International Journal of Innovative Computing, Information and Control
Volume **8**, Number **3(B)**, March **2012**

ICIC International ©2012 ISSN 1349-4198
pp. 2261–2274

A NUMERICAL METHOD FOR CONTINUOUS-DISCRETE UNSCENTED KALMAN FILTER

Michiaki Takeno and Tohru Katayama

Faculty of Culture and Information Science
Doshisha University
Kyotanabe 610-0394, Japan
eii1001@mail4.doshisha.ac.jp

Received March 2011; revised July 2011

Abstract. *In this paper, we consider a numerical method for the continuous-discrete (CD) unscented Kalman filter (UKF), composed of differential equations for the time update of conditional mean and covariance matrix of the state vector and the discrete measurement update equations. To solve the differential equations for the time update algorithm, we propose a Heun scheme-based procedure that has higher accuracy than the Euler scheme-based CD algorithm. We show the applicability of the proposed approximate procedure by simulation studies for several nonlinear models.*

Keywords: Unscented Kalman filter, Continuous-discrete nonlinear filtering, Stochastic differential equation, Heun scheme

1. Introduction.

Nonlinear filtering problems have been extensively studied in the past [1, 4, 7, 8, 9, 15], and have received renewed interest with the advent of particle filters [5, 12, 19], unscented Kalman filters (UKFs) [10, 11] and ensemble Kalman filters [6], etc.

Although standard filtering problems are usually formulated in discrete-time, real stochastic dynamical systems are continuous in time, so that they are described by stochastic differential equations. In fact, there exist many phenomena that can be modeled by means of a stochastic system where a continuous-time signal is measured discretely in time; thus we often encounter continuous-discrete (CD) nonlinear filtering problems [7, 9]. Examples of such applications include robotics [18], finance [2, 23], GPS/INS [8, 14], target tracking [4, 19], MRI [17], hybrid measurements [24, 25], etc.

The UKF in discrete-time setting has been developed by Julier et al. [10, 11] in order to improve the performance of the extended Kalman filter (EKF) by introducing the unscented transformation (UT) to approximately evaluate the conditional means and covariances under nonlinear transformations without using Jacobian matrices. A key to the UT is a selection of deterministic sample points, called σ points, that approximate the mean and covariance properties of conditional probability density function of the state vector given output observations. Moreover, Särkkä [22] had developed a CD unscented Kalman filter (UKF) algorithm, where two differential equations for time update of the conditional mean and covariance matrix are derived from the discrete UKF algorithm, by using a limiting procedure.

To implement the CD-UKF on a digital computer, we again discretize the time update equations, but in [22], numerical procedures for implementing the CD-UKF are not stated in detail. Thus, in this paper, we present a numerical method for implementing the CD-UKF based on the Heun scheme, which has higher-order accuracy than the Euler scheme [13, 16]. A special structure of the time update equations, involving σ points, in the CD-UKF algorithm prohibits the direct application of the Heun scheme, so that we present a

Heun scheme-based CD-UKF algorithm. We expect that since the discrete-time UKF is a second-order nonlinear filter [10], the derived CD algorithm has higher accuracy than the Euler scheme-based CD algorithm. Advantage of the present method over the standard Euler approximation-based CD-EKF algorithm [9] is shown by simulation studies.

The paper is organized as follows. In Section 2, the CD nonlinear system is described and the CD nonlinear filtering problem is stated. Section 3, as a preliminary, presents a numerical result that shows the difference of the Euler-Maruyama and the Heun schemes in solving a second-order nonlinear stochastic differential equation. Then, the CD-UKF algorithm is outlined in Section 4. Section 5 presents a Heun scheme-based method of computing the time update equations for the conditional mean and covariance matrix, together with a simple numerical example. In Section 6, we show results of simulations for a van der Pol model in an electrical circuit and a two-dimensional tracking problem based on the range and bearing information. Section 7 concludes this paper.

2. **System and Problem Formulation.** Consider a nonlinear stochastic system described by a stochastic differential equation with observations taken at discrete time instants t_k, i.e.,

$$dx(t) = f(x(t), t)dt + Ld\beta(t) \tag{1}$$

$$y_k = h_d(x(t_k)) + v_k, \quad k = 0, 1, \ldots; \quad 0 = t_0 < \cdots < t_{k-1} < t_k \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^p$ is the output vector, $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $h_d : \mathbb{R}^n \to \mathbb{R}^p$ are nonlinear functions, and $L \in \mathbb{R}^{n\times l}$ is a constant matrix. Also, $d\beta(t) \in \mathbb{R}^l$ is the increment of Brownian motion with mean 0 and covariance matrix $Qdt \in \mathbb{R}^{l\times l}$, and $v_k \in \mathbb{R}^p$ is a white noise with mean zero and covariance matrix $R \in \mathbb{R}^{p\times p}$. Since L is constant, the system (1) is called a Langevin-type stochastic differential equation.

Let $Y^k = \{y_0, y_1, \ldots, y_k\}$ be the observations up to time t_k. Then, the problem in this paper is to compute the conditional mean estimate $m(t_k|t_k)$ of the state $x(t_k)$ and the covariance matrix $P(t_k|t_k)$ based on the observations Y^k, where the conditional mean estimate and covariance matrix are defined by

$$m(t|t_{k-1}) = E[x(t) \mid Y^{k-1}] \tag{3}$$

$$P(t|t_{k-1}) = E\big[(x(t) - m(t|t_{k-1}))(x(t) - m(t|t_{k-1}))^{\mathrm{T}} \mid Y^{k-1}\big] \tag{4}$$

where $t_{k-1} \le t \le t_k$, $k = 1, 2, \ldots$. The dynamical system (1) is continuous in time and the observations of (2) are discrete in time, so that the filtering problem is called a continuous-discrete (CD) nonlinear filtering problem [7, 9].

In general, the CD filtering algorithm consists of the forward differential equations for the prediction, or the time update, and the algebraic equations for the measurement update. Thus, the CD nonlinear filtering problem can be solved by iterating the following two steps [9].

