International Journal of Innovative
Computing, Information and Control
Volume 8, Number 3(B), March 2012

ICIC International ©2012 ISSN 1349-4198
pp. 2313–2328

OPTIMIZATION OF THE OBSERVATIONS FOR STATIONARY LQG STOCHASTIC CONTROL SYSTEMS UNDER A QUADRATIC CRITERION

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Received February 2011; revised July 2011

Abstract. *In this paper, we study an optimization problem for the observations of the stationary LQG stochastic control systems that employ the stationary Kalman filter. The performance criterion for the selection of the gain matrix is assumed to be the sum of the LQG regulator performance value, which is a linear function of the estimation error covariance matrix, and a quadratic function of the observation gain matrix. By introducing the eigenvalues-eigenvectors representation of a nonnegative definite symmetric matrix that is a function of the gain matrix, we reformulate the problem as that of optimization with respect to a pair of orthogonal and diagonal matrices. The condition of optimality for this problem is derived under a weaker assumption than already known. For easy numerical calculations, we represent the orthogonal matrix in a multi-dimensional polar coordinates system with angular parameters. Moreover, we apply the connection rule of the angular parameters that was previously shown by the author. Using this rule, we can always find a point in the domain of the angular parameters, with the same value of the performance criterion, from any point outside the domain. A numerical example is provided for better understanding of the results of this paper.*

Keywords: Gaussian processes, Kalman filter, LQG regulator, optimization of observation

1. **Introduction.** In this paper, we consider an optimization problem for the observations of the stationary LQG optimal control systems. The performance index of the optimal control, which is quadratic in the state and input, becomes a linear function of the error covariance matrix of the state estimation when we apply the optimal control to the system [12]. Therefore, the performance of the control system is strongly dependent on the gain matrix in the linear observation. To improve the performance, it is clearly better to make the dimension and the value of this gain matrix as large as possible. However, generally these values are limited by certain physical or economical restrictions. For this reason, we usually use a criterion, for the optimization of the gain matrix, that is the sum of the performance index of the LQG control system and a quadratic function of the gain matrix. In fact, most of the previous studies on the optimization of the observations employed the criteria that are quadratic in both the estimation error and the gain matrix [1]-[4].

The estimation error covariance matrix, as it is well-known, is given by a solution of the matrix Riccati equation and is nonlinearly dependent on the observation gain matrix. Therefore, it is very difficult to obtain the solution of this problem when we select the gain matrix as the variable to be optimized. Namely, we cannot obtain the analytical solution or the condition of optimality, that are easily applicable in numerical computations.

In view of this situation, we previously employed a new approach for optimizing the observations [12]. In that approach, we removed the quadratic term of the gain matrix in the criterion and imposed the information theoretic criterion in advance [12]. Namely, in [12], we proposed an optimization method by the following two steps:

(i) Information theoretic optimization in order to maximize the mutual information between the state process and the observations subject to a power constraint concerned with the innovations process;
(ii) Optimization of the performance of the LQG control system, i.e., minimization of the performance index.

We derived the condition of optimality and also constructed a numerical algorithm to easily obtain the observation gain matrix. In particular, we obtained a simple optimization scheme without restrictions by introducing a representation of an orthogonal matrix through a multi-dimensional polar coordinates system and by proving the connection rule of the angular parameters [12]. However, it transpired that, by the optimization described by (i), the matrix Riccati equation reduces to a quasi-Lyapunov-type quadratic equation for which we cannot always obtain a solution [12].

Thus, in this paper, we reconsider the optimization of the observation of the LQG optimal control systems under the quadratic criterion for both the estimation error and the gain matrix. Although, the approach is very different from the previous works, but it is similar to that of [12]. Namely, using the following procedures, we construct and propose an easily calculable numerical algorithm of the optimization of the observation:

(a) The problem is converted to one with a symmetric-matrix-valued variable that is a function of the gain matrix and from which we can determine the gain matrix;
(b) By introducing the eigenvalues-eigenvectors representation of the symmetric matrix, we convert the problem to the one with a pair of orthogonal and diagonal matrices as the set of variables;
(c) To remove the constraints of orthogonality and normality, the orthogonal matrix is represented by a multi-dimensional polar coordinates system with angular parameters (change of variables);
(d) We apply the connection rule of the angular parameters [12] by which we can always find a point, with the same value of the symmetric matrix, in the domain from any point outside the domain.

There is not much difference between the mathematical features of the problem discussed in this paper and the one in Takeuchi [13] which considers the optimization of the observation for the stationary Kalman filter. However, a new proof of the condition of optimality is provided in this paper under a weaker assumption than the one in [13]. A numerical example is provided for better understanding the results of this paper.

Mathematical symbols, in this paper, are used in the following way. $\mathbf{R}$ is the space of all real numbers, i.e., $\mathbf{R} \triangleq (-\infty,\ \infty)$. For positive integers m and n, $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$ denote the spaces of n-dimensional vectors and $m \times n$-dimensional matrices whose components take values in $\mathbf{R}$. The prime denotes the transpose of a vector or a matrix and the Euclidean norm is $|\cdot|$. Thus, for $x \in \mathbf{R}^n$, $|x| = \sqrt{x'x}$. The identity matrix of any dimension is denoted by I. The components of a matrix are denoted by using subscripts. Thus, $[A]_{ij}$ is the (i,j)-component of A. In the case where no confusion may arise, we denote $[A]_{ij}$ simply by a_{ij}. If A is a square matrix, $\det|A|$ and $\mathrm{tr}[A]$ respectively denote the determinant and the trace of A. We use $A > 0$ and $A \geq 0$ to denote that A is positive definite and nonnegative definite, respectively. For any pair of matrices A and B, $A \otimes B$ denotes the Kronecker product of A and B, and $vec(A)$ is the vector formed by stacking the columns of A into a single column vector. The triplet $(\Omega,\ \mathcal{F},\ P)$ is a complete probability space, where Ω is a sample space with elementary events ω. $\mathcal{F}$ is a σ-field of subsets of Ω, and P is a probability measure. $E\{\ \cdot\ \}$ denotes the expectation and $E\{\ \cdot\ |\mathcal{G}\}$, $\mathcal{G} \subset \mathcal{F}$ the conditional expectation, given $\mathcal{G}$, with respect to P. $\sigma\{\ \cdot\ \}$ is the minimal sub-σ-field of $\mathcal{F}$ with respect to which the family of $\mathcal{F}$-measurable sets or random variables $\{\ \cdot\ \}$ is measurable.

2. **Problem Formulation.**

2.1. **Stationary optimal LQG regulator system.** Let $\boldsymbol{x} \equiv \{x_t(\omega);\ t = 0, 1, \cdots\}$ denote the state process of a control system which is an n-dimensional Gaussian stochastic process described by

$$\begin{cases} x_{t+1}(\omega) = Ax_t(\omega) + Cu(t) + Gw_t(\omega), \\ x_0(\omega) \quad\ = x^0(\omega), \end{cases} \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $C \in \mathbf{R}^{n\times r}$, $G \in \mathbf{R}^{n\times d_1}$, $x^0(\omega)$ is a Gaussian random vector with mean $\hat{x}^0$ and covariance Q^0, $\boldsymbol{u} \equiv \{u(t);\ t = 0, 1, \cdots\}$ is a r-dimensional control input, and $\boldsymbol{w} \equiv \{w_t(\omega);\ t = 0, 1, \cdots\}$ is a d_1-dimensional standard white Gaussian noise sequence. Suppose that the value of $\boldsymbol{x}$ is not available but we have m-dimensional observations described by

$$y_t(\omega) = Hx_t(\omega) + Rv_t(\omega), \tag{2}$$

where $\boldsymbol{y} \equiv \{y_t(\omega);\ t = 1, 2, \cdots\}$ is an m-dimensional observation process, $H \in \mathbf{R}^{m\times n}$, $R \in \mathbf{R}^{m\times d_2}$, and $\boldsymbol{v} \equiv \{v_t(\omega);\ t = 1, 2, \cdots\}$ is a d_2-dimensional standard white Gaussian noise sequence. We will assume the following two conditions throughout this paper.

