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PORTFOLIO SELECTION MODEL WITH INTERVAL VALUES BASED ON FUZZY PROBABILITY DISTRIBUTION FUNCTIONS

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Abstract. *In order to analyze uncertain phenomena in real world, the concept of fuzzy random variables is widely employed in model building. In dealing with fuzzy data, defuzzification plays a central role. In this paper, portfolio selection problems are dealt as interval values. We calculate the expected values, variance and covariance by using the estimated parameters of underlying probability distribution function. The estimated values enable us to build up a portfolio selection model with estimated parameters on the basic of Markowitz's mean-variance model. The result exemplified that we have different choices of k which can decide the best expected return and less risk level in our model, also that we can provide not only one choice of portfolio selection but also two or more for decision makers.*

Keywords: Portfolio selection, Optimization, Fuzzy probability distributions, Fuzzy statistics and data analysis

1. Introduction.

Portfolio selection problem has been well developed on the basis of a mean-variance approach proposed by Markowitz [11, 12, 14], who combines probability theory and optimization theory to model the behavior of the economic agents. The key principle of the mean-variance model is to use the expected return of a portfolio as the investment return and to use the expected variance of the portfolio as the investment risk. Mean-variance portfolio selection problem has been studied by Sharpe [18], Merton [13], Perold [16], Pang [15], Voros [21], Best [1, 2], and the others. In order to analyze uncertain phenomena in real world, also research works used multivariate data analysis and investigated fuzzy portfolio selection problems such as Wang et al. [29], Zhang et al. [34], Tanaka and Guo [19, 20], Wang and Zhu [28], Lai et al. [8], Zhang and Wang [31], Zhang et al. [32], Watada [24], Hasuike and Ishii [5, 6], Ramaswamy [17] and Leon et al. [7]. Usually, the works did not consider any kind of probability distribution function with fuzzy random variables. The objective of this paper is to combine the concept of distribution function and fuzzy random variables with portfolio selection model.

In order to find the probability distribution function with continuous fuzzy data, we have to take fuzzy statistics into consideration. Fundamental statistics, such as mean, median and mode, are useful measurements in illustrating some characteristics of a sample distribution. More research has placed focus on the fuzzy statistical analysis and applications in the social science fields, as Wu and Hwang [22] proposed fuzzy statistical test to discuss the stationary of Taiwan short-term money demand function; Wu and

Chen [25] identified a model structure through qualitative simulation; Casalino et al. [3], Esogbue and Song [4], Wu [23], and Wu and Sun [26] demonstrated the concepts of fuzzy statistics and applied them to social survey; Wu and Tseng [27] used fuzzy regression method of coefficient estimation to analyze Taiwan monitoring index of economic. All above-mentioned studies dealt with problems by means of central point values. Lin et al. [9] defined a new weight function of fuzzy numbers using central point and radius, so as to more effectively observe original fuzzy data. Moreover, Lin et al. [10] also propose a method to recognize the underlying distribution function using central point and radius. It gives us more information about original fuzzy data.

The objective of this paper is to define a portfolio selection model with interval values and reduce the calculation load in building optimization model on the basis of the definition of mean-variance model [11]. In the first step, we need to find out fuzzy probability distribution function (it means distribution function with fuzzy data) for each return. When we find out the fuzzy distribution function of each return, we can easily evaluate the expected return. Moreover, we can calculate the variance, too. The values enable us to define a portfolio selection model with interval values.

The rest of the paper consists of the following. Section 2 gives the brief review of related studies. The main method is described in Section 3. Section 4 illustrates some empirical studies to show that portfolio selection model with interval values in our model can get large return than Zhang's method. Finally, the concluding remarks and the topics of further studies are summarized in Section 5.

