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DEFAULTABLE BONDS VIA HKA

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Abstract. *To construct a no-arbitrage defaultable bond market under the physical measure* P*, we work in the state price density framework. Using the heat kernel approach (HKA for short) and the killing of a Markov process, we construct a single defaultable bond market that enables an explicit expression of the price of a defaultable bond and the credit spread. In the quadratic Gaussian settings, we show that the model is not only tractable but realistic via some simulation results.*
Keywords: (Non-)Systematic risk, State price density, Killed HKA, Markov functional model, Quadratic Gaussian

1. **Introduction.** Considering the (non-)systematic risk in credit markets, it is reported in [1] that the risk neutral default probability for BBB-rated bonds is eight times greater than the physical one. It should be noted that under the risk neutral measure Q we leave behind the systematic risk since the market price is something that has already been eliminated. The difference between P and Q, however, is sometimes very critical. Therefore, it is reasonable, or should we say, desirable, to construct a credit market model under the physical measure P and explore modeling the market risk.

In this paper, we rely on the **heat kernel approach** introduced by one of the author and his collaborators in [2]. The HKA, an abbreviation of "Heat Kernel Approach", was introduced by one of the authors and his collaborators in [2]. Briefly speaking, HKA is a systematic method to produce a tractable interest rate model which is "Markov functional" in the sense of Hunt-Kennedy-Pelsser [3]. In the fundamental paper [2], four different types of implementation methods are introduced, namely, 1) Eigenfunction models, 2) Weighted HKA, 3) Killed HKA and 4) the Trace Approach. As is pointed out in [2], the eigenfunction models are tailor-made for swaption pricing, and a deeper understanding for its mathematical structure leads to the trace approach, which is mathematically the most involved. The weighted HKA is extended to a time-inhomogeneous setting and applied to information-based models by J. Akahori and A. Macrina [4].

In the present paper, we will demonstrate how the Killed HKA is applied to the modelling of defaultable bonds by constructing a market where the market price of risk and the default probability are "built in the same block" (the precise meaning will be given later). We stress that the HKA is basically a state-price density approach where everything is written under the physical = statistical measure. Since the HKA furthermore gives an analytically tractable model, the framework proposed in this paper is promising in respect of modelling defaultable markets.

The organization of the present paper is as follows. After recalling the *plain-vanilla* HKA in Section 2.1 and the killed HKA in Section 2.2, we shall give the main result, a framework within the Killed HKA to model a defaultable bond market in Section 3. In Section 4, we will give some simulation results of an explicit example with a quadratic form of a Wiener process. Conclusions are presented in Section 5.

2. **Heat Kernel Approach.** Here we briefly recall the approach.

2.1. **Plain-vanilla HKA.** We work on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Now we consider a general Markov process $\{X_t^x\}_{t\geq 0, x\in\mathcal{S}}$ on a Polish space $\mathcal{S}$.

Definition 2.1. *Let X be an $\mathcal{S}$-valued Markov process. We shall say that a function p satisfies the* propagation property *if*

$$\mathbf{E}[p(t, X_s^x)] = p(t+s, x) \tag{1}$$

holds for any $t, s \geq 0$ and $x \in \mathcal{S}$.

The following fact is initialized by [5] and developed in [2].

Proposition 2.1 (Akahori et al. [2])**.** *Let X be an $\mathcal{S}$-valued Markov process, λ be a positive function on the half line, and p be a function with the propagation property. The bond market given by*

$$P_f(t, T) = \frac{p(\lambda_T + T - t, X_t^x)}{p(\lambda_t, X_t^x)} \tag{2}$$

is an arbitrage free market.

Example 2.1 (Generic example)**.** *Take a measurable, bounded $h : \mathcal{S} \to \mathbb{R}_{\geq 0}$, then*

$$p(t, x) := \mathbf{E}[h(X_t^x)] \tag{3}$$

satisfies the propagation property (1). In fact, by the Markov property, we have

$$\begin{aligned}\mathbf{E}\left[p(t, X_s)\right] &= \mathbf{E}\left[\mathbf{E}\left[h(X_t^{X_s^x})\right]\right] = \mathbf{E}\left[\mathbf{E}\left[h(X_{t+s}^x)|\mathcal{F}_s^X\right]\right] \\ &= \mathbf{E}\left[h(X_{t+s}^x)\right] = p(t+s, x).\end{aligned}$$

Proof: Let $\pi_t = p(\lambda_t, X_t^x)$. By the propagation property of p and the Markov property of X, we have

$$\begin{aligned}P_f(t, T) &= \frac{\mathbf{E}\left[\pi_T|\mathcal{F}_t\right]}{\pi_t} = \frac{\mathbf{E}\left[p(\lambda_T, X_T^x)|\mathcal{F}_t\right]}{p(\lambda_t, X_t^x)} \\ &= \frac{\mathbf{E}\left[p(\lambda_T, X_{T-t}^{X_t^x})\right]}{p(\lambda_t, X_t^x)} = \frac{p(\lambda_T + T - t, X_t^x)}{p(\lambda_t, X_t^x)}.\end{aligned}$$

This means that π is the state price density of the market.