$(C-1)$ $R_0 \triangleq RR' > 0$,
$(C-2)$ $x^0(\omega)$, $\boldsymbol{w}$ and $\boldsymbol{v}$ are mutually independent.

It is well-known that the least-squares estimate $\hat{x}_{t|t}(\omega) \triangleq E\{x_t(\omega)\,|\mathcal{Y}_t\}$ of $x_t(\omega)$ based on $\mathcal{Y}_t \triangleq \sigma\{y_s(\omega);\ s = 1,\ 2, \cdots,\ t\}$ is given by the Kalman filter:

$$\begin{cases} \hat{x}_{t|\,t-1}(\omega) = A\hat{x}_{t-1|\,t-1}(\omega) + Cu(t-1) \\ \hat{x}_{t|\,t}(\omega) \quad = \hat{x}_{t|\,t-1}(\omega) + Q^-H'\{HQ^-H' + R_0\}^{-1}\tilde{y}_t(\omega), \end{cases} \tag{3}$$

$$\begin{cases} Q^- = AQA' + GG' \\ Q \quad = Q^- - Q^-H'\{HQ^-H' + R_0\}^{-1}HQ^-, \end{cases} \tag{4}$$

where

$$\hat{x}_{t|\,t-1} \triangleq E\{x_t(\omega)\,|\mathcal{Y}_{t-1}\}\ , \tag{5}$$

$$Q^- \triangleq E\{[x_t(\omega) - \hat{x}_{t|\,t-1}(\omega)][x_t(\omega) - \hat{x}_{t|\,t-1}(\omega)]'\}, \tag{6}$$

$$Q \triangleq E\{[x_t(\omega) - \hat{x}_{t|\,t}(\omega)][x_t(\omega) - \hat{x}_{t|\,t}(\omega)]'\}. \tag{7}$$

Also, $\tilde{\boldsymbol{y}} \equiv \{\tilde{y}_t(\omega);\ t = 1,\ 2,\ \cdots\}$ in (3) is the innovations process:

$$\begin{aligned} \tilde{y}_t(\omega) &\triangleq y_t(\omega) - H\hat{x}_{t|\,t-1}(\omega) \\ &= H\{x_t(\omega) - \hat{x}_{t|\,t-1}(\omega)\} + Rv_t(\omega). \end{aligned} \tag{8}$$

The stationary optimal control input $\boldsymbol{u} \equiv \{u(t);\ t = 0, 1, \cdots\}$ is determined based on the well-known solution of the LQG regulator problem with the performance criterion:

$$\bar{J} \triangleq \lim_{T\to\infty} \frac{1}{T} E\left\{\sum_{t=1}^{T} [x_t'(\omega)Mx_t(\omega) + u'(t-1)Nu(t-1)]\right\}, \tag{9}$$

where $M \in \mathbf{R}^{n\times n}$ and $N \in \mathbf{R}^{r\times r}$ are non-negative definite and positive definite symmetric matrices, respectively. As it is well-known, for (1), (2) and (9), the optimal control is given by [12]

$$u^*(t) \triangleq -\{C'WC + N\}^{-1}C'WA\hat{x}_{t|t}(\omega), \tag{10}$$

where $W \in \mathbf{R}^{n\times n}$ is given by the positive definite solution of the matrix Riccati equation:

$$\begin{cases} W = A'YA + M \\ Y \ = W - WC\{C'WC + N\}^{-1}C'W. \end{cases} \tag{11}$$

From (9)-(11), the minimal value of $\bar{J}$ is given by [12]

$$\bar{J}^* = \mathrm{tr}[A'(W - Y)AQ] + \mathrm{tr}[W GG']. \tag{12}$$

2.2. **Optimization of the observations.** As we see from (4) and (12), the performance of the LQG regulator is strongly dependent on the observation gain matrix that is denoted by H. Since a larger H is necessary in order to decrease Q, and since the second term in $\bar{J}^*$ is independent of H, it may be reasonable to select the performance criterion of $H \in \mathbf{R}^{m\times n}$ as the quadratic form:

$$\hat{J} \triangleq \mathrm{tr}[A'(W - Y)AQ] + \mathrm{tr}[H\tilde{N}H'], \tag{13}$$

where $\tilde{N} \in \mathbf{R}^{n\times n}$ is a positive definite symmetric matrix. Here, the second term denotes, for example, the cost or the energy consumed by the observation. Now, we are concerned with

[Problem 1] Find $H \in \mathbf{R}^{m\times n}$ such that (13) is minimized subject to (4).

The formulation of Problem 1, which is based on (13), as an optimization problem with respect to the observation gain matrix H is well-known, and there have been many studies on this problem [1]-[4]. The main difference between the approaches of the present and the previous studies is that in this study, we do not obtain the condition of optimality for H itself, but for the eigenvectors and eigenvalues of the nonnegative definite symmetric matrix: $\tilde{N}^{1/2}H'R_0^{-1}H\tilde{N}^{1/2}$. In this approach, Theorem 2.1 below plays an important role.

Since $\tilde{N}^{1/2}H'R_0^{-1}H\tilde{N}^{1/2}$ is a symmetric matrix, we have

$$\tilde{N}^{1/2}H'R_0^{-1}H\tilde{N}^{1/2} = \tilde{U}\tilde{\Xi}\tilde{U}', \tag{14}$$

where

$$\tilde{\Xi} \triangleq \mathrm{diag}(\xi_1,\ \xi_2,\ \cdots,\ \xi_{\tilde{m}}), \quad \xi_i > 0,\ i = 1,\ 2,\ \cdots,\ \tilde{m}, \tag{15}$$

$$\tilde{m} \triangleq \mathrm{rank}\,[H]\ \ (\leq m)\,, \tag{16}$$

and $\tilde{U} = [u_1\ u_2\ \cdots\ u_{\tilde{m}}]$ is the set of eigenvectors of the symmetric matrix $\tilde{N}^{1/2}H'R_0^{-1}H\tilde{N}^{1/2}$ in (14) corresponding to the positive eigenvalues $\xi_i,\ i = 1,\ 2,\ \cdots,\ \tilde{m}$. Note that we have $\tilde{U}'\tilde{U} = I$. Without loss of generality, we can assume that

$$\xi_1 \geq \xi_2 \geq\ \cdots\ \geq \xi_{\tilde{m}} > 0. \tag{17}$$

Note that since $\tilde{N} > 0$, (14) implies

$$H'R_0^{-1}H = \tilde{N}^{-1/2}\tilde{U}\tilde{\Xi}\tilde{U}'\tilde{N}^{-1/2}. \tag{18}$$

Then, we have the following theorem which guarantees that we can take H in the form:

$$H = R_0^{1/2}\tilde{\Gamma}\tilde{\Xi}^{1/2}\tilde{U}'\tilde{N}^{-1/2}, \tag{19}$$

where $\tilde{\Gamma} \in \mathbf{R}^{m\times\tilde{m}}$ denotes the first $\tilde{m}$ columns of an orthogonal matrix $\Gamma \in \mathbf{R}^{m\times m}$ such that $\Gamma\Gamma' = \Gamma'\Gamma = I$.