2. Problem Statement and Preliminaries.

Definition 2.1. *An interval value is denoted as* $A = [a, b]$ *with central point* $o = \dfrac{b+a}{2}$ *and radius* $l = \dfrac{b-a}{2}$*. We use the notation as* $A \equiv (o, l)$ *without any confusion.*

Let us recall that Markowitz's mean-variance model is based on probability distribution where uncertainty is equated with randomness [11, 12, 14]. That is, the rate of return on the ith asset, r_i, will be regarded as a random variable.

Consider a market with n risky assets. An investor's position in this market is described by means of a portfolio $\mathbf{x} = [x_1, x_2, \cdots, x_n]'$, where the ith component x_i represents the proportion invested in asset i. The return vector on portfolio $\mathbf{x}$ is described by $\mathbf{r} = [r_1, r_2, \cdots, r_n]'$, where r_i represents the return rate of asset i. In conventional mean-variance methodology for portfolio selection, r_i is regarded as a random variable, $\forall\ i = 1, 2, \cdots, n$. Assume $\bar{\mathbf{r}} = [\bar{r}_1, \bar{r}_2, \cdots, \bar{r}_n]'$ and $\mathbf{V} = [\sigma_{ij}]_{n\times n}$ be the expected return vector and covariance matrix, respectively [28]. The return R on the portfolio $\mathbf{x}$ is given by $R = \sum_{i=1}^{n} r_i x_i$. Set $\mathbf{I} = [1, 1, \cdots, 1]'$. The objective of the investor is to choose a portfolio that maximizes the return on the investment subject to some constraints on the risk of the investment. A mean-variance model for portfolio selection can be formulated mathematically as:

$$\left.\begin{array}{ll} \max & \bar{\mathbf{r}}\mathbf{x} \\ \text{s.t.} & \sqrt{\mathbf{x}'\mathbf{V}\mathbf{x}} \le \sigma \\ & \mathbf{I}'\mathbf{x} \le 1 \\ & x_i \ge 0 \quad \forall i = 1, 2, \cdots, n \end{array}\right\}, \tag{1}$$

where σ ($\sigma \ge 0$) represents the tolerated risk level.

We rewrite Formula (1) in the following simple mathematical programming problem:

$$\left.\begin{array}{ll} \max & E\left(\sum_{i=1}^{n} r_i x_i\right) \\ \min & Var\left(\sum_{i=1}^{n} r_i x_i\right) \\ \text{s.t.} & \sum_{i=1}^{n} x_i \leq 1 \\ & x_i \geq 0 \quad \forall i = 1, 2, \cdots, n \end{array}\right\}. \tag{2}$$

We need to solve the programming problem as follows:

$$\left.\begin{array}{ll} \max & \sum_{i=1}^{n} E(r_i) x_i \\ \min & \mathbf{x}'\mathbf{V}\mathbf{x} \\ \text{s.t.} & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \quad \forall i = 1, 2, \cdots, n \end{array}\right\}, \tag{3}$$

where $\mathbf{x}'\mathbf{V}\mathbf{x} = \sum_{i=1}^{n} \sigma_{ii}^2 x_i^2 + \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} \sigma_{ij} x_i x_j$, a_i and a_j are distributed to some distribution functions. Let $\sigma_{ij} = cov(a_i, a_j)$ be covariance.

In this view, if we can know the expected value, variance and covariance of each r_i, $\forall i = 1, 2, \cdots, n$, then we can easily solve the portfolio selection model in Formula (3).

3. **Fuzzy Portfolio Selection Model with Interval Values.** In this section, let us solve a portfolio selection problem with fuzzy numbers (interval data). Suppose that there are n distinct tradable assets in the market. The terminal rate of return for asset i, denoted as r_i, is assumed as a fuzzy random variable. By the widely accepted definition, the expectation of a fuzzy random variable is a fuzzy variable. We give the following combination to express the total fuzzy return on a portfolio $R(x)$.

Let A_i be a fuzzy continuous random variable on the probability space (Ω, F, P). We define the fuzzy expected return and fuzzy expected variance in the following.