- **Prediction step** For $t_{k-1} \le t < t_k$, integrate two forward differential equations in order to obtain the predicted estimate $m(t_k|t_{k-1})$ and covariance matrix $P(t_k|t_{k-1})$, where the initial conditions are the filtered estimate $m(t_{k-1}|t_{k-1})$ and covariance matrix $P(t_{k-1}|t_{k-1})$.
- **Filtering step** At $t = t_k$, update $m(t_k|t_{k-1})$ and $P(t_k|t_{k-1})$ based on the observation y_k to obtain the new filtered estimate $m(t_k|t_k)$ and covariance matrix $P(t_k|t_k)$.

Except for special cases [3, 4], the nonlinear filtering problem has no optimal solution, so that there have been published a large number of papers for deriving approximate nonlinear filtering algorithms, including extended Kalman filters (EKFs) [7, 8, 9, 19].

Särkkä [22] has recently developed a CD-UKF by using a limiting procedure in the discrete-time UKF, but a numerical procedure for solving the time update differential equations in the CD-UKF algorithm is not presented. In this paper, we derive a numerical method for the CD-UKF by applying the Heun scheme, where the use of the Heun scheme is motivated by the fact that it yields better numerical results for solving the stochastic differential equation of (1) than the Euler-Maruyama scheme [13, 16, 20].

Before stating the CD-UKF algorithm due to Särkkä [22], we consider how we simulate the continuous-time system (1) on a digital computer.

3. **Euler-Maruyama and Heun Schemes.** In this section, we briefly review numerical methods of solving the stochastic differential Equation (1). Let $\varDelta_p$ be a sampling interval for the numerical integration[1]. Then, the Euler-Maruyama scheme for numerical solutions of (1) is given by

$$x(t+\varDelta_p) = x(t) + f(x(t),t)\varDelta_p + \sqrt{\varDelta_p}Lw(t) \tag{5}$$

where $w(t) \in \mathbb{R}^l$ is a white noise with mean 0 and covariance matrix Q defined above. We see that (5) is a discrete-time nonlinear stochastic system driven by a white noise $w(t)$. It is well known [13] that the Euler-Maruyama scheme has strong order of convergence 0.5.

Also, the higher-order Heun scheme is described by

$$x(t+\varDelta_p) = x(t) + \frac{\varDelta_p}{2}(c_1 + c_2) + \sqrt{\varDelta_p}Lw(t) \tag{6}$$

where

$$c_1 = f(x(t),t), \quad c_2 = f\left(x(t) + \varDelta_p c_1 + \sqrt{\varDelta_p}Lw(t), t + \varDelta_p\right)$$

and where $w(t)$ is the same white noise appearing in (5), and c_2 is called the supporting value. It follows from [13] that the Heun scheme has strong order of convergence 1.0, so that the Heun scheme (6) will produce a better trajectory of the stochastic differential equation than the Euler scheme for a fixed $\varDelta_p$. There exist some higher-order schemes, but they are much more complicated than the above schemes; see [21] for comparison of numerical results using different schemes for Langevin-type stochastic differential equations.

To demonstrate the difference in two schemes, we present a simulation result for a van der Pol equation of the form

$$\frac{dx_1(t)}{dt} = x_2(t) + w_1(t) \tag{7}$$

$$\frac{dx_2(t)}{dt} = \epsilon\left(1 - x_1^2(t)\right)x_2(t) - x_1(t) + w_2(t) \tag{8}$$

where $x_1(t)$ is the voltage across a capacitance, and $\epsilon = 0.8$ is a constant. Also, $w_1(t) \sim N(0, q_1)$ and $w_2(t) \sim N(0, q_2)$ are independent white noises, uncorrelated with the initial conditions. Figure 1 depicts the trajectories of state x_1 by the Euler-Maruyama and Heun schemes with the initial states $x_1(0) = 0.2$, $x_2(0) = 0.1$ and $q_1 = 0.04$, $q_2 = 0$, where the sampling interval is $\varDelta_p = 0.1$. For comparison, two trajectories obtained by the Runge-Kutta method are included; one is the trajectory x_1 of the deterministic van der Pol equation with $w_1 = 0$ and $w_2 = 0$ in (7) and (8) and the other is obtained by adding the white noise w_1 to the Runge-Kutta solution at each time step $t_k = k\varDelta_p$, $k = 0, 1, \ldots$[2].

We observe that the trajectory by the Euler-Maruyama scheme considerably deviates from the trajectory generated by the Heun scheme, but two trajectories due to the Heun scheme and the Runge-Kutta method with an additive noise are rather close each other. Also, the well-known deterministic solution [R-K (noise 0) in Figure 1] deviates from these

[1] The subscript "p" denotes prediction in this paper.

[2] The convergence of this simple Runge-Kutta based scheme is not guaranteed [13, 16], so that the Runge-Kutta scheme is not used for in later sections.

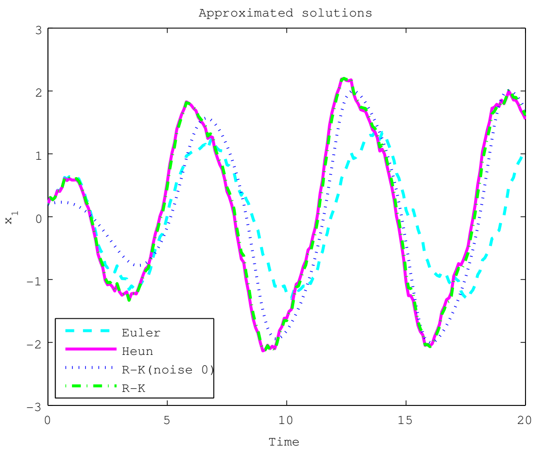

FIGURE 1. Trajectories $x_1(t)$ by Euler-Maruyama and Heun schemes

trajectories. This shows the advantage of the Heun scheme over the Euler-Maruyama scheme in solving a nonlinear stochastic differential equation.