Theorem 2.1. ([13], [14]). *Assume (C-1) and (C-2). Then, any* $H \in \mathbf{R}^{m\times n}$ *which satisfies (14) for a fixed set of values* $\tilde{U}$, $\tilde{\Xi}$ *and* $\tilde{N}$ *yields the same value of* Q.

Proof: The assertion immediately follows since the second relation of (4) can be rewritten as

$$Q^{-1} = \left(Q^{-}\right)^{-1} + H' R_0^{-1} H. \tag{20}$$

Thus, without loss of generality, we can take H in the form given by (19) that is an expression of H with property (14) and/or (18). Thus, the problem has been converted to the optimization with respect to $\tilde{\Gamma} \in \mathbf{R}^{m\times\tilde{m}}$, $\tilde{U} \in \mathbf{R}^{n\times\tilde{m}}$ and $\tilde{\Xi} \triangleq \mathrm{diag}(\xi_1,\ \xi_2,\ \cdots,\ \xi_{\tilde{m}})$.

3. **The condition of Optimality.** As we see from (18), (19) and (20), Q is independent of $\tilde{\Gamma} \in \mathbf{R}^{m\times\tilde{m}}$ and is determined by (18), (20) and the first part of (4). Hence, the optimal value of $\tilde{\Gamma}$ should be determined in such a way that $\mathrm{tr}\left[H\tilde{N}H'\right]$, the second term in (13), is minimized.

Theorem 3.1. *Assume (C-1) and (C-2). Then the optimal value of* $\tilde{\Gamma} \in \mathbf{R}^{m\times\tilde{m}}$ *is given by the set of eigenvectors of* R_0 *corresponding to the first* $\tilde{m}$ *eigenvalues in ascending order, i.e.,*

$$\begin{gathered} R_0 = \left[\tilde{\Gamma}\ \ \bar{\Gamma}\right] \Psi \begin{bmatrix} \tilde{\Gamma}' \\ \bar{\Gamma}' \end{bmatrix}, \quad \Psi = \mathrm{diag}(\psi_1,\ \psi_2,\ \cdots,\ \psi_{\tilde{m}},\ \cdots,\ \psi_m), \\ \psi_1 \le \psi_2 \le \cdots \le \psi_{\tilde{m}} \le \cdots \le \psi_m. \end{gathered} \tag{21}$$

Remark 3.1. *If we take* $\tilde{\Gamma} \in \mathbf{R}^{m\times\tilde{m}}$ *according to Theorem 3.1, we have*

$$H = \tilde{\Gamma}\,\tilde{\Psi}^{1/2}\tilde{\Xi}^{1/2}\tilde{U}'\,\tilde{N}^{-1/2}, \tag{22}$$

where

$$\tilde{\Psi} \triangleq \mathrm{diag}(\psi_1,\ \psi_2,\ \cdots,\ \psi_{\tilde{m}}). \tag{23}$$

The form of H *that is given by (22) is very similar to the result of the optimization by an information theoretic criterion [12; Theorem 3.2] which has the following form.*

$$H = \tilde{\Gamma}\tilde{\Xi}^{1/2}\tilde{U}'(Q^{-})^{-1/2}$$

In both cases, $\tilde{\Gamma}$ *is the set of eigenvectors of* R_0. *The main differences between these two expressions of* H *are:*

(i) $\tilde{N}^{-1/2}$ *is a constant matrix in the present problem whereas it is replaced by* $(Q^{-})^{-1/2}$ *which is a variable to be determined in [12].*

(ii) $\tilde{\Xi}$ *is, here, determined to achieve the minimal value of* $\hat{J}$ *given by (13), whereas it is determined by a Generalized Water Filling Theorem in [12], i.e.,* $\tilde{\Xi} = \alpha I - \tilde{\Psi}$ *for some positive constant* α.

Thus, we have converted Problem 1 to the following form.

[Problem 2] Find $\tilde{U} \in \mathbf{R}^{n\times\tilde{m}}$ and $\tilde{\Xi} = \mathrm{diag}(\xi_1,\ \xi_2,\ \cdots,\xi_{\tilde{m}})$ such that

$$\hat{J} = \mathrm{tr}[A'(W-Y)AQ] + \mathrm{tr}\left[\tilde{\Psi}\tilde{\Xi}\right] \to \min., \tag{24}$$

subject to (4), (17) , (22) and

$$\tilde{U}'\tilde{U} = I. \tag{25}$$

For Problem 2, let us define the Lagrangean by

$$L(\tilde{\Xi},\ \tilde{U},\ \tilde{\Lambda}) \triangleq \mathrm{tr}[A'(W-Y)AQ] + \mathrm{tr}\left[\tilde{\Psi}\tilde{\Xi}\right] + \mathrm{tr}\left[\tilde{\Lambda}(\tilde{U}'\tilde{U}-I)\right], \tag{26}$$

where $\tilde{\Lambda} \in \mathbf{R}^{\tilde{m}\times\tilde{m}}$ is a symmetric matrix whose (i,j)-component is a Lagrange multiplier for the same component of (25), i.e.,

$$\begin{aligned}\mathrm{tr}\left[\tilde{\Lambda}(\tilde{U}'\tilde{U}-I)\right] &= \sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{m}} \lambda_{ij}\left[(\tilde{U}'\tilde{U}-I)\right]_{ji} \\ &= \sum_{i=1}^{\tilde{m}}\sum_{j=1}^{\tilde{m}} \lambda_{ij}\left[(\tilde{U}'\tilde{U}-I)\right]_{ij}.\end{aligned} \tag{27}$$

For Problem 2 and (26), we have the following result.

Theorem 3.2. (Condition of Optimality). *Assume (C-1), (C-2) and*

$(C-3)$ For $F \triangleq Q(Q^-)^{-1}A$, the set:

$$\mathcal{H} \triangleq \left\{(\tilde{U},\ \tilde{\Xi});\ \det|F\otimes F - I| \neq 0\ \right\},$$

is not empty.