It should be noted that we do not assume π_t to be a supermartingale, as in [2], i.e., in economic terms *we do not assume positive short rates.* The four implementation methods mentioned in the introduction is introduced in [2] to obtain supermartingales out of a propagator, or equivalently to obtain positive rate models.

2.2. **The killed HKA.** We now recall, and give a more detailed description of the Killed HKA [1]. Let V be a non-negative measurable function on $\mathcal{S}$. Put $Y_t^y = y + \int_0^t V(X_s^x)ds$ for $y \in \mathbf{R}$. Let us define

$$q(t, x) = \mathbf{E}\left[\exp\left(-\int_0^t V(X_s^x)ds\right)\right]. \tag{4}$$

[1]This part is shared with [6].

Then the function

$$q(t,x,y) = e^{-y}q(t,x),$$

satisfies the propagation property with respect to (X^x, Y^y);

$$\mathbf{E}[q(s, X_t^x, Y_t^y)] = q(t+s, x, y). \tag{5}$$

In fact, by the Markov property of X, we have

$$\mathbf{E}\left[e^{-Y_{t+s}}|\mathcal{F}_t\right] = \mathbf{E}\left[e^{-Y_s(\theta_t)}|\mathcal{F}_t\right] \times e^{-Y_t} = \mathbf{E}\left[e^{-Y_s}|X_t\right] \times e^{-Y_t},$$

where θ is the shift operator. Thus we obtain

$$\mathbf{E}\left[q(s, X_t^x, Y_t^y)\right] = \mathbf{E}\left[e^{-Y_t^y} q(s, X_t^x)\right] = \mathbf{E}\left[e^{-Y_t^y} e^{-Y_s^y \circ \theta_t}\right] = q(t+s, x, y).$$

This fact ensures that the bond market model constructed as

$$P(t,T) = \frac{q\left(\lambda_T + T - t, X_t^x\right)}{q\left(\lambda_t, X_t^x\right)}, \tag{6}$$

where λ is an increasing function, is arbitrage-free since we can choose

$$\pi_t = q(\lambda_t, X_t) \exp\left(-\int_0^t V(X_s)ds\right)$$

as a state price density of the market. In fact,

$$\begin{aligned}
\mathbf{E}[\pi_T|\mathcal{F}_t] &= \mathbf{E}\left[\mathbf{E}\left[\exp\left(-\int_T^{T+\lambda_T} V(X_u^x)du\right)\middle|\mathcal{F}_T\right] \exp\left(-\int_0^T V(X_s)ds\right)\middle|\mathcal{F}_t\right] \\
&= \mathbf{E}\left[\exp\left(-\int_t^{T+\lambda_T} V(X_u^x)du\right)\middle|X_t\right] \exp\left(-\int_0^t V(X_s)ds\right) \\
&= q(\lambda_T + T - t, X_t) \exp\left(-\int_0^t V(X_s)ds\right) = \pi_t \frac{q(\lambda_T + T - t, X_t)}{q(\lambda_t, X_t)}.
\end{aligned}$$

Note that the bond price P is decreasing in T since q is increasing in t, which is ensured by the positivity of V. Thus we obtain a positive rate model.

3. **HKA to Defaultable Bond.** This section is the main part of the present paper. Let us now consider a defaultable bond in the following situation:

1. The bond pays a unit amount at the maturity time T unless it defaults.
2. At the default time τ, nothing will be recovered.
3. The state variable is a Markov process $\{X_t^x; t \geq 0\}$, which is observable in the market.
4. The default probability is completely determined through the information of X in the following manner; the hazard rate of the default time on the filtration $\mathcal{F}^X$ is given by

$$\mathbf{E}\left[1_{\{\tau > t\}}|\mathcal{F}_t^X\right] = \exp\left(-\int_0^t V(X_u^x)du\right)$$

   where $\mathcal{F}^X$ is the natural filtration on X and V is a non-negative measurable function.
5. The default come as a "surprise" to the market. To be precise, the market filtration $\{\mathcal{G}_t\}$ is defined as $\mathcal{G}_t = \sigma(X_s, \{\tau \leq s\}; s \leq t)$ and we assume that $\mathcal{F}_0^X = \{\Omega, \emptyset\} = \mathcal{G}_0$.
6. A state price density of the market is given by

$$\pi_t := q(\lambda_t, x)|_{x=X_t} = \mathbf{E}\left[\exp\left(-\int_0^{\lambda_t} V(X_u^x)du\right)\right]|_{x=X_t}$$

   where q is defined as (4) and λ is a non-decreasing function.