4. **Continuous-Discrete UKF.** According to [22], we summarize the CD-UKF algorithm. Let the σ-points at t $(t_{k-1} \le t < t_k)$ be given by $\mathcal{X}_0(t|t_{k-1})$, $\mathcal{X}_1(t|t_{k-1}), \cdots,$ $\mathcal{X}_{2n}(t|t_{k-1}) \in \mathbb{R}^n$, and define the matrix

$$\mathcal{X}(t|t_{k-1}) = [\mathcal{X}_0(t|t_{k-1}) \quad \mathcal{X}_1(t|t_{k-1}) \quad \cdots \quad \mathcal{X}_{2n}(t|t_{k-1})] \in \mathbb{R}^{n\times(2n+1)}$$

Also, let the weights for the UT be given by

$$W_0 = \frac{\lambda}{n+\lambda}, \quad W_i = W_{n+i} = \frac{1}{2(n+\lambda)} \quad (i = 1, \dots, n)$$

where λ is a parameter [10]. As in [22], let the vector $\boldsymbol{w}_m \in \mathbb{R}^{2n+1}$ and matrix $\mathcal{Z} \in \mathbb{R}^{(2n+1)\times(2n+1)}$ be defined by

$$\boldsymbol{w}_m = \begin{bmatrix} W_0 \\ \vdots \\ W_{2n} \end{bmatrix}, \quad \mathcal{Z} = I_{2n+1} - [\boldsymbol{w}_m \ \cdots \ \boldsymbol{w}_m] \tag{9}$$

Further, we define

$$\mathcal{W} = \mathcal{Z} \begin{bmatrix} W_0 & & \\ & \ddots & \\ & & W_{2n} \end{bmatrix} \mathcal{Z}^{\mathrm{T}} \in \mathbb{R}^{(2n+1)\times(2n+1)} \tag{10}$$

In terms of $\boldsymbol{w}_m$ and $\mathcal{W}$, the conditional mean $m(t|t_{k-1})$ and the conditional covariance matrix $P(t|t_{k-1})$ are compactly expressed as

$$\begin{aligned} m(t|t_{k-1}) &= \sum_{i=0}^{2n} \mathcal{X}_i(t|t_{k-1}) W_i = \mathcal{X}(t|_t k-1)\boldsymbol{w}_m \\ P(t|t_{k-1}) &= \sum_{i=0}^{2n} [\mathcal{X}_i(t|t_{k-1}) - m(t|t_{k-1})] W_i [\mathcal{X}_i(t|t_{k-1}) - m(t|t_{k-1})]^{\mathrm{T}} \\ &= \mathcal{X}(t|t_{k-1}) \mathcal{W} \mathcal{X}^{\mathrm{T}}(t|t_{k-1}) \end{aligned}$$

For simplicity, we write $m(t) := m(t|t_{k-1})$, $P(t) := P(t|t_{k-1})$ and $\mathcal{X}(t) := \mathcal{X}(t|t_{k-1})$ below if there is no confusion. We also define

$$f(\mathcal{X}(t)) = [f(\mathcal{X}_0, t) \ \cdots \ f(\mathcal{X}_{2n}, t)] \in \mathbb{R}^{n\times(2n+1)}$$
$$\bar{f}(\mathcal{X}(t)) = \sum_{i=0}^{2n} f(\mathcal{X}_i, t)W_i = f(\mathcal{X}(t))\boldsymbol{w}_m$$

Noting that $d\beta(t)$ in (1) is the increment of the Brownian motion, it follows from [22] that

$$\frac{dm(t)}{dt} = f(\mathcal{X}(t))\boldsymbol{w}_m \tag{11}$$
$$\frac{dP(t)}{dt} = f(\mathcal{X}(t))\mathcal{W}\mathcal{X}^{\mathrm{T}}(t) + \mathcal{X}(t)\mathcal{W}f^{\mathrm{T}}(\mathcal{X}(t)) + LQL^{\mathrm{T}} \tag{12}$$

where

$$f(\mathcal{X}(t))\mathcal{W}\mathcal{X}^{\mathrm{T}}(t) = \sum_{i=0}^{2n}[f(\mathcal{X}_i, t) - \bar{f}(\mathcal{X}(t))]W_i[\mathcal{X}_i(t) - m(t)]^{\mathrm{T}} \tag{13}$$

In particular, if f is linear, i.e., $f = Ax$, we have

$$f(\mathcal{X}(t))\boldsymbol{w}_m = A\mathcal{X}(t)\boldsymbol{w}_m = Am(t)$$
$$f(\mathcal{X}(t))\mathcal{W}\mathcal{X}^{\mathrm{T}}(t) = A\mathcal{X}(t)\mathcal{W}\mathcal{X}^{\mathrm{T}}(t) = AP(t)$$

so that the above differential Equations (11) and (12) reduce to the well-known prediction equations in the continuous-time Kalman filter, i.e.,

$$\frac{dm(t)}{dt} = Am(t)$$
$$\frac{dP(t)}{dt} = AP(t) + P(t)A^{\mathrm{T}} + LQL^{\mathrm{T}}$$

It should be noted that the introduction of the vector $\boldsymbol{w}_m$ and matrix $\mathcal{W}$ greatly simplifies the forms of the prediction, or time update, equations of the CD-UKF [22]. Also, we see that unlike the linear case, the right-hand sides of (11) and (12) are defined by functions of σ points $\mathcal{X}_i(t)$, and not by $m(t)$ and $P(t)$. By solving (11) and (12), we have the predicted mean and covariance matrix; see Step 2 in the CD-UKF algorithm below.

The other equations needed for the CD-UKF are the equations that update the predicted estimate and covariance matrix based on the observation y_k to get the filtered estimate $m(t_k|t_k)$ and covariance matrix $P(t_k|t_k)$, which are the same as those of the observation update equations in the discrete-time UKF; see Step 3 in the CD-UKF algorithm.

4.1. **Continuous-discrete UKF algorithm.** In the following, we define the sampling interval for observations as Δ_o, implying that we receive observations at every Δ_o seconds, and the sampling interval for prediction step as Δ_p ($\Delta_p \ll \Delta_o$), which is the basic step for the numerical integration of the prediction equations of (11) and (12). The relation between Δ_p and Δ_o is depicted in Figure 2, where the interval Δ_p is much smaller than the observation interval Δ_o.

A numerical procedure of the CD-UKF algorithm due to Särkkä [22] is described by using two sampling intervals defined in Figure 2.

Step 1: (Initial values) Let the initial values for the filtered estimate and covariance matrix be $m(t_0|t_0)$ and $P(t_0|t_0)$, and put $k = 1$.

Step 2: (Time update) Let $j = 0$.