Then, the condition of optimality of $\tilde{U}$ and $\tilde{\Xi}$ is given by

$$\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}\tilde{U} = \tilde{U}\tilde{\Psi}, \tag{28}$$

where $X \in \mathbf{R}^{n\times n}$ is a solution of

$$X = F'XF + A'(W-Y)A. \tag{29}$$

Since $\tilde{\Psi}$ is a diagonal matrix given by (23), (28) implies

Corollary 3.1. *Assume (C-1)-(C-3). The optimal $(\tilde{U},\ \tilde{\Xi})$ is such that*

(i) *Each column vector of $\tilde{U}$ is an eigenvector of $\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}$.*
(ii) *The order of the column vectors in $\tilde{U}$ is such that the corresponding eigenvalues are in ascending order.*
(iii) *The $\tilde{m}$ eigenvalues of $\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}$ corresponding to $\tilde{U}$ coincide with the first $\tilde{m}$ eigenvalues of R_0.*

Remark 3.2. *Clearly, the condition* $\det|F\otimes F - I| \neq 0$ *is equivalent to the one that no eigenvalue of $F\otimes F$ is equal to 1. Let $\mu_i,\ i=1,\ 2,\cdots,\ n$ denote the eigenvalues of $F \triangleq Q(Q^-)^{-1}A$. Then, the condition holds if and only if the following two conditions are fully satisfied.*

(i) $\mu_i \neq 1,\ \ i=1,\ 2,\cdots,\ n.$
(ii) $\mu_i\mu_j \neq 1,\ \ i<j,\ \ i,j=1,\ 2,\cdots,\ n.$

Remark 3.3. *As we see, the set of optimal values $(Q,\ Q^-,\ \tilde{U},\ \tilde{\Xi})$ is given by a solution of the set of equations (4), (22), (25), (28) and (29). Although it is not easy to obtain an analytical solution of these equations, these relations together with the properties given in Corollary 3.1 are applicable in constructing recursive numerical algorithms.*

4. **Proofs of Theorems.** In this section, we will give proofs of the theorems presented in the previous section.

(Proof of Theorem 3.1) From (19), it is seen that

$$\begin{aligned} \mathrm{tr}\left[H\tilde{N}H'\right] &= \mathrm{tr}\left[R_0^{1/2}\tilde{\Gamma}\tilde{\Xi}^{1/2}\tilde{U}'\tilde{N}^{-1/2}\cdot\tilde{N}\cdot\tilde{N}^{-1/2}\tilde{U}\tilde{\Xi}^{1/2}\tilde{\Gamma}'R_0^{1/2}\right] \\ &= \mathrm{tr}\left[R_0^{1/2}\tilde{\Gamma}\tilde{\Xi}\tilde{\Gamma}'R_0^{1/2}\right] = \mathrm{tr}\left[\tilde{\Gamma}'R_0\tilde{\Gamma}\tilde{\Xi}\right], \end{aligned} \tag{30}$$

where we used (25) and the relation $\mathrm{tr}\,[XY] = \mathrm{tr}\,[YX]$. From (17) and the last expression in (30), we see that $\mathrm{tr}\left[H\tilde{N}H'\right]$ is minimized when the diagonal components of $\tilde{\Gamma}'R_0\tilde{\Gamma}$ are minimized and are in ascending order. Since $R_0 > 0$, we can conclude that $\tilde{\Gamma}$ is optimal when the diagonal components of $\tilde{\Gamma}'R_0\tilde{\Gamma}$ are the first $\tilde{m}$ eigenvalues of R_0 which satisfy the relations in (21). Hence, the column vectors of $\tilde{\Gamma}$ are the corresponding eigenvectors. This completes the proof.

For the proof of Theorem 3.2, we need the following lemma.

Lemma 4.1. *Assume (C-1)-(C-3). Then, the solutions of the matrix Lyapunov equations:*

$$X = F'XF + Z, \tag{31}$$

and

$$\hat{X} = F\hat{X}F' + Z, \tag{32}$$

are respectively given by

$$vec(X) = (I - F' \otimes F')^{-1}vec(Z), \tag{33}$$

and

$$vec(\hat{X}) = (I - F \otimes F)^{-1}vec(Z). \tag{34}$$

(Proof) Noting the well-known relation $vec(XYZ) = (Z' \otimes X)vec(Y)$, (31) implies

$$(I - F' \otimes F')vec(X) = vec(Z). \tag{35}$$

Thus, we have (33). We also have (34) from (32) in the same way. This completes the proof.

Now, let us proceed to the proof of Theorem 3.2.

(Proof of Theorem 3.2) Substituting (18) into (20), we have

$$Q^{-1} = \left(Q^-\right)^{-1} + \tilde{N}^{-1/2}\tilde{U}\tilde{\Xi}\tilde{U}'\tilde{N}^{-1/2}. \tag{36}$$

First, let us note that

$$\frac{\partial Q}{\partial u_{ij}} = \frac{\partial (Q^{-1})^{-1}}{\partial u_{ij}} = -Q\frac{\partial Q^{-1}}{\partial u_{ij}}Q. \tag{37}$$

Then, substitution of (36) into (37) yields

$$\frac{\partial Q}{\partial u_{ij}} = Q(Q^-)^{-1}\frac{\partial Q^-}{\partial u_{ij}}(Q^-)^{-1}Q - Q\tilde{N}^{-1/2}\{E_{ij}\tilde{\Xi}\tilde{U}' + \tilde{U}\tilde{\Xi}E_{ji}\}\tilde{N}^{-1/2}Q, \tag{38}$$

where

$$E_{ij} \triangleq \begin{array}{cc} j & \\ \left[\begin{array}{ccccccc} & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \\ 0 & \cdots\cdots & 0 & 1 & 0 & \cdots & 0 \\ & & & 0 & & & \\ & 0 & & \vdots & & 0 & \\ & & & 0 & & & \end{array}\right] & \begin{array}{c} \\ \\ \\ i \\ \\ \\ \\ \end{array} \end{array} . \tag{39}$$

Note that from the first relation of (4), we have

$$\frac{\partial Q^-}{\partial u_{ij}} = A \frac{\partial Q}{\partial u_{ij}} A'. \tag{40}$$

Substituting (40) into (38), we have

$$\frac{\partial Q}{\partial u_{ij}} = F \frac{\partial Q}{\partial u_{ij}} F' - Q\tilde{N}^{-1/2}\{E_{ij}\tilde{\Xi}\tilde{U}' + \tilde{U}\tilde{\Xi}E_{ji}\}\tilde{N}^{-1/2}Q. \tag{41}$$

Hence, using (32) and (34) in Lemma 4.1, we have

$$vec\left(\frac{\partial Q}{\partial u_{ij}}\right) = -\left(I - F \otimes F\right)^{-1} vec\left(Q\tilde{N}^{-1/2}\{E_{ij}\tilde{\Xi}\tilde{U}' + \tilde{U}\tilde{\Xi}E_{ji}\}\tilde{N}^{-1/2}Q\right). \tag{42}$$

For simplicity, let

$$\tilde{M} \triangleq M + A'(W - Y)A, \tag{43}$$

and

$$S \triangleq Q\tilde{N}^{-1/2}\{E_{ij}\tilde{\Xi}\tilde{U}' + \tilde{U}\tilde{\Xi}E_{ji}\}\tilde{N}^{-1/2}Q. \tag{44}$$