Note that the Assumptions 1-5 may be natural (except Assumption 2, which assumes zero recovery) and very generic, while the last assumption is very specific in that the function V controls both the market price of risk as well as the default probability of a bond. Very heuristically speaking, this market is fully subject to the risk of a defaultable bond.

We stress that this is just a toy model, which shows: how the killed HKA can be applied to defaultable market modeling. The following is established in [6]:

Theorem 3.1. *Under the above Assumptions 1-6,*

(i) *the price $P_d(t,T)$ of a defaultable zero coupon bond is given by*

$$P_d(t,T) = 1_{\{\tau>t\}} \frac{q(\lambda_T + T - t, X_t^x)}{q(\lambda_t, X_t^x)}, \tag{7}$$

(ii) *the price $P_f(t,T)$ of a default-free bond is given by*

$$P_f(t,T) = \frac{\hat{q}(\lambda_T + T - t, T - t, X_t^x)}{q(\lambda_t, X_t^x)}, \tag{8}$$

where

$$\hat{q}(t,s,x) = \mathbf{E}\left[\exp\left(-\int_s^t V(X_u^x)du\right)\right], \tag{9}$$

(iii) *and then the "credit spread" is given by*

$$\partial_T \log \frac{\hat{q}(\lambda_T + T - t, T - t, X_t^x)}{\hat{q}(\lambda_T + T - t, 0, X_t^x)}. \tag{10}$$

Remark 3.1. *Note that the "credit spread" makes no sense when $\tau \le t$, so we can only think of the case that $\tau > t$.*

Proof: The proof is based on the following fundamental lemma due to Dellacherie (see [8]): For any $\mathcal{F}_T^X$-integrable random variable Z and $0 < t < T$, we have

$$\mathbf{E}[1_{\{\tau>T\}} Z | \mathcal{G}_t] = \frac{1_{\{\tau>t\}}}{E[1_{\{\tau>t\}} | \mathcal{F}_t^X]} \mathbf{E}[1_{\{\tau>T\}} Z | \mathcal{F}_t^X].$$

Hence, we have

$$P_d(t,T) = \frac{1}{\pi_t} \frac{1_{\{\tau>t\}}}{\mathbf{E}\left[1_{\{\tau>t\}} | \mathcal{F}_t^X\right]} \mathbf{E}\left[1_{\{\tau>T\}} \pi_T | \mathcal{F}_t^X\right].$$

Then by a Markov property and a tower property,

$$\mathbf{E}\left[1_{\{\tau>T\}} \pi_T | \mathcal{F}_t^X\right] = \mathbf{E}\left[\mathbf{E}\left[1_{\{\tau>T\}} | \mathcal{F}_T^X\right] q(\lambda_T, X_T^x) | \mathcal{F}_t^X\right],$$

and by the assumption that $\mathbf{E}\left[1_{\{\tau>t\}} | \mathcal{F}_t^X\right] = \exp\left(-\int_0^t V(X_u^x)du\right)$

$$\mathbf{E}\left[\mathbf{E}\left[1_{\{\tau>T\}} | \mathcal{F}_T^X\right] q(\lambda_T, X_T^x) | \mathcal{F}_t^X\right] = \exp\left(-\int_0^t V(X_u^x)du\right) \mathbf{E}\left[e^{-\int_t^T V(X_s^x)ds} q(\lambda_T, X_T^x) | \mathcal{F}_t^X\right].$$

Here using Equation (5), we have

$$\mathbf{E}\left[e^{-\int_t^T V(X_s^x)ds} q(\lambda_T, X_T^x) | \mathcal{F}_t^X\right] = q(\lambda_T + T - t, X_t^x),$$

so that

$$P_d(t,T) = 1_{\{\tau>t\}} \frac{q(\lambda_T + T - t, X_t^x)}{q(\lambda_t, X_t^x)}.$$