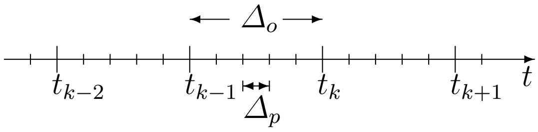

FIGURE 2. Sampling intervals Δ_p and Δ_o

Step 2a: Let $t = t_{k-1} + j\Delta_p$. If $t > t_k$, then go to Step 3 with the predicted estimate $m(t_k|t_{k-1})$ and σ points $\mathcal{X}(t_k|t_{k-1})$. Otherwise, compute

$$\begin{aligned}\mathcal{X}(t|t_{k-1}) &= [\mathcal{X}_0(t|t_{k-1}) \ \cdots \ \mathcal{X}_{2n}(t|t_{k-1})] \\ &= \Big[m(t|t_{k-1}) \quad m(t|t_{k-1}) + \Big(\sqrt{cP(t|t_{k-1})}\Big)_1 \ \cdots \ m(t|t_{k-1}) + \Big(\sqrt{cP(t|t_{k-1})}\Big)_n \\ &\qquad m(t|t_{k-1}) - \Big(\sqrt{cP(t|t_{k-1})}\Big)_1 \ \cdots \ m(t|t_{k-1}) - \Big(\sqrt{cP(t|t_{k-1})}\Big)_n\Big]\end{aligned}$$

where $\sqrt{(\cdot)}$ denotes the matrix square root, $(\,\cdot\,)_i$ is the ith column vector of the matrix square root, and $c = n + \lambda$.

Step 2b: Integrate (11) and (12) to get $m(t + \Delta_p|t_{k-1})$ and $P(t + \Delta_p|t_{k-1}))$ from $m(t|t_{k-1})$ and $P(t|t_{k-1})$, respectively. See, Subsection 5.2 for the detailed numerical algorithm.

Step 2c: Choose the following σ points:

$$\begin{aligned}\mathcal{X}\,(t + \Delta_p|t_{k-1}) &= [\mathcal{X}_0(t + \Delta_p|t_{k-1}) \ \cdots \ \mathcal{X}_{2n}\,(t + \Delta_p|t_{k-1})] \\ &= \Big[m\,(t + \Delta_p|t_{k-1}) \quad m\,(t + \Delta_p|t_{k-1}) + \Big(\sqrt{cP(t + \Delta_p|t_{k-1})}\Big)_1 \\ &\qquad \cdots \ m(t + \Delta_p|t_{k-1}) + \Big(\sqrt{cP(t + \Delta_p|t_{k-1})}\Big)_n \\ &\qquad m(t + \Delta_p|t_{k-1}) - \Big(\sqrt{cP(t + \Delta_p|t_{k-1})}\Big)_1 \\ &\qquad \cdots \ m(t + \Delta_p|t_{k-1}) - \Big(\sqrt{cP(t + \Delta_p|t_{k-1})}\Big)_n\Big]\end{aligned}$$

Step 2d: Let $j := j + 1$, and go to Step 2a.

Step 3: (Measurement update) Given $\mathcal{X}(t_k) \coloneqq \mathcal{X}(t_k|t_{k-1})$, we define $\mathcal{Y}(t_k) = h_d(\mathcal{X}(t_k))$. Then, the Kalman gain is obtained as

$$K(t_k) = \mathcal{X}(t_k)\mathcal{W}\mathcal{Y}^{\mathrm{T}}(t_k)[\mathcal{Y}(t_k)\mathcal{W}\mathcal{Y}^{\mathrm{T}}(t_k) + R]^{-1}$$

so that the filtered estimate and covariance matrix are given by

$$m(t_k|t_k) = m(t_k|t_{k-1}) + K(t_k)\big[y(t_k) - \mathcal{Y}(t_k)\boldsymbol{w}_m\big] \tag{14}$$

$$P(t_k|t_k) = P(t_k|t_{k-1}) - K(t_k)[\mathcal{Y}(t_k)\mathcal{W}\mathcal{Y}^{\mathrm{T}}(t_k) + R]K^{\mathrm{T}}(t_k) \tag{15}$$

Step 4: Let $k := k + 1$, and then go to Step 2.

The filtered and predicted estimates and covariance matrices are recursively computed by using Step 1 $\sim$ Step 4 as above.

5. Prediction Step in Continuous-Discrete UKF.

5.1. **Time update equations by Euler scheme.** We consider the integration of (11) and (12) satisfied by the conditional mean estimate and error covariance matrix. Let the sampling interval be Δ_p as shown in Figure 2. Then, integrating (11) and (12), we have

$$m(t+\Delta_p|t_{k-1}) = m(t|t_{k-1}) + \int_t^{t+\Delta_p} f(\mathcal{X}(\tau),\tau)\boldsymbol{w}_m d\tau \tag{16}$$

$$\begin{aligned} P(t+\Delta_p|t_{k-1}) = P(t|t_{k-1}) + \int_t^{t+\Delta_p} \Big[& f(\mathcal{X}(\tau),\tau)\mathcal{W}\mathcal{X}^{\mathrm{T}}(\tau) \\ & + \mathcal{X}(\tau)\mathcal{W}f^{\mathrm{T}}(\mathcal{X}(\tau),\tau) + LQL^{\mathrm{T}}\Big]d\tau \end{aligned} \tag{17}$$

where $t = t_{k-1} + j\Delta_p$ with j the number of iteration in Step 2 of the CD-UKF algorithm. Approximating the value of integrands of (16) and (17) by the values at the left end point t of integration, we have the following schemes:

$$m(t+\Delta_p|t_{k-1}) = m(t|t_{k-1}) + f(\mathcal{X}(t),t)\boldsymbol{w}_m\Delta_p \tag{18}$$

$$\begin{aligned} P(t+\Delta_p|t_{k-1}) = P(t|t_{k-1}) + \Big[& f(\mathcal{X}(t),t)\mathcal{W}\mathcal{X}^{\mathrm{T}}(t) \\ & + \mathcal{X}(t)\mathcal{W}f^{\mathrm{T}}(\mathcal{X}(t),t) + LQL^{\mathrm{T}}\Big]\Delta_p \end{aligned} \tag{19}$$

These are the Euler approximation-based update equations of the conditional mean and error covariance matrix, which are to be used in Step 2b to obtain σ points $\mathcal{X}(t+\Delta_p|t_{k-1})$ in Step 2c. Numerical results, however, show that the Euler approximation-based method is not very accurate, so that we need to introduce a higher-order approximation in stead of (18) and (19).