Then, it follows from the relation $\mathrm{tr}\,[AB'] = vec\,(A)'\,vec\,(B)$ and (42) that

$$\begin{aligned} \frac{\partial}{\partial u_{ij}} \mathrm{tr}[\tilde{M}Q] &= vec(\tilde{M})'vec\left(\frac{\partial Q}{\partial u_{ij}}\right) \\ &= -vec(\tilde{M})'\left(I - F \otimes F\right)^{-1} vec\,(S) \\ &= -\left[\left(I - F' \otimes F'\right)^{-1} vec(\tilde{M})\right]' vec\,(S) \\ &= -vec(X)'vec\,(S) \\ &= -\,\mathrm{tr}\,[XS]\,, \end{aligned}$$

where we used the fact $(F \otimes F)' = F' \otimes F'$ in the third equality, and applied (31) and (33) with $Z = \tilde{M}$ in the fourth equality. Thus, we see that

$$\begin{aligned} \frac{\partial}{\partial u_{ij}} \mathrm{tr}[\tilde{M}Q] &= -\,\mathrm{tr}\,[XS] \\ &= -\,\mathrm{tr}\left[XQ\tilde{N}^{-1/2}\{E_{ij}\tilde{\Xi}\tilde{U}' + \tilde{U}\tilde{\Xi}E_{ji}\}\tilde{N}^{-1/2}Q\right] \\ &= -2\,\mathrm{tr}\left[\tilde{N}^{-1/2}QXQ\,\tilde{N}^{-1/2}\tilde{U}\tilde{\Xi}E_{ji}\right] \\ &= -2\left[\tilde{N}^{-1/2}QXQ\,\tilde{N}^{-1/2}\tilde{U}\tilde{\Xi}\right]_{ij}, \end{aligned}$$

where in the last equality, we used the fact $\mathrm{tr}\,[XE_{ji}] = [X]_{ij}$. Thus, we have shown that

$$\frac{\partial}{\partial \tilde{U}} \mathrm{tr}[\tilde{M}Q] = -2\,\tilde{N}^{-1/2}QXQ\,\tilde{N}^{-1/2}\tilde{U}\tilde{\Xi}. \tag{45}$$

Using (45) and (26), we have the condition

$$\frac{\partial L(\tilde{\Xi},\ \tilde{U},\ \tilde{\Lambda})}{\partial \tilde{U}} = -2\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}\tilde{U}\tilde{\Xi} + 2\tilde{U}\tilde{\Lambda} = 0, \tag{46}$$

which implies

$$\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}\tilde{U}\tilde{\Xi} = \tilde{U}\tilde{\Lambda}. \tag{47}$$

Here, note that we can take $\tilde{\Lambda}$ as a diagonal matrix because $\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}$ is symmetric and $\tilde{\Xi}$ is diagonal [10; Lemma 4.2]. Similarly, the derivatives with respect to ξ_i are computed as

$$vec\left(\frac{\partial Q}{\partial \xi_i}\right) = -\left(I - F\otimes F\right)^{-1} vec\left(Q\tilde{N}^{-1/2}\tilde{U}E_{ii}\tilde{U}'\tilde{N}^{-1/2}Q\right), \tag{48}$$

$$\frac{\partial}{\partial \xi_i}\,\mathrm{tr}[\tilde{M}Q] = -\left[\tilde{U}'\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}\tilde{U}\right]_{ii}, \tag{49}$$

$$\frac{\partial}{\partial \tilde{\Xi}}\,\mathrm{tr}[\tilde{M}Q] = -\tilde{U}'\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}\tilde{U}, \tag{50}$$

and

$$\frac{\partial}{\partial \tilde{\Xi}}\,\mathrm{tr}[\tilde{\Psi}\tilde{\Xi}] = \tilde{\Psi}, \tag{51}$$

where (49) implies (50) because of (47) and the fact that $\tilde{\Lambda}$ is diagonal. Now, using (50) and (51), we have the condition

$$\frac{\partial L(\tilde{\Xi},\ \tilde{U},\ \tilde{\Lambda})}{\partial \tilde{\Xi}} = -\tilde{U}'\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}\tilde{U} + \tilde{\Psi} = 0, \tag{52}$$

and, hence, we have

$$\tilde{U}'\tilde{N}^{-1/2}QXQ\tilde{N}^{-1/2}\tilde{U} = \tilde{\Psi}. \tag{53}$$

By applying (53) and (25) to (47), we have

$$\tilde{\Lambda} = \tilde{\Psi}\tilde{\Xi}, \tag{54}$$

and which, together with (47), implies (28). This completes the proof.

5. **Representation of $\tilde{U}$ by a multi-dimensional Polar Coordinates System.** In this section, we convert the constrained problem given by (24) and (25) to an unconstrained one by introducing a multi-dimensional polar coordinates system in $\mathbf{R}^n$. Let us denote

$$\tilde{U} = T(n)\left\{\prod_{\ell=1}^{\bar{k}}\begin{bmatrix} I_{\ell\times\ell} & 0 \\ 0 & T(n-\ell)\end{bmatrix}\right\}\begin{bmatrix} I_{\tilde{m}\times\tilde{m}} \\ 0\end{bmatrix}, \tag{55}$$

where

$$\bar{k} \triangleq \min\left(\tilde{m}-1,\ n-2\right), \tag{56}$$

and, for $k = n,\ n-1,\ \cdots,\ n-\bar{k}$,

$$\begin{aligned} T(k) &\triangleq T(k;\ \theta_{k1},\ \theta_{k2},\ \cdots,\theta_{k\,k-1}) \\ &= [z_{k1}\ z_{k2}\ \cdots\ z_{kk}], \end{aligned} \tag{57}$$

$$z_{k1} = \begin{bmatrix} \prod_{i=1}^{k-1} \cos\theta_{ki} \\ \sin\theta_{k1} \prod_{i=2}^{k-1} \cos\theta_{ki} \\ \vdots \\ \sin\theta_{k\,j-1} \prod_{i=j}^{k-1} \cos\theta_{ki} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \sin\theta_{k\,k-2} \cos\theta_{k\,k-1} \\ \sin\theta_{k\,k-1} \end{bmatrix}, \quad z_{k\ell} = \begin{bmatrix} -\sin\theta_{k\,k-\ell+1} \prod_{i=1}^{k-\ell} \cos\theta_{ki} \\ -\sin\theta_{k\,k-\ell+1} \sin\theta_{k1} \prod_{i=2}^{k-\ell} \cos\theta_{ki} \\ \vdots \\ -\sin\theta_{k\,k-\ell+1} \sin\theta_{k\,j-1} \prod_{i=j}^{k-\ell} \cos\theta_{ki} \\ \vdots \\ -\sin\theta_{k\,k-\ell+1} \sin\theta_{k\,k-\ell} \\ \cos\theta_{k\,k-\ell+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{58}$$

$$\ell = 2,\ 3,\ \cdots,\ k,$$

and

$$0 \le \theta_{k1} \le 2\pi, \quad -\tfrac{\pi}{2} \le \theta_{k\,i} \le \tfrac{\pi}{2}, \quad i = 2,\ 3, \cdots,\ k-1. \tag{59}$$

Then, it can be easily seen that $T'(k)\,T(k) = T(k)\,T'(k) = I_{k\times k}$ and, hence, we have (25) for $\tilde{U}$ which is given by (55). The fact that all $\tilde{U} \in \mathbf{R}^{n\times\tilde{m}}$ for which we have (25) can be expressed by (55)-(59) is shown in [12], [13].

Thus, we have converted Problem 2 with constraint (25) to the one with the unconstrained angular variables given by (59) for $k = n,\ n-1,\ \cdots,\ n-\bar{k}$.