Definition 3.1. Fuzzy Expected Return

If $A_i \equiv (o_i, l_i)$ is interval data, $\forall i = 1, 2, \cdots, n$, then the fuzzy expected return of A_i is defined as

$$E(R(x)) = \sum_{i=1}^{n} E(A_i) x_i = \left(\sum_{i=1}^{n} E(o_i) x_i, \sum_{i=1}^{n} E(l_i) x_i\right).$$

Definition 3.2. Fuzzy Expected Variance

If $A_i \equiv (o_i, l_i)$ is interval data, $\forall i = 1, 2, \cdots, n$, then the fuzzy expected variance of A_i is defined as

$$var(R(x)) = var\left(\sum_{i=1}^{n} A_i x_i\right) = \left(var\left(\sum_{i=1}^{n} o_i x_i\right), var\left(\sum_{i=1}^{n} l_i x_i\right)\right),$$

where $var\left(\sum_{i=1}^{n} o_i x_i\right) = \sum_{i=1}^{n} \sigma_{o_i}^2 x_i^2 + \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} cov(o_i, o_j) x_i x_j$ and $var\left(\sum_{i=1}^{n} l_i x_i\right) = \sum_{i=1}^{n} \sigma_{l_i}^2 x_i^2 + \sum_{j=1}^{n} \sum_{i=1, i\neq j}^{n} cov(l_i, l_j) x_i x_j$, $\forall i, j = 1, 2, \cdots, n$.

When we know the probability distribution function of o_i and l_i, we can easily find out the fuzzy expected return and fuzzy expected variance. Moreover, we can solve Formula (3). Now, let us give the portfolio selection model with interval values as follows.

Definition 3.3. Portfolio Selection Model with Interval Values
If $A_i \equiv (o_i, l_i)$ is interval value, $\forall i = 1, 2, \cdots, n$, and we do some simulation for o_i and l_i, then we have probability distribution function of o_i and l_i, $\forall i = 1, 2, \cdots, n$. We say that o_i distributes to the distribution function D_{o_i}, denoted as $o_i \sim D_{o_i}$, and l_i distributes to the distribution function D_{l_i}, denoted as $l_i \sim D_{l_i}$. Our goal is to solve the following optimization model.

$$\left.\begin{array}{ll} \max & \sum_{i=1}^{n} E(o_i)x_i \\ \min & \mathbf{x}'\mathbf{V}_o\mathbf{x} \\ \text{s.t.} & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \quad \forall i = 1, 2, \cdots, n \end{array}\right\}, \tag{4}$$

and

$$\left.\begin{array}{ll} \min & \sum_{i=1}^{n} E(l_i)x_i \\ \min & \mathbf{x}'\mathbf{V}_l\mathbf{x} \\ \text{s.t.} & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \quad \forall i = 1, 2, \cdots, n \end{array}\right\}, \tag{5}$$

where $\mathbf{V}_o = [cov(o_i, o_j)]_{n\times n}$ and $\mathbf{V}_l = [cov(l_i, l_j)]_{n\times n}$, $\forall i, j = 1, 2, \cdots, n$.

When we find out the optimal solution for the model of (4) and (5), we can get the optimal solution vector $\mathbf{x}^* = [\mathbf{o}^*, \mathbf{l}^*]'$, where $\mathbf{o}^*$ is the optimal solution for model in (4) and $\mathbf{l}^*$ is the optimal solution for model in (5).

Moreover, when we have the solution $\mathbf{x}^* = [\mathbf{o}^*, \mathbf{l}^*]'$, we can get the total expected return

$$E(R(\mathbf{x}^*)) = \left(\sum_{i=1}^{n} E(o_i)\mathbf{o}^*, \sum_{i=1}^{n} E(l_i)\mathbf{l}^*\right).$$

Now, we give the procedure of solving portfolio selection model with interval values in the following.