5.2. **Time update equations by Heun scheme.** In order to improve the Euler-based approximation of (18) and (19), we shall apply the Heun scheme shown in Section 3 to the computation of (16) and (17), deriving the update equations for the conditional mean and error covariance matrix with a higher-order approximation than the Euler approximation.

Recall that $t = t_{k-1} + j\Delta_p$ with j the number of iteration in Step 2. Let the σ point at t be given by

$$\mathcal{X}(t) = \begin{bmatrix} \mathcal{X}_{1,1} & \cdots & \mathcal{X}_{1,2n+1} \\ \vdots & & \vdots \\ \mathcal{X}_{n,1} & \cdots & \mathcal{X}_{n,2n+1} \end{bmatrix} \in \mathbb{R}^{n\times(2n+1)} \tag{20}$$

where the argument t in the right-hand side is suppressed. By applying the Heun scheme to $f(\mathcal{X},\tau)$ in (16), we define

$$\begin{aligned} c_{fi}^{(s)}(1) &= f_i(\mathcal{X}_{1,s},\cdots,\mathcal{X}_{n,s},t) \\ c_{fi}^{(s)}(2) &= f_i\left(\mathcal{X}_{1,s} + \Delta_p c_{f1}^{(s)}(1),\cdots,\mathcal{X}_{n,s} + \Delta_p c_{fn}^{(s)}(1), t+\Delta_p\right) \\ & \quad (i = 1,\dots,n;\, s = 1,\dots,2n+1) \end{aligned}$$

where $c_{fi}^{(s)}(2)$ are supporting values. Moreover, let

$$f_{i,s}^{+} = \frac{\Delta_p}{2}\left[c_{fi}^{(s)}(1) + c_{fi}^{(s)}(2)\right]$$

and define

$$F^{+} = \begin{bmatrix} f_{1,1}^{+} & \cdots & f_{1,2n+1}^{+} \\ \vdots & & \vdots \\ f_{n,1}^{+} & \cdots & f_{n,2n+1}^{+} \end{bmatrix} \in \mathbb{R}^{n\times(2n+1)} \tag{21}$$

It therefore follows from (16) that a time update equation for the conditional mean estimate is given by

$$m(t+\Delta_p|t_{k-1}) = m(t|t_{k-1}) + F^{+}\boldsymbol{w}_m \tag{22}$$

where the order of approximation of (22) based on the Heun scheme is higher than that of (18).

Now we consider the time update equation of error covariance matrix by applying the Heun scheme to (17), whose integrand includes a matrix product $\varphi(\mathcal{X},\tau) = f(\mathcal{X},\tau)\mathcal{W}\mathcal{X}^{\mathrm{T}}(\tau) \in \mathbb{R}^{n\times n}$ and its transpose. We must first vectorize the matrix $\varphi(\mathcal{X},\tau)$ to apply the Heun scheme. In view of the definition of (13), it is quite difficult, though not impossible, to vectorize it due to its rather complicated form. Therefore, here we replace $f(\mathcal{X},\tau)$ in (17) by F^{+} defined by (21), approximating the other factor $\mathcal{X}(\tau)$ by $\mathcal{X}(t)$ as in the Euler scheme. Hence, by using σ points $\mathcal{X}$ of (20) and F^{+} of (21), we arrive at a time update equation for the error covariance matrix of the form

$$P(t+\Delta_p|t_{k-1}) = P(t|t_{k-1}) + F^{+}\mathcal{W}\mathcal{X}^{\mathrm{T}} + \mathcal{X}\mathcal{W}F^{+\mathrm{T}} + \Delta_p LQL^{\mathrm{T}} \tag{23}$$

where all the elements in the right-hand side are evaluated at time $t = t_{k-1} + j\Delta_p$. We note that (23) is more accurate than (19) as the time update equation of error covariance matrix. It should be however noted that since the factors $f(\mathcal{X},\tau)$ and $\mathcal{X}(\tau)$ are treated separately in the product $\varphi(\mathcal{X},\tau)$, the present scheme is not of the exact Heun scheme.

Thus the procedure of discretization of (11) and (12) is summarized as

$$m(t+\Delta_p|t_{k-1}) = m(t|t_{k-1}) + F^{+}\boldsymbol{w}_m \tag{24}$$

$$P(t+\Delta_p|t_{k-1}) = P(t|t_{k-1}) + F^{+}\mathcal{W}\mathcal{X}^{\mathrm{T}} + \mathcal{X}\mathcal{W}F^{+\mathrm{T}} + \Delta_p LQL^{\mathrm{T}} \tag{25}$$

where $\mathcal{X}$ and F^{+} are given by (20) and (21), respectively.

5.3. **A simple example.** To explain our procedure based on the Heun scheme in detail, we consider a first-order stochastic system

$$\frac{dx_1(t)}{dt} = -ax_1(t) + bu(t) + w(t) \tag{26}$$

$$y_k = x_1(t_k) + v_k \tag{27}$$

where a is the unknown parameter to be estimated, u is a known input, and w and v_k are zero mean white noises with variances q and r, respectively. Define the extended state vector as

$$x(t) = \begin{bmatrix} x_1(t) \\ a \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Then, we have

$$\frac{dx_1(t)}{dt} = -x_2(t)x_1(t) + bu(t) + w(t)$$

$$\frac{dx_2(t)}{dt} = 0$$

where $f_1 = -x_1x_2 + bu$, and $f_2 = 0$. Since $n = 2$, there exist five σ points, i.e., $\mathcal{X} \in \mathbb{R}^{2\times 5}$, and the vector $\boldsymbol{w}_m \in \mathbb{R}^5$ and matrix $\mathcal{W} \in \mathbb{R}^{5\times 5}$ for UT are given by (9) and (10), respectively.

