = 1 - \prod_{j=1}^{n} (1 - \bar{R}_j + \Delta_{2j}),\ b_2 = 1 - \prod_{j=1}^{n} (1 - \bar{R}_j),\ c_2 = 1 - \prod_{j=1}^{n} (1 - \bar{R}_j - \Delta_{3j}),$$
$$p_2 = 1 - \prod_{j=1}^{n} (1 - \bar{R}_j + \Delta_{1j}),\ r_2 = 1 - \prod_{j=1}^{n} (1 - \bar{R}_j - \Delta_{4j}).$$

4. **Numerical Example.**

Example 4.1. *In [8,11], they considered the following problem. Two grinding machines are working next to each other. What is the possibility that people coming into the vicinity of the machines will be injured mainly by getting a chip into their eyes? The most endangered persons are the operators who are obliged to wear safety glasses but often fail to do this. Further endangered are persons coming into the vicinity of the machines, who are persons who bring and carry away items, and others entering the area for other reasons.*

The basic events contributing to the accident are as shown in Table 1.

$U = F + G + H$, $V = C + D$, $Z = E \times U \times V$, $X = A + B + Z$.

Let $\alpha = 0.02$, $\alpha_1 = 0.011$, $\alpha_2 = 0.009$, $\beta = 0.2$, $\beta_1 = 0.12$, $\beta_2 = 0.08$. $t_9(\alpha_1) = 2.7017$, $t_9(\alpha_2) = 2.9068$, $t_9(\beta_1) = 1.2698$, $t_9(\beta_2) = 1.5630$.

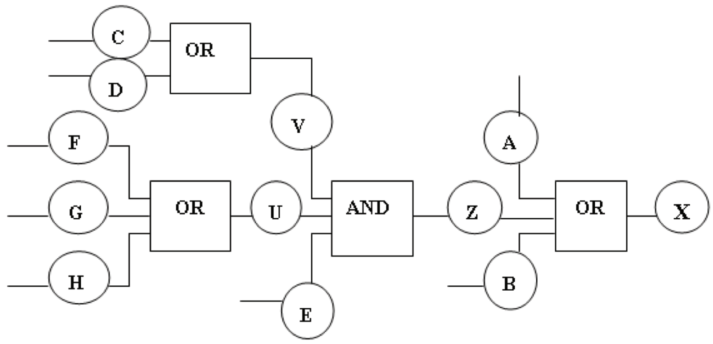

FIGURE 4. Fault tree of the example

TABLE 1. The basic events contributing to the accident [8,11]

j	Symbol	Populations reliability	Sample size	Sample mean	Sample standard deviation
1	A	R_A	$n_A = 10$	$\bar{R}_A = 0.1$	$S_A = 0.004$
2	B	R_B	$n_B = 10$	$\bar{R}_B = 0.2$	$S_B = 0.004$
3	C	R_C	$n_C = 10$	$\bar{R}_C = 0.8$	$S_C = 0.010$
4	D	R_D	$n_D = 10$	$\bar{R}_D = 0.6$	$S_D = 0.010$
5	E	R_E	$n_E = 10$	$\bar{R}_E = 0.9$	$S_E = 0.020$
6	F	R_F	$n_F = 10$	$\bar{R}_F = 0.5$	$S_F = 0.004$
7	G	R_G	$n_G = 10$	$\bar{R}_G = 0.6$	$S_G = 0.004$
8	H	R_H	$n_H = 10$	$\bar{R}_H = 0.7$	$S_H = 0.001$

TABLE 2. Two endpoints in Figure 4

j	Symbol	$\bar{R}_j - \Delta_{1j}$	$\bar{R}_j + \Delta_{4j}$	$\bar{R}_j - \Delta_{2j}$	$\bar{R}_j + \Delta_{3j}$
1	A	0.0966	0.1037	0.0984	0.1020
2	B	0.1966	0.2037	0.1984	0.2020
3	C	0.7915	0.8092	0.7960	0.8049
4	D	0.5915	0.6092	0.5960	0.6049
5	E	0.8829	0.9184	0.8920	0.9099
6	F	0.4966	0.5037	0.4984	0.5062
7	G	0.5966	0.6037	0.5984	0.6062
8	H	0.6991	0.7069	0.6996	0.7005

From Table 2, we get the fuzzy reliability of C and D:

$$\tilde{R}_C = [(0.7960, 0.8, 0.8049; 0.8), (0.7915, 0.8, 0.8092; 0.98)];$$

$$\tilde{R}_D = [(0.5960, 0.6, 0.6049; 0.8), (0.5915, 0.6, 0.6092; 0.98)].$$

According to Theorem 3.1, we obtain the fuzzy reliability of the parallel system $V = C + D$, as follows:

$$\begin{aligned}
&\left(\tilde{1} \ominus \tilde{R}_C\right) \otimes \left(\tilde{1} \ominus \tilde{R}_D\right) \\
&= [(0.1951, 02, 0.2040; 0.8), (0.1908, 0.2, 0.2085; 0.98)] \\
&\quad \otimes [(0.3951, 0.4, 0.4040; 0.8), (0.3908, 0.4, 0.4085; 0.98)] \\
&= [(0.0771, 0.08, 0.0824; 0.8), (0.0746, 0.08, 0.0852; 0.98)]; \\
\tilde{R}_V &= [(0.9176, 0.92, 0.9229; 0.8), (0.9148, 0.92, 0.9254; 0.98)].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tilde{R}_U &= \tilde{1} \ominus \left[\left(\tilde{1} \ominus \tilde{R}_F\right) \otimes \left(\tilde{1} \ominus \tilde{R}_G\right) \otimes \left(\tilde{1} \ominus \tilde{R}_H\right)\right] \\
&= [(0.9395, 0.94, 0.9418; 0.8), (0.9389, 0.94, 0.9412; 0.98)].
\end{aligned}$$

Due to Note 1, we get the fuzzy reliability of the serial system $Z = E \times U \times V$ as follows:

$$\begin{aligned}
\tilde{R}_E &= [(0.8920, 0.9, 0.9099; 0.8), (0.8829, 0.9, 0.9184; 0.98)]; \\
\tilde{R}_Z &= \tilde{R}_E \otimes \tilde{R}_U \otimes \tilde{R}_V \\
&= [(0.7690, 07783, 0.7909; 0.8), (0.7583, 0.7783, 0.7999; 0.98)].
\end{aligned}$$

According to Theorem 3.1, we get the fuzzy reliability of the parallel system $X = A + B + Z$, as follows:

$$\begin{aligned}
\tilde{R}_A &= [(0.0984, 0.1, 0.1020; 0.8), (0.0966, 0.1, 0.1037; 0.98)]; \\
\tilde{R}_B &= [(0.1984, 0.2, 0.202; 0.8), (0.1966, 0.2, 0.2037; 0.98)]; \\
\tilde{R}_X &= \tilde{1} \ominus \left[\left(\tilde{1} \ominus \tilde{R}_A\right) \otimes \left(\tilde{1} \ominus \tilde{R}_B\right) \otimes \left(\tilde{1} \ominus \tilde{R}_Z\right)\right] \\
&= [(0.1434, 0.1596, 0.1669; 0.8), (0.1428, 0.1596, 0.1754; 0.98)].
\end{aligned}$$

By Definition 2.4, we defuzzify $\tilde{R}_X$ and get the estimate reliability of the system in the fuzzy sense

$$\frac{1}{2} d\left(\tilde{R}_X, \tilde{0}\right) = 0.1588.$$

5. **Discussions.**

(A) The crisp case is a special case of fuzzy case.

If $\alpha_k \to 0.5$, $\beta_k \to 0.5$, $k = 1, 2$, then $\alpha \to 1$, $\beta \to 1$, $t_{n_j-1}(\alpha_k) \to 0$, $t_{n_j-1}(\beta_k) \to 0$, $k = 1, 2$. The result in Theorem 3.1 will then be reduced to

$$\begin{aligned}
&\tilde{1} \ominus \left[\left(\tilde{1} \ominus \tilde{R}_1\right) \otimes \left(\tilde{1} \ominus \tilde{R}_2\right) \otimes \cdots \otimes \left(\tilde{1} \ominus \tilde{R}_n\right)\right] \\
\to &\left[\left(1 - \prod_{j=1}^{n} \left(1 - \bar{R}_j\right), 1 - \prod_{j=1}^{n} \left(1 - \bar{R}_j\right), 1 - \prod_{j=1}^{n} (1 - \bar{R}_j); \lambda\right),\right. \\
&\left.\left(1 - \prod_{j=1}^{n} (1 - \bar{R}_j), 1 - \prod_{j=1}^{n} (1 - \bar{R}_j), 1 - \prod_{j=1}^{n} (1 - \bar{R}_j); \rho\right)\right].
\end{aligned}$$

(B) The comparison of this article and [11].

In [11], the author did not use the statistical method. He used L-R type fuzzy number to consider the fuzzy reliability problem. The L-R type fuzzy number operations are as follows:

$$(m,\alpha,\beta)_{LR} \oplus (n,\gamma,\delta)_{LR} = (m+n, \alpha+\gamma, \beta+\delta)_{LR};$$
$$(m,\alpha,\beta)_{LR} \ominus (n,\gamma,\delta)_{LR} = (m-n, \alpha+\delta, \beta+\gamma)_{LR};$$
$$(m,\alpha,\beta)_{LR} \bullet (n,\gamma,\delta)_{LR} = (mn, m\gamma+n\alpha, m\delta+n\beta)_{LR}.$$

The fuzzy reliability of the subsystem P_j is $\tilde{R}_j = (m_j, \alpha_j, \beta_j)_{LR}$.

The equation above only denotes $n-1$ approximations. He did not defuzzify the system and did not find the estimated reliability of the parallel system in the fuzzy sense.

(C) The comparison of this article and [5].