Procedure of Solving Portfolio Selection Model with Interval Values:
Step 1. Collect the interval data.
Step 2. Compute o_i and l_i, $\forall i = 1, 2, \cdots, n$.
Step 3. Identify the underlying distribution by simulating o_i and l_i, $\forall i = 1, 2, \cdots, n$.
Step 4. Calculate the parameters for the expected value, variance and covariance in the model of (4) and (5).
Step 5. Solve the optimization model of (4) and (5) and get the optimal solution $\mathbf{o}^*$ and $\mathbf{l}^*$.
Step 6. Set the $\mathbf{x}^*$ into fuzzy vector.
Step 7. Compute the possibility distribution of total return $R(\mathbf{x}^*)$.

In order to illustrate our proposed effective meanings and approaches of the efficient portfolios, we exemplify a real portfolio selection problem with interval values in the following section.

4. **Application.**

Example 4.1. *We select five exchange currencies (USD, EUR, AUD, GBP and CHF) from Bank of Tokyo-Mitsubishi. Original data come from every day's closed prices from*

January 2010 to November 2010. There are 224 interval values in this period $[a, b]$, where a is the minimum price and b is the maximum price in one day. We give some interval values in Table 1.

Table 1. Interval values of each exchange currency

	$[a,b]$			
USD	[83.40, 84.30]	[83.80, 84.40]	$\cdots$	[83.01, 83.54]
EUR	[108.46, 110.01]	[109.98, 112.00]	$\cdots$	[111.96, 113.65]
AUD	[79.83, 80.82]	[80.52, 81.65]	$\cdots$	[81.03, 82.51]
GBP	[129.54, 130.47]	[130.74, 131.74]	$\cdots$	[132.69, 133.48]
CHF	[83.32, 84.32]	[83.83, 84.44]	$\cdots$	[84.27, 85.12]

First, we calculate the central point $o = \dfrac{a+b}{2}$ and radius $l = \dfrac{b-a}{2}$. We give the data in Table 2.

Table 2. Central point and radius of each interval values $[a, b]$

	(o,l)			
USD	(83.85, 0.45)	(84.10, 0.30)	$\cdots$	(83.275, 0.265)
EUR	(109.235, 0.775)	(100.99, 1.01)	$\cdots$	(112.805, 0.845)
AUD	(80.325, 0.495)	(81.085, 0.565)	$\cdots$	(81.77, 0.74)
GBP	(130.005, 0.465)	(131.24, 0.50)	$\cdots$	(133.085, 0.395)
CHF	(83.82, 0.5)	(84.135, 0.305)	$\cdots$	(84.695, 0.425)

We simulate the values of o and l respectively. We get the probability distributions o and l respectively in exchange currency. We give the result in Table 3.

Table 3. Parameters of probability distribution functions for interval values

	o	l
USD	$W(29.91, 89.73)$	$W(3.07, 0.51)$
EUR	$\Gamma(270.87, 0.43)$	$\Gamma(7.38, 0.09)$
AUD	$\Gamma(563.13, 0.14)$	$\Gamma(4.99, 0.11)$
GBP	$\Gamma(601.20, 0.23)$	$\Gamma(7.37, 0.10)$
CHF	$N(83.95, 2.63^2)$	$\Gamma(8.97, 0.05)$

When we know the distribution function, we use moment method estimator (MME) to estimate our parameter in each distribution function. We can find out the expected values and variances by using those parameters. Table 4 shows the results.