We now consider the discretization of $f_1 = -x_1x_2 + bu$ to obtain F^{+} of (21). It follows from (20) that

$$\mathcal{X}(t) = \begin{bmatrix} \mathcal{X}_{1,1} & \cdots & \mathcal{X}_{1,5} \\ \mathcal{X}_{2,1} & \cdots & \mathcal{X}_{2,5} \end{bmatrix} \in \mathbb{R}^{2\times 5}$$

Thus, for $s = 1, \ldots, 5$, the Heun scheme gives

$$\begin{aligned} c_{f1}^{(s)}(1) &= -\mathcal{X}_{2,s}(t)\mathcal{X}_{1,s}(t) + bu(t) \\ c_{f1}^{(s)}(2) &= -\mathcal{X}_{2,s}(t)\left(\mathcal{X}_{1,s}(t) + \Delta_p c_{f1}^{(s)}(1)\right) + bu(t+\Delta_p) \\ f_{1,s}^{+} &= \frac{\Delta_p}{2}\left[c_{f1}^{(s)}(1) + c_{f1}^{(s)}(2)\right] \\ f_{2,s}^{+} &= 0 \end{aligned}$$

so that we have

$$F^{+} = \begin{bmatrix} f_{1,1}^{+} & \cdots & f_{1,5}^{+} \\ 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{2\times 5}$$

Thus the time update equations are given by (24) and (25) with F^+ obtained above.

We assume that $q = 0.01$, $q_a = 0.0001$ and $r = 0.01$, where q_a is employed for accelerating the parameter estimation of the unknown parameter a. Also, the true values of parameters are $a = 1.0$, $b = 5.0$ (known), the input function is $u(t) = \sin(2\pi f t)$, $f = 1/200$, and the initial state is $x_1(0) = 0.01$. The sampling intervals are $\Delta_p = 0.025$ and $\Delta_o = 0.1$, respectively, and the initial values for the nonlinear filter are $m(0|0) = [0.01,\ 0.01]^{\mathrm{T}}$ and $P(0|0) = 0.01I_2$, where I_2 is the 2×2 unit matrix.

Figures 3 and 4 display the simulation results by the Heun scheme-based CD-UKF with $\lambda = 2$ and by the Euler CD-EKF. We see from Figure 3 that the estimate of unknown parameter a is quite good for both methods. Also, from Figure 4, the absolute state estimation error $|x_1(t_k) - m_1(t_k|t_k)|$ by both methods are nearly the same.

This simulation result shows that for a linear first-order system with a linear observation equation, the estimation results are nearly the same. In the next section, we show simulation results for a nonlinear system and for a linear system with nonlinear observation equations to show the applicability of the proposed Heun scheme-based CD-UKF algorithm.

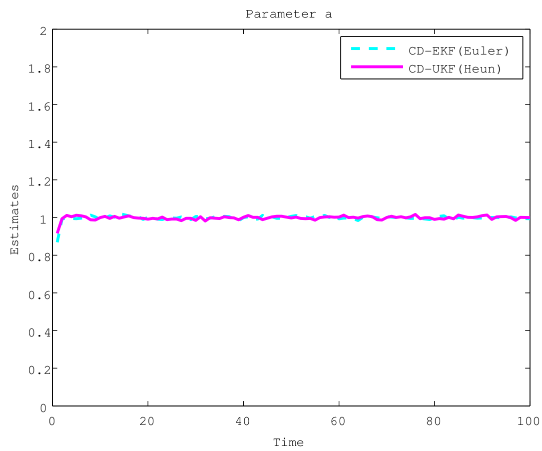

FIGURE 3. Estimation of parameter a

6. **Further Numerical Results.** Two simulation results are shown to illustrate the applicability of the present CD-UKF method based on the Heun scheme, comparing with the performance by the CD-EKF method. We first consider a state and parameter estimation problem of a van der Pol model, and then a two-dimensional tracking problem based on the range and bearing information.

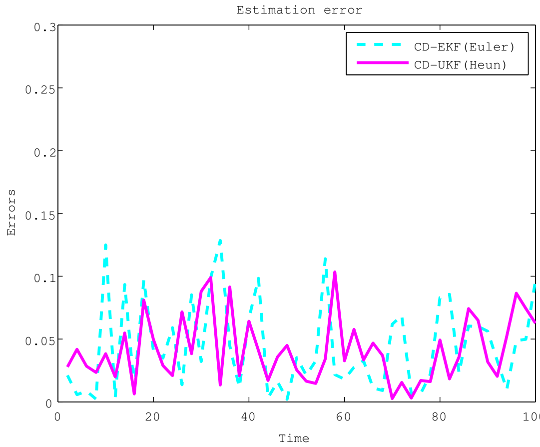

FIGURE 4. State estimation error $|x_1(t_k) - m_1(t_k|t_k)|$

6.1. **Van der Pol model.** Consider the van der Pol model treated in Section 3, i.e.,

$$\frac{dx_1(t)}{dt} = x_2(t) + w_1(t) \tag{28}$$

$$\frac{dx_2(t)}{dt} = \varepsilon\left(1 - x_1^2(t)\right)x_2(t) - x_1(t) + w_2(t) \tag{29}$$

$$y_k = x_1(t_k) + v_k \tag{30}$$

where $w_1(t) \sim N(0, q_1)$, $w_2(t) \sim N(0, q_2)$ and $v_k \sim N(0, r)$ are Gaussian white noises. We assume that the time interval for simulation is $T = 100$, the true value of the parameter is $\varepsilon = 1.0$, and that $q_1 = 0.001$, $q_2 = 0.001$, $q_\varepsilon = 0.00001$ (the acceleration parameter for ε), $r = 0.01$. Also, the initial states are $x_1(0) = 0.1$ and $x_2(0) = 0.1$. Let the sampling intervals be $\varDelta_p = 0.025$ and $\varDelta_o = 0.5$, so that we have 20 times updates between output observations. The initial values are $m(0|0) = [0.01,\ 0.01,\ 0.01]^{\mathrm{T}}$ and $P(0|0) = \mathrm{diag}(0.1,\ 0.1,\ 0.1)$. Also, the state estimation error is defined by

$$E_k = \sqrt{(x_1(t_k) - m_1(t_k|t_k))^2 + (x_2(t_k) - m_2(t_k|t_k))^2} \tag{31}$$