The α-level set of triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3)$ is

$$\tilde{A}_\alpha = [a_1^\alpha, a_3^\alpha] = [(a_2-a_1)\alpha + a_1, -(a_3-a_2)\alpha + a_3], \quad \alpha \in [0,1].$$

In [5], the fuzzy reliability of the parallel system is

$$\begin{aligned}\tilde{P} &= 1 - \prod_{i=1}^{n}(1-\tilde{P}_i) = [1,1] - \prod_{i=1}^{n}([1,1] - [a_{i1}^\alpha, a_{i3}^\alpha]) \\ &= \left[1 - \prod_{i=1}^{n}\left(-(a_{i2}-a_{i1})\alpha + 1 - a_{i1}\right), 1 - \prod_{i=1}^{n}\left((a_{i3}-a_{i2})\alpha + 1 - a_{i3}\right)\right].\end{aligned}$$

To consider the ith fuzzy reliability, we let $\tilde{P}_i = (a_{i1}, a_{i2}, a_{i3})$. Using the decomposition theorem, we have

$$\tilde{P}_i = \bigcup_{0\le\alpha\le1} [(a_{i2}-a_{i1})\alpha + a_{i1}, -(a_{i3}-a_{i2})\alpha + a_{i3}; \alpha], \quad i = 1, 2, \ldots, n$$

and

$$\tilde{P}_1 \otimes \tilde{P}_2 \otimes \cdots \otimes \tilde{P}_n = \bigcup_{0\le\alpha\le1}\left[\prod_{i=1}^{n}((a_{i2}-a_{i1})\alpha + a_{i1}), \prod_{i=1}^{n}(-(a_{i3}-a_{i2})\alpha + a_{i3}); \alpha\right].$$

Therefore, in [5], the author used only one α for the decomposition to treat the problem.

6. Conclusions.

(A) If we want to consider the reliability of a factory production process, experimental work is required. We can get the reliability statistical data R_{jq}, $q = 1, 2, \ldots, n_j$ of the subsystem P_j in the parallel system (see Figure 2). It is difficult to obtain significant results with this reliability problem if we consider only a model that does not include experiments.

For the subsystem P_j, from the statistical data R_{jq}, $q = 1, 2, \ldots, n_j$ we can find their average value. Let $\bar{R}_j = \frac{1}{n_j}\sum_{q=1}^{n_j} R_{jq} \in [0,1]$. This is a point estimate of R_j in the statistical sense. We use it as the point estimate of the reliability of P_j. Because the probability of the error between the point estimate $\bar{R}_j$ and R_j is unknown, we use the confidence intervals Equation (12) and Equation (15) in Section 3.2 instead.

Since the interval is not a value, it is not convenient in the calculation. Therefore, we convert the confidence interval into a level (λ, ρ) *i-v* fuzzy number Equation (17) in Section 3.2. We use the signed distance to defuzzify and then have the estimate of the reliability of P_j in the fuzzy sense.

(B) This paper is better than articles in [5,6] in the real application.

(B-1) In [6], for the reliability of the subsystem P_j, the author used the triangular fuzzy number $\tilde{R}_j = (m_j - \alpha_j, m_j, m_j + \beta_j)$. He did not discuss how to determine m_j, α_j and β_j. Therefore, we cannot apply it to a real problem. In our paper, the level (λ, ρ) *i-v* fuzzy numbers $\tilde{R}_j = \left[\tilde{R}_j^L, \tilde{R}_j^U\right]$ in Equation (17) are derived from statistical data and confidence intervals, where $\bar{R}_j$ is the average value of statistical data R_{jq}, $q = 1, 2, \ldots, n_j$, S_j^2 is the sample variance. Both of them are from statistical data with α, α_1 and α_2 provided by the decision maker. The values m_j, α_j and β_j in [6] correspond to $\bar{R}_j$, $t_{n_j-1}(\alpha_1)\frac{s_j}{\sqrt{n_j}}$ and $t_{n_j-1}(\alpha_2)\frac{s_j}{\sqrt{n_j}}$ in this paper. Therefore, this paper is better than [6] in reality applications.

(B-2) In [5], they used the α-level set $[(a_{j2} - a_{j1})\alpha] + a_{j1}, -(a_{j3} - a_{j2})\alpha + a_{j3}]$ of the triangular fuzzy number (a_{j1}, a_{j2}, a_{j3}) to consider the reliability of subsystem P_j, where $(a_{j2} - a_{j1})\alpha + a_{j1}$ is the left hand side of α-cut and $-(a_{j3} - a_{j2})\alpha + a_{j3}$ is the right hand side of α-cut, $0 < \alpha \leq 1$, $j = 1, 2, \ldots, n$, as shown before. They did not discuss how to determine a_{j1}, a_{j2} and a_{j3} as stated in (B-1). Therefore, it cannot be applied to a real problem. Similarly, as shown in (B-1), a_{j1}, a_{j2} and a_{j3} correspond to $\bar{R}_j$, $t_{n_j-1}(\alpha_1)\frac{s_j}{\sqrt{n_j}}$ and $t_{n_j-1}(\alpha_2)\frac{s_j}{\sqrt{n_j}}$ in this paper. Therefore, this paper is better than [5] in reality applications.

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