Table 4. Expected value and variance for interval values

	o_1	o_2	o_3	o_4	o_5
Expected value	88.09	116.47	78.84	138.28	83.95
Variance	13.62	50.08	11.04	31.80	6.92
	l_1	l_2	l_3	l_4	l_5
Expected value	0.46	0.66	0.55	0.74	0.45
Variance	0.03	0.06	0.06	0.07	0.02

Moreover, we give the values of $\mathbf{V}_o = [cov(o_i, o_j)]_{5\times5}$ and $\mathbf{V}_l = [cov(l_i, l_j)]_{5\times5}$, $\forall i, j = 1, 2, \cdots, 5$ are in the following.

$$\mathbf{V}_o = \begin{bmatrix} 13.62 & 0.92 & 0.04 & 0.77 & 0.32 \\ 0.92 & 50.08 & -1.15 & 1.83 & 0.33 \\ 0.04 & -1.15 & 11.04 & 0.05 & -0.64 \\ 0.77 & 1.83 & 0.05 & 31.80 & -0.48 \\ 0.32 & 0.33 & -0.64 & -0.48 & 6.92 \end{bmatrix}$$

and

$$\mathbf{V}_l = \begin{bmatrix} 0.0257 & -0.0004 & -0.0008 & 0.0011 & 0.0001 \\ -0.0004 & 0.0602 & 0.0019 & 0.0029 & -0.0012 \\ -0.0008 & 0.0019 & 0.0625 & 0.0006 & -0.0004 \\ 0.0011 & 0.0029 & 0.0006 & 0.0661 & -0.0020 \\ 0.0001 & -0.0012 & -0.0004 & -0.0020 & 0.0236 \end{bmatrix}.$$

Now, we have all the data we need in model (4) and (5). We rewrite our model with estimated parameters as follows.

$$\left.\begin{array}{ll} \max & 88.09x_1 + 116.47x_2 + 78.84x_3 + 138.28x_4 + 83.95x_5 \\ \min & 13.62x_1^2 + 50.08x_2^2 + 11.04x_3^2 + 31.80x_4^2 + 6.92x_5^2 \\ & +2(0.92x_1x_2 + 0.04x_1x_3 + 0.77x_1x_4 + 0.32x_1x_5 - 1.15x_2x_3 \\ & +1.83x_2x_4 + 0.33x_2x_5 + 0.05x_3x_4 - 0.64x_3x_5 - 0.48x_4x_5) \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ & x_i \geq 0 \quad \forall i = 1, 2, \cdots, n \end{array}\right\}, \tag{6}$$

and

$$\left.\begin{array}{ll} \min & 0.46x_1 + 0.66x_2 + 0.55x_3 + 0.74x_4 + 0.45x_5 \\ \min & 0.03x_1^2 + 0.06x_2^2 + 0.06x_3^2 + 0.07x_4^2 + 0.02x_5^2 + 2(-0.0004x_1x_2 \\ & -0.0008x_1x_3 + 0.0011x_1x_4 + 0.0001x_1x_5 + 0.0019x_2x_3 + 0.0029x_2x_4 \\ & -0.0012x_2x_5 + 0.0006x_3x_4 - 0.0004x_3x_5 - 0.002x_4x_5) \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 = 1 \\ & x_i \geq 0 \quad \forall i = 1, 2, \cdots, n \end{array}\right\}. \tag{7}$$

In fact, that it is not easy to solve the model (6) and (7). In order to reduce the calculation load, we simplify the minimum goal in (6) and (7) as follows.
We set

$$\begin{array}{rl} G_1 = \min & 13.62x_1^2 + 50.08x_2^2 + 11.04x_3^2 + 31.80x_4^2 + 6.92x_5^2 \\ & +2(0.92x_1x_2 + 0.04x_1x_3 + 0.77x_1x_4 + 0.32x_1x_5 - 1.15x_2x_3 \\ & +1.83x_2x_4 + 0.33x_2x_5 + 0.05x_3x_4 - 0.64x_3x_5 - 0.48x_4x_5). \end{array}$$

Hence, we have

$$G_1 \leq (\sqrt{13.62}x_1 + \sqrt{50.08}x_2 + \sqrt{11.04}x_3 + \sqrt{31.80}x_4 + \sqrt{6.92}x_5)^2 \leq k_1^2,$$

where k_1 ($k_1 \geq 0$) represents the tolerated risk level.