6. **A method of Connections of the Angular Parameters at the Boundary of the Domain.** For simplicity, let

$$\Theta \triangleq \left\{\theta_{k1},\ \theta_{k\,i},\ i = 2,\ 3,\ \cdots,\ k-1,\ \ k = n,\ \cdots,\ n-\bar{k}\right\}. \tag{60}$$

Clearly, $\tilde{U} = \tilde{U}(\Theta)$ is a periodic function of Θ. Hence, for an exterior point of the domain given by (59), there always exists an interior point (of the domain) for which $\hat{J}$ has the same value as the exterior point. If we can find such a pair of values of Θ, the optimization of searching the minimal point of $\hat{J}$ becomes much simpler. Namely, we can replace the exterior point generated by the algorithm with the corresponding interior point, and continue searching. In usual cases of optimization, we must stop searching when the algorithm generates an exterior point. In such a case, we usually take a nearest boundary point and memorize the value of the objective function at that point as a local minimal point. However, if we can replace the exterior point with the interior point at which $\hat{J}$ has the same value as that at the exterior point, we can continue searching until we find a minimal point. By the following theorem, we show that it suffices to search over the following set of the angular parameters with $1\big/2^{\bar{k}+1}$ size:

$$\begin{aligned} &0 \le \theta_{k1} \le \pi, \quad -\tfrac{\pi}{2} \le \theta_{k\,i} \le \tfrac{\pi}{2}, \\ &i = 2,\ 3, \cdots,\ k-1, \quad k = n,\ n-1, \cdots,\ n-\bar{k}, \end{aligned} \tag{61}$$

and also the rule by which we can replace any exterior point of (61) with the corresponding interior point.

Theorem 6.1. (Connections of Angular Parameters [12]). *For $\tilde{U}$ given by (55), we have the following rules for the case when only one parameter violates (61). For $k = n,\ n-1, \cdots,\ \bar{k}$, let*

$$\theta_k \triangleq [\theta_{k1}\ \ \theta_{k2}\ \ \cdots\ \ \theta_{k\,k-1}]', \tag{62}$$

$$q \in \left\{n,\ n-1,\ \cdots,\ n-\bar{k}\right\}, \tag{63}$$

and assume that only one parameter in θ_q violates (61), and the other parameters in θ_q and θ_k, $k \neq q$ all satisfy (61). Then, by (i)-(iii) below, we can find the corresponding value of Θ which is in the domain given by (61), and for which $\hat{J}$ given by (24) has the same value.

(i) *When $\theta_{q1} \in [-\pi,\ 0)$ or $\theta_{q1} \in (\pi,\ 2\pi]$ and $\theta_{q\,i} \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$, $i = 2,\ 3, \cdots,\ q-1$, $\hat{J}$ keeps the same value after the following replacements.*

(a) *q: odd,*

$$\theta_{q1} \to \theta_{q1} \pm \pi, \quad \theta_{q\,i} \to -\theta_{q\,i},\ i = 2,\ 3,\ \cdots,\ q-1,$$

$$\left\{\begin{array}{c} \theta_{(2j)\,(2j-1)} \to -\theta_{(2j)\,(2j-1)} \\ \theta_{(2j-1)\,1} \to \pi - \theta_{(2j-1)\,1} \end{array}\right. ,\ \ j = 2,\ 3,\ \cdots,\ (q-1)/2,$$

$$\theta_{2\,1} \to \pi - \theta_{2\,1},$$

(b) *q: even,*

$$\theta_{q1} \to \theta_{q1} \pm \pi, \quad \theta_{q\,i} \to -\theta_{q\,i},\ i = 2,\ 3,\ \cdots,\ q-1,$$

$$\left\{\begin{array}{c} \theta_{(2j+1)\,(2j)} \to -\theta_{(2j+1)\,(2j)} \\ \theta_{(2j)\,1} \to \pi - \theta_{(2j)\,1} \end{array}\right. ,\ \ j = 1,\ 2,\ \cdots,\ (q-2)/2,$$

(ii) *When $\theta_{q\,q-1} \in \left[-\frac{3\pi}{2},\ -\frac{\pi}{2}\right)$ or $\theta_{q\,q-1} \in \left(\frac{\pi}{2},\ \frac{3\pi}{2}\right]$, and $\theta_{q\,1} \in [0,\ \pi]$ and $\theta_{q\,i} \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$, $i = 2,\ 3, \cdots,\ q-2$, $\hat{J}$ keeps the same value after the following replacements.*

(a) *q: odd,*

$$\theta_{q\,q-1} \to \theta_{q\,q-1} \pm \pi,$$

$$\left\{\begin{array}{ll} \theta_{(2j)\,1} \to \pi - \theta_{(2j)\,1}, & j = 1,\ 2,\ \cdots,\ (q-1)/2, \\ \theta_{(2j-1)\,(2j-2)} \to -\theta_{(2j-1)\,(2j-2)}, & j = 2,\ 3,\ \cdots,\ (q-1)/2, \end{array}\right.$$

(b) *q: even,*

$$\theta_{q\,q-1} \to \theta_{q\,q-1} \pm \pi,$$

$$\left\{\begin{array}{ll} \theta_{(2j+1)\,1} \to \pi - \theta_{(2j+1)\,1}, & j = 1,\ 2,\ \cdots,\ (q-2)/2, \\ \theta_{(2j)\,(2j-1)} \to -\theta_{(2j)\,(2j-1)}, & j = 2,\ 3,\ \cdots,\ (q-2)/2, \end{array}\right.$$

$$\theta_{2\,1} \to \pi - \theta_{2\,1}.$$

(iii) *When $\theta_{q\,\tau} \in \left[-\frac{3\pi}{2},\ -\frac{\pi}{2}\right)$ or $\theta_{q\,\tau} \in \left(\frac{\pi}{2},\ \frac{3\pi}{2}\right]$, $\tau = 2, 3,\ \cdots,\ q-2$, and $\theta_{q1} \in [0,\ \pi]$ and $\theta_{q\,i} \in \left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$, $i \neq \tau$, $i = 2,\ 3,\ \cdots,\ q-1$, $\hat{J}$ keeps the same value after the following replacements.*

(a) $2\tau < q$,

$$\theta_{q\,\tau} \to \theta_{q\,\tau} \pm \pi, \quad \theta_{q\,i} \to -\theta_{q\,i},\ i = \tau+1,\ \tau+2,\ \cdots,\ q-1,$$

$$\left\{\begin{array}{ll} \theta_{(q-2j+1)\,(q-\tau-j)} \to -\theta_{(q-2j+1)\,(q-\tau-j)}, & j = 1,\ 2,\ \cdots,\ \tau-1, \\ \theta_{(q-2j)\,(\tau-j)} \to -\theta_{(q-2j)\,(\tau-j)}, & j = 1,\ 2,\ \cdots,\ \tau-2, \end{array}\right.$$

$$\theta_{(q-2\tau+2)\,1} \to \pi - \theta_{(q-2\tau+2)\,1}, \quad \theta_{(q-2\tau+2)\,i} = -\theta_{(q-2\tau+2)\,i},$$
$$i = 2,\ 3,\ \cdots,\ q-2\tau+1,$$

(b) $2\tau \geq q$,

$$\theta_{q\tau} \to \theta_{q\tau} \pm \pi, \quad \theta_{q\,i} \to -\theta_{q\,i},\ i = \tau+1,\ \tau+2,\ \cdots,\ q-1,$$

$$\left\{\begin{array}{c} \theta_{(q-2j+1)\,(q-\tau-j)} \to -\theta_{(q-2j+1)\,(q-\tau-j)}, \\ \theta_{(q-2j)\,(\tau-j)} \to -\theta_{(q-2j)\,(\tau-j)}, \end{array}\right. \quad j = 1,\ 2,\ \cdots,\ q-\tau-2\,,$$

$$\theta_{(-q+2\tau+3)\,1} \to \pi - \theta_{(-q+2\tau+3)\,1}, \quad \theta_{(-q+2\tau+3)\,i} = -\theta_{(-q+2\tau+3)\,i},$$
$$i = 2,\ 3,\ \cdots,\ -q+2\tau+2.$$

(Proof) The outline of the proof is given in [12].