On the other hand, (ii) follows from a Markov property and a tower property. And it is known that the "credit spread" is given by

$$-\partial_T \log \frac{P_d(t,T)}{P_f(t,T)}.$$

Here since $P_d(t,T)$ and $P_f(t,T)$ are (i) and (ii) respectively,

$$-\partial_T \log \frac{P_d(t,T)}{P_f(t,T)} = \partial_T \log \frac{\mathbf{E}\left[q(\lambda_T, X_T^x)|\mathcal{F}_t^X\right]}{q(\lambda_T + T - t, X_t^x)}$$

when $\tau > t$. Then by a Markov property and a tower property,

$$\mathbf{E}\left[q(\lambda_T, X_T^x)|\mathcal{F}_t^X\right] = \mathbf{E}\left[e^{-\int_{T-t}^{\lambda_T+T-t} V(X_s^{X_t^x})ds}\right] = \hat{q}(\lambda_T + T - t, T - t, X_t^x).$$

Hence, we obtain

$$-\partial_T \log \frac{P_d(t,T)}{P_f(t,T)} = \partial_T \log \frac{\hat{q}(\lambda_T + T - t, T - t, X_t^x)}{q(\lambda_T + T - t, X_t^x)}.$$

4. **Quadratic Example.** Now we give some simulation results of an explicit example, where X is a d-dimensional Wiener process and $V(x) = \frac{\beta^2|x|^2}{2}$ $(\beta > 0)$. Let $q(t,x)$ and $\hat{q}(t,x)$ be as in Equations (4) and (9). Then they are explicitly given by

$$q(t,x) = (\cosh\beta t)^{-d/2} \exp\left(-\frac{\beta x^2}{2}\frac{\sinh\beta t}{\cosh\beta t}\right), \tag{11}$$

and

$$\hat{q}(t,x) = (\cosh\beta(t-s) + \beta s \sinh\beta(t-s))^{-d/2} \exp\left(-\frac{\beta x^2}{2}\frac{\tanh\beta(t-s)}{1+\beta s\tanh\beta(t-s)}\right), \tag{12}$$

which results from Lemma 1 and Corollary 1 in the Appendix. Hence, we obtain the analytic expression of the bond prices. The simulated yield curves in (Figure 1) and (Figure 2), implied by a default-free bond as

$$-\frac{1}{T}\log P_f(t,T)$$

are obtained by using Equations (8) and (12). Here the parameters are set to be $\beta = 0.1$, $x = 0.01, 10, 20, 30$, $\lambda_t = e^t/10$, and the present time $t = 0$ in (Figure 1), $\beta = 1.8$ and $\lambda_t = e^t/100$ in (Figure 2). The x-axis is the maturity, ranging from one year to ten years, and y-axis is the price of a default-free bond.

We see from the curves (Figure 1) and (Figure 2) that increasing the value of x does not only shift upward, but also causes a "hump" in the curve, which can not be observed in the normal affine model, with the proper choice of λ. Moreover, using Formulae (10) and (12), we obtain the simulated credit spread curves in (Figure 3). Here the parameters are set to be $\lambda_t = \sqrt{t}$, $\beta = 0.1, 0.2, \ldots, 1$, $x = 0$, and the present time $t = 0$.

As usual, the lower the credit rating of a defaultable bond is, the wider the spread is, and it is non-decreasing in time to maturity. Moreover, the spread of a defaultable bond with a low rating is much wider in the time to maturity than the one of a bond with a hight rating. This fact is illustrated in (Figure 3).

5. **Conclusions.** We have introduced a way of constructing a single defaultable bond market model under the physical measure P by applying the killed HKA. We have also presented some simulation results in the quadratic case. Comparing with the well-known Hull-White model, we have observed a complex "hump" in the yield implied by a default-free bond, which is due to the parameter λ.

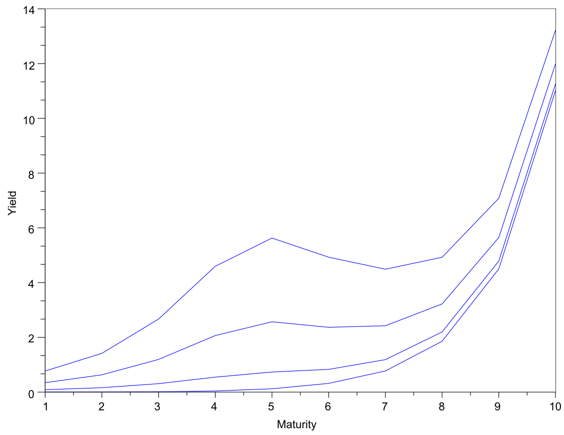

FIGURE 1. Simulated yield curves implied by a default-free bond when $\lambda_t = e^t/10$