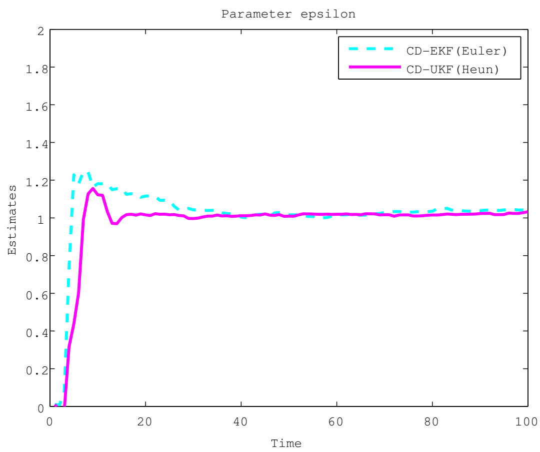

FIGURE 5. Estimation of parameter ε

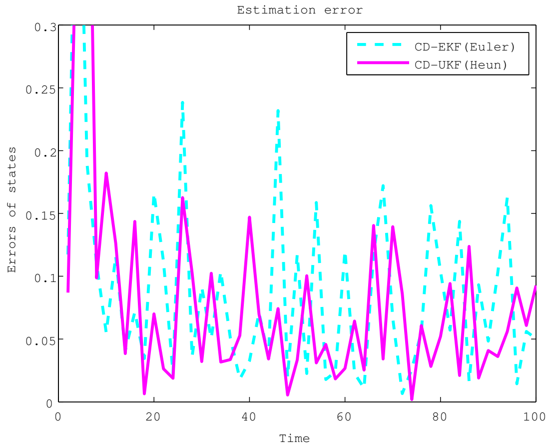

FIGURE 6. State estimation error E_k

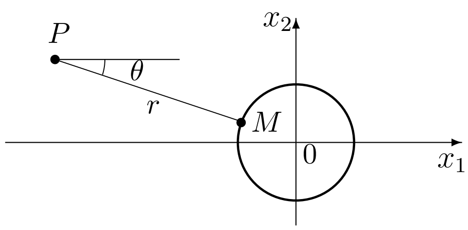

FIGURE 7. A two-dimensional tracking model

Figure 5 shows the estimation results of unknown parameter ε by the present method with $\lambda = 2$, together with the Euler-based CD-EKF method [9]. We see that the performance of the parameter estimation by the present method is better than the CD-EKF method; also similar results are obtained even if the sampling intervals Δ_p and Δ_o are slightly changed. We see from Figure 6 that the state estimation error by the present method is smaller than that by the Euler-based CD-EKF method.

6.2. **A two-dimensional motion model.** Consider an object M moving along a circle centered at the origin as shown in Figure 7. Let (x_1, x_2) and $(\dot{x}_1, \dot{x}_2)$ be the position and the velocity of M, respectively. Then, the equations of motion are described by

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ w_1(t) \end{bmatrix} \tag{32}$$

and

$$\frac{d}{dt}\begin{bmatrix} x_2(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_2(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ w_2(t) \end{bmatrix} \tag{33}$$

where $w_1(t) \sim N(0, q_1)$, $w_2(t) \sim N(0, q_2)$ are Gaussian white noises.

Moreover, as shown in Figure 7, the observation point $P = (\alpha_1, \alpha_2)$ is located outside the circle, and we observe the distance $r = y_1$ from P to the target M, and the angle $\theta = y_2$ measured from the horizontal line along x_1 axis. Then, the observation equations are given by

$$y_1(t_k) = \sqrt{(x_1(t_k) - \alpha_1)^2 + (x_2(t_k) - \alpha_2)^2} + v_1(t_k) \tag{34}$$

$$y_2(t_k) = \tan^{-1}\left(\frac{x_2(t_k) - \alpha_1}{x_1(t_k) - \alpha_1}\right) + v_2(t_k), \quad k = 0, 1, \dots \tag{35}$$

We consider the estimation of the state vector and the parameters α_1, α_2, so that the extended state vector becomes $x = [x_1, \dot{x}_1, x_2, \dot{x}_2, \alpha_1, \alpha_2]^{\mathrm{T}} \in \mathbb{R}^6$. We assume that the true values of parameters are $\alpha_1 = 5$, $\alpha_2 = 3$, and that $Q = \mathrm{diag}(0, 10^{-3}, 0, 10^{-3}, 10^{-4}, 10^{-4})$, $R = \mathrm{diag}(0.05, 0.05)$. Also, the true initial state is $x(0) = [1, 1, 1, -1, 5, 3]^{\mathrm{T}} \in \mathbb{R}^6$.

Let the time interval for simulation be $T = 100$, and let the observation interval be $\Delta_o = 0.2$, and the prediction interval $\Delta_p = 0.01$, so that we have 20 times updates between observations. The initial estimate and covariance matrix are given by $m(0|0) = [10^{-2}, 10^{-2}, 10^{-2}, 10^{-2}, 10^{-1}, 10^{-1}]^{\mathrm{T}}$ and $P(0|0) = 10^{-1} I_6$, and the UT parameter is $\lambda = 1$. Also, the estimation error for the position of M is defined by (31).

Simulation results are depicted in Figures 8 and 9, which clearly show that the present Heun scheme-based CD-UKF yields better performance for the parameter estimation of (α_1, α_2) as well as for the state estimation of (x_1, x_2) than the Euler scheme-based CD-EKF.

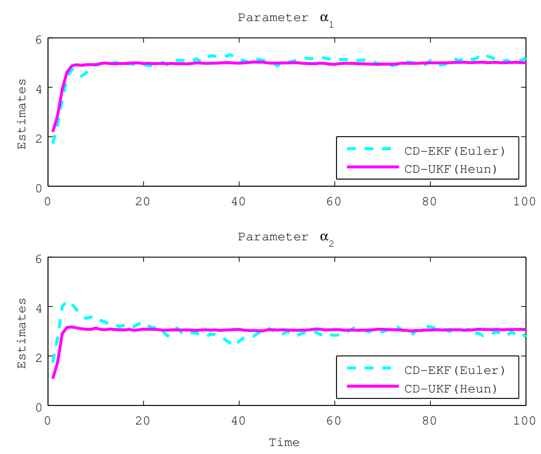

FIGURE 8. Estimation of parameters α_1 and α_2 by CD-UKF (solid line) and CD-EKF (dashed line)