It implies that $3.69x_1 + 7.08x_2 + 3.22x_3 + 5.64x_4 + 2.63x_5 \leq k_1$.

Let

$$\begin{array}{rl} G_2 = \min & 0.03x_1^2 + 0.06x_2^2 + 0.06x_3^2 + 0.07x_4^2 + 0.02x_5^2 + 2(-0.0004x_1x_2 \\ & -0.0008x_1x_3 + 0.0011x_1x_4 + 0.0001x_1x_5 + 0.0019x_2x_3 + 0.0029x_2x_4 \\ & -0.0012x_2x_5 + 0.0006x_3x_4 - 0.0004x_3x_5 - 0.002x_4x_5). \end{array}$$

Hence, we have

$$G_2 \leq (\sqrt{0.03}x_1 + \sqrt{0.06}x_2 + \sqrt{0.06}x_3 + \sqrt{0.07}x_4 + \sqrt{0.02}x_5)^2 \leq k_2^2,$$

where k_2 ($k_2 \geq 0$) represents the tolerated risk level.

It implies that $0.17x_1 + 0.24x_2 + 0.24x_3 + 0.26x_4 + 0.14x_5 \leq k_2$.

Therefore, we rewrite model (6) and (7) in the following.

$$\left.\begin{array}{ll} \max & 88.09x_1 + 116.47x_2 + 78.84x_3 + 138.28x_4 + 83.95x_5 \\ \text{s.t.} & 3.69x_1 + 7.08x_2 + 3.22x_3 + 5.64x_4 + 2.63x_5 \leq k_1 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 \\ & x_i \geq 0 \qquad \forall i = 1, 2, \cdots, n \end{array}\right\}, \tag{8}$$

and

$$\left.\begin{array}{ll} \min & 0.46x_1 + 0.66x_2 + 0.55x_3 + 0.74x_4 + 0.45x_5 \\ \text{s.t.} & 0.17x_1 + 0.24x_2 + 0.24x_3 + 0.26x_4 + 0.14x_5 \leq k_2 \\ & x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 \\ & x_i \geq 0 \qquad \forall i = 1, 2, \cdots, n \end{array}\right\}. \tag{9}$$

We solve the model (8) and (9) by using GP-IGP (Linear and Integer Goal Programming). The result depends on the selection of k_i, $i = 1, 2$. Table 5 shows the result of model (8) and Table 6 shows the result of model (9).

TABLE 5. Optimal solution of $\mathbf{o}^*$ for different conditions and the result of parameters

k_1	0.5	1	1.5	2	2.5	3
$\sum_{i=1}^{5} x_i$	0.19	0.38	0.57	0.76	0.95	1
$\mathbf{o}^*$	$[0,0,0,0,.19]'$	$[0,0,0,0,.38]'$	$[0,0,0,0,.57]'$	$[0,0,0,0,.76]'$	$[0,0,0,0,.95]'$	$[0,0,0,.12,.88]'$
max	15.96	31.92	47.88	63.84	79.80	90.63
k_1	3.5	4	4.5	5	5.5	6
$\sum_{i=1}^{5} x_i$	1	1	1	1	1	1
$\mathbf{o}^*$	$[0,0,0,.29,.71]'$	$[0,0,0,.46,.54]'$	$[0,0,0,.62,.38]'$	$[0,0,0,.79,.21]'$	$[0,0,0,.95,.05]'$	$[0,0,0,1,0]'$
max	99.65	108.68	117.70	126.73	135.75	138.28