7. **A Numerical Example.** In this section, we will give an illustrative example for a 3-dimensional LQG system with 3-dimensional observations ($n = m = 3$).

Example 7.1. *Let us consider a 3-dimensional system with*

$$A = \begin{bmatrix} 0.5 & 0.3 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.1 & 0.5 & 0.6 \end{bmatrix}, \quad G = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix},$$

$$C = \begin{bmatrix} 1.2 & 0.1 & 0.2 \\ 0.2 & 1.4 & 0.2 \\ 0.1 & 0.5 & 1.1 \end{bmatrix},$$

$$M = diag(40.0,\ 100.0,\ 55.0), \quad N = I,$$

$$R_0 = \begin{bmatrix} -\frac{1}{8} & -\frac{3\sqrt{3}}{8} & \frac{3}{4} \\ -\frac{3\sqrt{3}}{8} & \frac{5}{8} & \frac{\sqrt{3}}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 0.49 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.44 \end{bmatrix} \begin{bmatrix} -\frac{1}{8} & -\frac{3\sqrt{3}}{8} & -\frac{3}{4} \\ -\frac{3\sqrt{3}}{8} & \frac{5}{8} & -\frac{\sqrt{3}}{4} \\ \frac{3}{4} & \frac{\sqrt{3}}{4} & -\frac{1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} 1.2395 & 0.1015 & -0.2128 \\ 0.1015 & 0.8673 & -0.3437 \\ -0.2128 & -0.3437 & 0.8231 \end{bmatrix},$$

i.e.,

$$\Psi = \mathrm{diag}(0.49,\ 1.0,\ 1.44), \quad \Gamma = \begin{bmatrix} -\frac{1}{8} & -\frac{3\sqrt{3}}{8} & \frac{3}{4} \\ -\frac{3\sqrt{3}}{8} & \frac{5}{8} & \frac{\sqrt{3}}{4} \\ -\frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{1}{2} \end{bmatrix}.$$

For the above system and observations, the solution of (11) is given by

$$W = \begin{bmatrix} 40.1699 & 0.0937 & 0.0117 \\ 0.0937 & 100.1910 & 0.1872 \\ 0.0256 & 0.1872 & 55.2648 \end{bmatrix},$$

and

$$Y = \begin{bmatrix} 0.7184 & -0.0857 & -0.1342 \\ -0.0857 & 0.6969 & -0.4178 \\ -0.1342 & -0.4178 & 0.9710 \end{bmatrix}.$$

By taking $\tilde{N} = (1/150.0) \cdot I$, we made numerical computations for the three cases: $m = 3$, $\tilde{m} = 2$ and $\tilde{m} = 1$. We carried out the optimization by a simple alternate search algorithm with respect to the angular parameters: θ_{31}, θ_{32} and θ_{21}, and $\tilde{\Xi} \triangleq \mathrm{diag}(\xi_1,\ \cdots,\ \xi_{\tilde{m}})$ by making use of the connection rule shown in Section 5. The results are summarized in Table 1.

In Figure 1, the result of the optimization of ξ_1, ξ_2 and ξ_3 by the alternate search for $\tilde{m} = 3$ with the initial values $\xi_1 = \xi_2 = 1.0$ and $\xi_3 = 2.0$ is shown. Figure 2 shows the corresponding change of the value of $\hat{J}$. As we see from Figures 1 and 2, we have good convergence. Also, for $\tilde{m} = 2$, the runs of θ_{31}, θ_{32} and θ_{21} are respectively shown in Figures 3, 4 and 5, for 27 different initial values of $\Theta = \{\theta_{31}, \ \theta_{32}, \ \theta_{21}\}$ and $\tilde{\Xi} = \mathrm{diag}(\xi_1, \ \xi_2)$. Thus, we obtained good convergence for all runs to the optimal value shown in Table 1.

TABLE 1. The optimal values of $\Theta = \{\theta_{31}, \ \theta_{32}, \ \theta_{21}\}$, $\tilde{\Xi}$, $\tilde{U}$, H, Q, Q^{-} and $\hat{J}$ for $\tilde{m} = 3, \ 2$ and 1

		$\tilde{m} = 3$	$\tilde{m} = 2$	$\tilde{m} = 1$
Θ	θ_{31}	1.1019	1.1016	1.1014
	θ_{32}	0.6179	0.6183	0.6177
	θ_{21}	0.7854	0.7854	
$\tilde{\Xi}$	ξ_1	0.8951	0.8955	0.8957
	ξ_2	0.2162	0.2163	
	ξ_3	0.0354		
$\tilde{U}$		$\begin{bmatrix} 0.3684 & -0.8159 & -0.4457 \\ 0.7271 & -0.0459 & 0.6850 \\ 0.5793 & 0.5764 & -0.5764 \end{bmatrix}$	$\begin{bmatrix} 0.3684 & -0.8160 \\ 0.7268 & -0.0459 \\ 0.5796 & 0.5762 \end{bmatrix}$	$\begin{bmatrix} 0.3688 \\ 0.7270 \\ 0.5791 \end{bmatrix}$
H		$\begin{bmatrix} 1.7210 & 0.8519 & -3.9141 \\ -5.3783 & -3.1747 & -1.6900 \\ 0.3868 & -5.2570 & -4.1493 \end{bmatrix}$	$\begin{bmatrix} 2.6454 & -0.5675 & -2.7197 \\ -4.8466 & -3.9934 & -1.0030 \\ -0.2292 & -4.3096 & -4.9481 \end{bmatrix}$	$\begin{bmatrix} -0.3741 & -0.7374 & -0.5874 \\ -1.9436 & -3.8317 & -3.0522 \\ -2.2443 & -4.4245 & -3.5244 \end{bmatrix}$
Q		$\begin{bmatrix} 0.0407 & -0.0300 & 0.0163 \\ -0.0300 & 0.0545 & -0.0403 \\ 0.0163 & -0.0403 & 0.0474 \end{bmatrix}$	$\begin{bmatrix} 0.0697 & -0.0745 & 0.0538 \\ -0.0745 & 0.1229 & -0.0978 \\ 0.0538 & -0.0978 & 0.0958 \end{bmatrix}$	$\begin{bmatrix} 0.2559 & -0.0626 & -0.0794 \\ -0.0626 & 0.1237 & -0.1064 \\ -0.0794 & -0.1064 & 0.1913 \end{bmatrix}$
Q^{-}		$\begin{bmatrix} 0.2558 & 0.0017 & 0.0005 \\ 0.0017 & 0.2523 & 0.0018 \\ 0.0005 & 0.0018 & 0.2559 \end{bmatrix}$	$\begin{bmatrix} 0.2566 & 0.0009 & 0.0010 \\ 0.0009 & 0.2530 & 0.0013 \\ 0.0010 & 0.0013 & 0.2562 \end{bmatrix}$	$\begin{bmatrix} 0.2939 & 0.0078 & -0.0238 \\ 0.0078 & 0.2543 & -0.0033 \\ -0.0238 & -0.0033 & 0.2727 \end{bmatrix}$
$\hat{J}$		1.4873	1.5565	3.8220

8. **Concluding Remarks.** In this paper, we have considered the optimization of the observations of the stationary LQG stochastic control systems under a quadratic criterion. The numerical algorithm used in Section 7 is simple but does not use the condition of optimality very efficiently. It may be possible to improve the algorithm by doing so. Furthermore, the gradient methods usually give better results than the algorithm used in this paper. For the gradient methods, however, we need explicit representations of the derivatives of $\hat{J}$ with respect to Θ and $\tilde{\Xi}$. This point will be discussed in the near future.