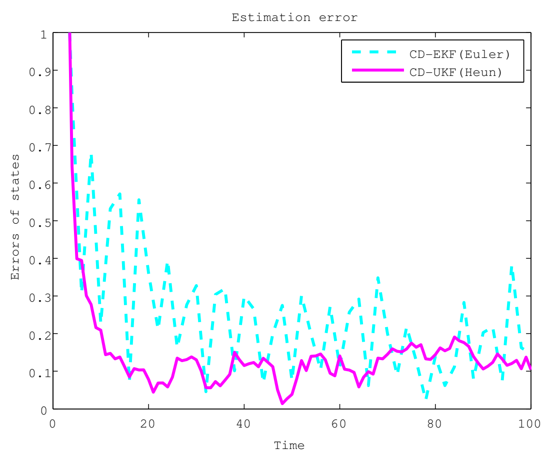

FIGURE 9. State estimation error E_k

7. **Conclusions.** This paper has developed a numerical integration method for the time update equations in the CD-UKF method due to Särkkä [22]. Motivated by the fact that the Heun scheme is superior to the Euler-Maruyama scheme for numerical solutions of stochastic differential equations, we have employed the idea of Heun scheme to integrate the time update equations satisfied by the conditional mean and covariance matrix. The present algorithm is explained in detail by using a state and parameter estimation problem for a first-order stochastic system. Numerical results of state and parameter estimation problems for a van der Pol model and a two-dimensional tracking problem based a nonlinear observation are included to show the applicability of the present Heun scheme-based UKF algorithm, producing better numerical results than Euler-based CD-EKF method [9]. It should be noted that the present Heun scheme-based method can be applied to any nonlinear systems, giving superiority to the existing CD-EKF method [9].

REFERENCES

[1] B. D. O. Anderson and J. B. Moore, *Optimal Filtering*, Prentice-Hall, 1979.

[2] S. I. Aihara, A. Bagchi and S. Saha, On parameter estimation of stochastic volatility models from stock data using particle filter – Application to AEX index –, *International Journal Innovative Computing, Information and Control*, vol.5, no.1, pp.17-27, 2009.

[3] M. Basin, J. Perez and D. Carderon-Alvarez, Optimal filtering for linear system over polynomial observations, *International Journal Innovative Computing, Information and Control*, vol.4, no.2, pp.313-320, 2008.

[4] F. E. Daum, Nonlinear filters: Beyond the Kalman filter, *IEEE Trans. Aerospace & Electronics Magazine*, vol.20, no.8, pp.57-69, 2005.

[5] A. Doucet, N. de Freitas and N. Gordon, *Sequential Monte Carlo Methods in Practice*, Springer, 2001.

[6] G. Evensen, The ensemble Kalman filter for combined state and parameter estimation – Monte Carlo techniques for data assimilation in large systems, *IEEE Control Systems Magazine*, vol.29, no.3, pp.83-104, 2009.

[7] A. Gelb, *Applied Optimal Estimation*, MIT Press, 1974.

[8] M. S. Grewal and A. P. Andrews, *Kalman Filtering: Theory and Practice Using Matlab*, 3rd Edition, John Wiley, 2008.

[9] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, Academic, 1970.

[10] S. Julier and J. Uhlmann, Unscented filtering and nonlinear estimation, *Proc. of IEEE*, vol.92, no.3, pp.401-421, 2004.

[11] S. J. Julier, J. K. Uhlmann and H. F. Durrant-Whyte, A new method for the nonlinear transformation of means and covariances in filters and estimators, *IEEE Trans. Automat. Control*, vol.45, no.3, pp.477-482, 2000.

[12] G. Kitagawa, Monte carlo filter and smoother for non-Gaussian, non-linear state space models, *J. Computational and Graphical Statistics*, vol.5, pp.1-25, 1996.

[13] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, 3rd Printing, Springer, 1999.

[14] Y. Kubo, S. Fujioka, M. Nishiyama and S. Sugimoto, Nonlinear filtering methods for the INS/GPS in-motion alignment and navigation, *International Journal Innovative Computing, Information and Control*, vol.2, no.5, pp.1137-1151, 2006.

[15] H. J. Kushner, On the differential equations satisfied by conditional probability densities of Markov processes, *SIAM J. Control*, vol.2, no.1, pp.106-119, 1964.

[16] T. Mitsui, T. Koto and Y. Saito, *Introduction to Computational Science*, Kyoritsu Shuppan, 2004 (in Japanese).

[17] L. Murray and A. Storkey, Continuous time particle filtering for fMRI, *Advances in Neural Information Processing Systems*, vol.19, pp.1049-1056, 2008.

[18] B. Ng, A. Pfeffer and R. Dearden, Continuous time particle filtering, *Proc. of Int. Joint Conf. on Artificial Intelligence (PIJCAI)*, pp.1360-1365, 2005.

[19] B. Ristic, S. Arulampalam and N. Gordon, *Beyond the Kalman Filter – Particle Filters for Tracking Applications*, Artech House, 2004.

[20] Y. Saito and T. Mitsui, Discrete approximations for stochastic differential equations, *RIMS Technical Report*, Kyoto University, vol.746, pp.251-260, 1991.

[21] Y. Saito and M. Nagaoka, On the numerical solutions of the Langevin-type stochastic differential equation, *RIMS Technical Report*, Kyoto University, vol.1127, pp.143-152, 2000 (in Japanese).

[22] S. Särkkä, On unscented Kalman filtering for state estimation of continuous-time nonlinear systems, *IEEE Trans. Automat. Control*, vol.52, no.9, pp.1631-1641, 2007.

[23] S. E. Shreve, *Stochastic Calculus for Finance II – Continuous-Time Models*, Springer, 2004.

[24] H. Zhang, M. Basin and M. Skliar, Optimal state estimation for continuous stochastic state-space system with hybrid measurements, *International Journal Innovative Computing, Information and Control*, vol.2, no.2, pp.357-370, 2006.

[25] H. Zhang, M. V. Basin and M. Skliar, Itô-Volterra optimal state estimation with continuous, multirate, randomly sampled, and delayed measurements, *IEEE Trans. Automat. Control*, vol.52, no.3, pp.401-416, 2007.