TABLE 6. Optimal solution of $\mathbf{l}^*$ for different conditions and the result of parameters

k_2	0.5	1	1.5	2	2.5	3
$\sum_{i=1}^{5} x_i$	0.19	0.38	0.57	0.76	0.95	1
$\mathbf{l}^*$	$[0,0,0,0,.19]'$	$[0,0,0,0,.38]'$	$[0,0,0,0,.57]'$	$[0,0,0,0,.76]'$	$[0,0,0,0,.95]'$	$[0,0,0,0,1]'$
max	0.09	0.17	0.26	0.34	0.43	0.45
k_2	3.5	4	4.5	5	5.5	6
$\sum_{i=1}^{5} x_i$	1	1	1	1	1	1
$\mathbf{l}^*$	$[0,0,0,0,1]'$	$[0,0,0,0,1]'$	$[0,0,0,0,1]'$	$[0,0,0,0,1]'$	$[0,0,0,0,1]'$	$[0,0,0,0,1]'$
max	0.45	0.45	0.45	0.45	0.45	0.45

In Table 5, we can see that when we choose largest k_1, we will get the largest return. Moreover, we get the same maximum 138.28 when $k_1 \geq 6$ and $\sum_{i=1}^{5} x_i = 1$. In Table 6, we can see that when $\sum_{i=1}^{5} x_i = 1$ and $k_2 \geq 3$, we get the same minimum 0.45. Suppose that we choose $\sum_{i=1}^{5} x_i = 1$ and $k_1 = k_2 = 6$, then the optimal solution of model (8) is $\mathbf{o}^* = [0, 0, 0, 1, 0]'$ and the optimal solution of model (9) is $\mathbf{l}^* = [0, 0, 0, 0, 1]'$.

Hence, we have our return of fuzzy vector

$$\mathbf{x}^* = [\mathbf{o}^*, \mathbf{l}^*]' = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{10}$$

Moreover, we get the total expected return

$$E(R(\mathbf{x}^*)) = \left(\sum_{i=1}^{n} E(o_i)\mathbf{o}^*, \sum_{i=1}^{n} E(l_i)\mathbf{l}^* \right) = (138.28, 0.45). \tag{11}$$

The interval value of the total expected return resulted in $[137.83, 138.73]$.

In the next example, let us calculate the same problem with another model proposed by W. G. Zhang [33].

Example 4.2. *In* [33]*, Zhang uses the concept with γ-level to deal with optimization model. He also gives an additional condition by using upper and lower bounds in the model proposed by Markowitz. We applied his model to solve our problem and gave the same upper bound and lower bound of his example.*

First, we calculated the expected value of 224 interval fuzzy data by means of fuzzy set theory [30]. We obtained 5 fuzzy interval numbers as follows: USD $= r_1 = (87.64, 88.57)$, EUR $= r_2 = (116.07, 117.41)$, AUD $= r_3 = (79.66, 80.76)$, GBP $= r_4 = (135.12, 136.58)$ and CHF $= r_5 = (83.48, 84.42)$. The lower bound and upper bounds of x_i are given by $[l_1, l_2, l_3, l_4, l_5] = [0.1, 0.1, 0.1, 0.1, 0.1]$ and $[u_1, u_2, u_3, u_4, u_5] = [0.4, 0.4, 0.4, 0.5, 0.6]$, respectively.

Hence, the lower possibilistic mean-standard deviation model is

$$\left.\begin{array}{ll} \max & 87.64x_1 + 116.07x_2 + 79.66x_3 + 135.12x_4 + 83.48x_5 \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 \\ & u_i \geq x_i \geq l_i \quad \forall i = 1, 2, \cdots, n \end{array}\right\}, \tag{12}$$

and then the upper possibilistic mean-standard deviation model is

$$\left.\begin{array}{ll} \max & 88.57x_1 + 117.41x_2 + 80.76x_3 + 136.58x_4 + 84.42x_5 \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 + x_5 \leq 1 \\ & u_i \geq x_i \geq l_i \quad \forall i = 1, 2, \cdots, n \end{array}\right\}. \tag{13}$$

Tables 7 and 8 show the result, where $\mathbf{L}^*$ denotes as the optimal solution in model (12) and $\mathbf{U}^*$ denotes as the optimal solution in model (13).