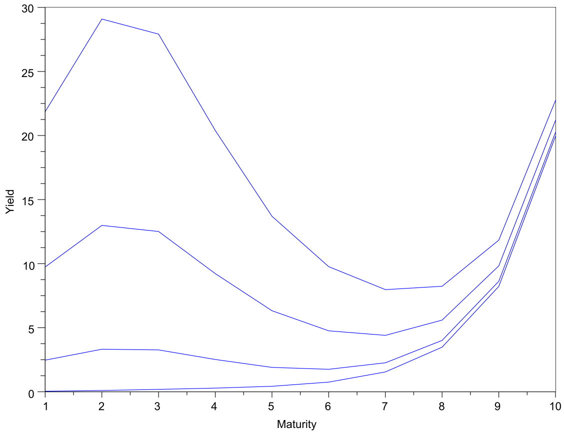

FIGURE 2. Simulated yield curves implied by a default-free bond when $\lambda_t = e^t/100$

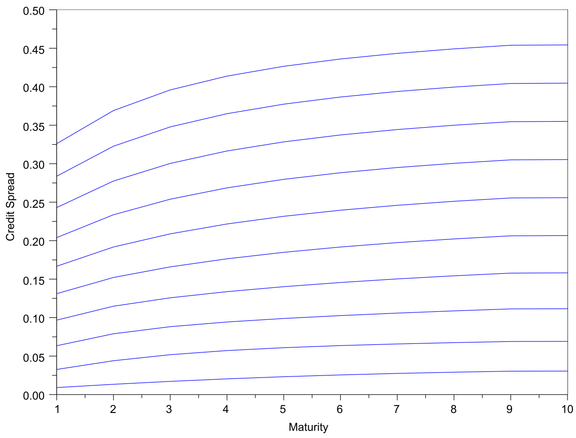

FIGURE 3. Simulated credit spread curves

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Appendix.

Lemma 1. *Let X be a d-dimensional Wiener process starting at x. For $\alpha, \beta \geq 0$, it holds that*

$$\begin{aligned}&\mathbf{E}\left[e^{-\alpha|X_t^x|^2-\frac{\beta^2}{2}\int_0^t|X_s^x|^2ds}\right]\\ &=\begin{cases}\left(\cosh\beta t+\dfrac{2\alpha}{\beta}\sinh\beta t\right)^{-d/2}\exp\left(-\dfrac{\beta x^2}{2}\dfrac{\beta\sinh\beta t-2\alpha\cosh\beta t}{\beta\cosh\beta t+2\alpha\sinh\beta t}\right) & \beta>0,\\ (2\alpha t+1)^{-d/2}\exp\left(-\frac{\alpha x^2}{2\alpha t+1}\right) & \beta=0.\end{cases}\end{aligned} \tag{13}$$

This is a well-known formula and there are many ways to prove it. One way is presented in [6].

The following is an immediate consequence of Lemma 1:

Corollary 1. *Let X be a d-dimensional Wiener process starting at x. For $\beta > 0$, it holds*

$$\begin{aligned}\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_s^t|X_v^x|^2dv}\right]&=(\cosh\beta(t-s)+\beta s\sinh\beta(t-s))^{-d/2}\\ &\quad\times\exp\left(-\frac{\beta x^2}{2}\frac{\sinh\beta(t-s)}{\cosh\beta(t-s)+\beta s\sinh\beta(t-s)}\right).\end{aligned}$$

Proof: By a Markov property, a tower property, and Lemma 1,

$$\begin{aligned}\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_s^t|X_v^x|^2dv}\right]&=\mathbf{E}\left[\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_s^t|X_v^x|^2dv}\middle|\mathcal{F}_s^X\right]\right]=\mathbf{E}\left[\mathbf{E}\left[e^{-\frac{\beta^2}{2}\int_0^{t-s}|X_v^x|^2dv}\middle|X_s^x\right]\right]\\ &=\cosh\beta(t-s)^{-d/2}\mathbf{E}\left[\exp\left(-\frac{\beta|X_s^x|^2}{2}\frac{\sinh\beta(t-s)}{\cosh\beta(t-s)}\right)\right].\end{aligned}$$

The proof is complete by replacing α by $\frac{\beta}{2}\frac{\sinh\beta(t-s)}{\cosh\beta(t-s)}$ in Lemma 1.