Acknowledgment. The author would like to express his sincere thanks to Mr. T. Teranishi for his help in the numerical computations.

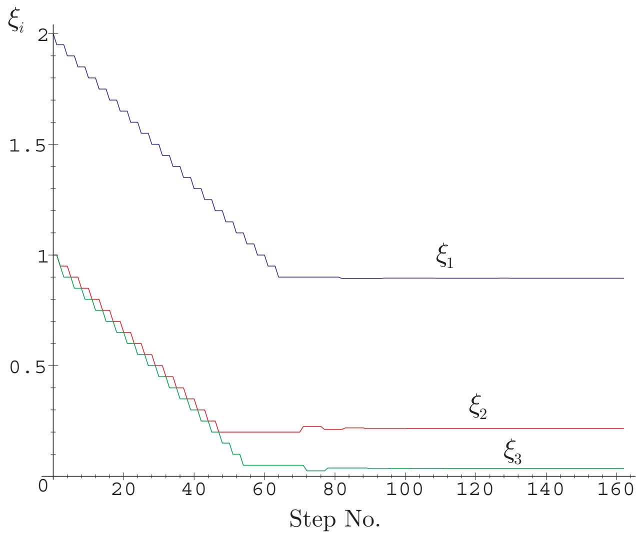

FIGURE 1. The results of the optimization of ξ_1, ξ_2 and ξ_3 by the alternate search for $\tilde{m} = 3$ with the initial values $\xi_1 = \xi_2 = 1.0$ and $\xi_3 = 2.0$

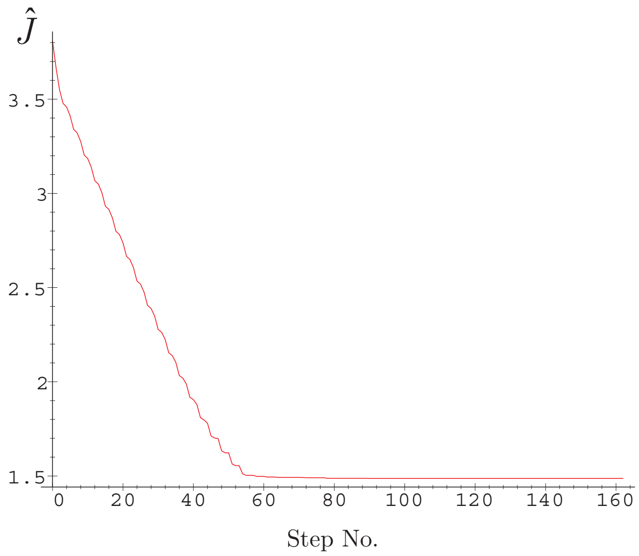

FIGURE 2. The results of the optimization by the alternate search for$\tilde{m} = 3$

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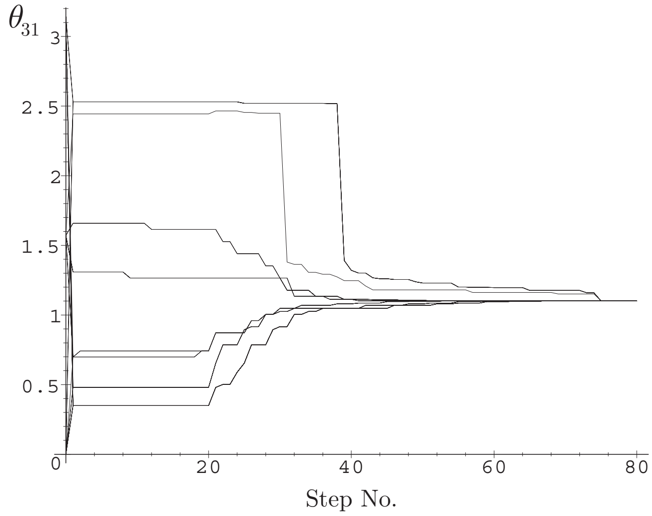

FIGURE 3. The value of θ_{31} for 27 initial values of $\Theta = \{\theta_{31},\ \theta_{32},\ \theta_{21}\}$ and $\tilde{\Xi} = \mathrm{diag}(\xi_1,\ \xi_2)$ ($\tilde{m} = 2$)

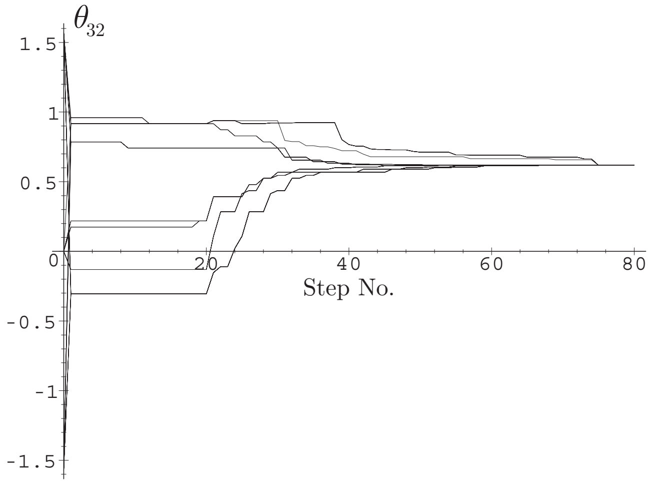

FIGURE 4. The value of θ_{32} for 27 initial values of $\Theta = \{\theta_{31},\ \theta_{32},\ \theta_{21}\}$ and $\tilde{\Xi} = \mathrm{diag}(\xi_1,\ \xi_2)$ ($\tilde{m} = 2$)

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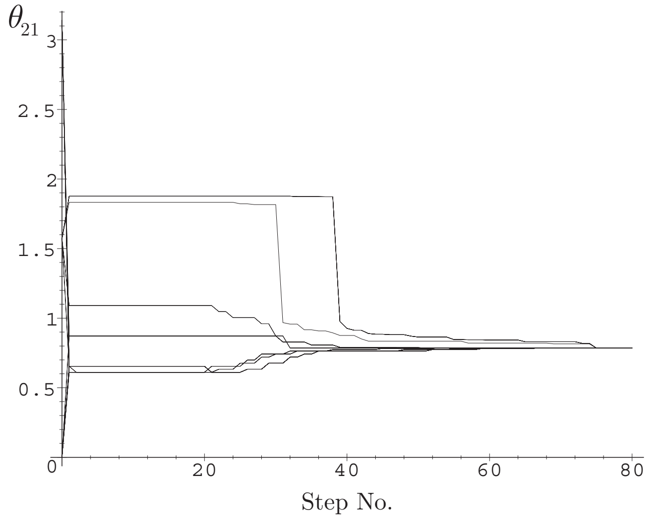

FIGURE 5. The value of θ_{21} for 27 initial values of $\Theta = \{\theta_{31},\ \theta_{32},\ \theta_{21}\}$ and $\tilde{\Xi} = \mathrm{diag}(\xi_1,\ \xi_2)$ $(\tilde{m} = 2)$

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