TABLE 7. Optimal solution of $\mathbf{L}^*$ for different conditions and the result of parameters

$\sum_{i=1}^{5} x_i$	0.51	0.58	0.65	0.72
$\mathbf{L}^*$	$[.10, .10, .10, .11, .10]'$	$[.10, .10, .10, .18, .10]'$	$[.10, .10, .10, .25, .10]'$	$[.10, .10, .10, .32, .10]'$
max	51.55	61.01	70.46	79.92

$\sum_{i=1}^{5} x_i$	0.79	0.87	0.94	1
$\mathbf{L}^*$	$[.10, .10, .10, .39, .10]'$	$[.10, .10, .10, .47, .10]'$	$[.10, .14, .10, .50, .10]'$	$[.10, .20, .10, .50, .10]'$
max	89.38	100.19	108.89	115.85

TABLE 8. Optimal solution of $\mathbf{U}^*$ for different conditions and the result of parameters

$\sum_{i=1}^{5} x_i$	0.51	0.58	0.65	0.72
$\mathbf{L}^*$	$[.10, .10, .10, .11, .10]'$	$[.10, .10, .10, .18, .10]'$	$[.10, .10, .10, .25, .10]'$	$[.10, .10, .10, .32, .10]'$
max	52.14	61.70	71.26	80.82

$\sum_{i=1}^{5} x_i$	0.79	0.87	0.94	1
$\mathbf{L}^*$	$[.10, .10, .10, .39, .10]'$	$[.10, .10, .10, .47, .10]'$	$[.10, .14, .10, .50, .10]'$	$[.10, .20, .10, .50, .10]'$
max	90.38	101.31	110.10	117.15

From Tables 7 and 8, we can see that when we choose $\sum_{i=1}^{5} x_i = 1$, the optimal solution of model (12) is $\mathbf{L}^* = [0.1, 0.2, 0.1, 0.5, 0.1]$. When we choose $\sum_{i=1}^{5} x_i = 1$, the optimal solution of model (13) is $\mathbf{U}^* = [0.1, 0.2, 0.1, 0.5, 0.1]$. Hence, we have expected return with fuzzy vector as follows:

$$\mathbf{x}^{**} = [\mathbf{L}^*, \mathbf{U}^*]' = \begin{bmatrix} 0.1 & 0.2 & 0.1 & 0.5 & 0.1 \\ 0.1 & 0.2 & 0.1 & 0.5 & 0.1 \end{bmatrix}. \tag{14}$$

Moreover, we get the total expected return

$$E(R(\mathbf{x}^{**})) = \left(\sum_{i=1}^{n} E(L_i)\mathbf{L}^*, \sum_{i=1}^{n} E(U_i)\mathbf{U}^* \right) = (115.85, 117.15), \tag{15}$$

i.e., the interval value of the total expected return resulted in $[115.85, 117.15]$.

5. **Conclusions.** In this paper, we proposed a method to estimate the probability distribution function with interval values. When we know the probability distribution function with interval values, we can calculate the expected value, variance and covariance by using the estimated parameters of the underlying distribution function. Therefore, we can easily build up and solve the portfolio selection model with interval values on the basis of the Markowitz's mean-variance model. We gave an empirical study by using portfolio selection model wit interval values in Example 4.1 and used the model proposed by W. G. Zhang [33] in Example 4.2. We can see that it is more meaningful to use our model to estimate the maximum return because of considering more information, which comes from original fuzzy data. We have different choices of k which can decide the best expected return and less risk level in our model. Through this method, we also can provide not only one choice of portfolio selection but also two or more for decision makers.

We have further points to improve in the future as follows:

1. In this paper, we just use 4 underlying probability distribution functions in industry to estimate the parameters. We can do more estimation with other underlying probability distribution functions and compare which one gives better results in our fuzzy portfolio selection problem.
2. Moreover, if we can get the triangular fuzzy numbers or trapezoid fuzzy numbers, it will be good for us to examine fuzzy portfolio selection model which we proposed is easily using for decision makers.

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