International Journal of Innovative Computing, Information and Control
Volume **8**, Number **3(B)**, March **2012**
 ISSN 1349-4198
pp. 2389–2397

AN IMPROVED RECURSIVE ALGORITHM OF OPTIMAL FILTER FOR DISCRETE-TIME LINEAR SYSTEMS SUBJECT TO COLORED OBSERVATION NOISE

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Received March 2011; revised July 2011

Abstract. *This paper describes an optimal state estimator for a class of discrete-time linear stochastic systems that are subject to both colored observation noise and unknown inputs. Previously, we proposed an optimal filter for such systems based on Chen and Patton's optimal disturbance decoupling observer (ODDO). More recently, we modified the ODDO by correcting the estimation error covariance matrix. Here, we combine these results and thus propose a new reliable optimal state estimator for systems having colored observation noise and unknown inputs.*
Keywords: Fault detection, Optimal filter, Stochastic systems, Colored noise, Robust filter, Unknown inputs, Discrete-time systems, Chi-square test

1. **Introduction.** Over the past thirty years, a number of model-based fault detection techniques have been investigated. Fault detection theory for linear systems and their applications were developed by both Frank [1] and Chen and Patton [2] (also [3, 4]), and Guo et al. [5] investigated the robust fault diagnosis problem for linear time invariant systems by using a linear matrix inequality approach. In contrast, Zhang et al. [6] studied the fault detection problem for nonlinear systems.

In studying the fault detection problem, there are two major difficulties [7, 8]. The first is that appropriate state estimators for fault diagnosis can be obtained only from accurate mathematical models. However, derived mathematical models often contain modeling errors, which significantly increase the state estimation error in a similar manner to the models having unknown inputs. If such inaccurate models are used, fault detection filters cannot be considered to be reliable. This issue, known as the modeling error problem, can be avoided by using algorithms that are robust with respect to unknown inputs [2, 3, 9]. The second difficulty is that in practice an observation system will not be free from colored random noise. This noise cannot be regarded as white Gaussian noise, and therefore fault diagnosis for such stochastic systems requires filters robust to colored observation noise. This type of problems appears in the state estimation problem of infinite-dimensional systems using finite-dimensional filters [10, 11]. These two practical demands lead to our major motivation for this paper.

A well-known approach for constructing state estimators is to consider an augmented system by combining the original stochastic system and the one affected by the colored noise (e.g., [9]). Another approach that has been discussed in detail is to transform systems with colored observation noise into those with white Gaussian noise [10-15]. Previously, we developed a robust optimal filter for stochastic systems having unknown inputs and colored observation noise [13]. This optimal filter was based on Chen and Patton's Optimal Disturbance Decoupling Observer (ODDO; [2, 3]). However, in another paper we also corrected the error covariance matrix for the ODDO [16-18]. In the present work, we apply our corrected recursive algorithm for the ODDO to our proposed optimal filter in order to diagnose faults in stochastic systems with unknown inputs. We thus obtain a reliable optimal state estimator with the primary advantage that it can be applied to systems with not only unknown inputs but also colored observation noise.

The remainder of this paper is organized as follows. In Section 2, we introduce a discrete-time stochastic system having both unknown inputs and colored observation noise sequences, and, under several simple assumptions, transform it into a system with white Gaussian noise sequences. We then apply the modified ODDO to the transformed system and derive the optimal filter for the described stochastic system in Sections 3 and 4. Numerical simulations and conclusions are presented in Section 5 and Section 6, respectively.

2. **Problem Formulation.** Consider the following linear discrete-time stochastic system, having an unknown input and colored observation noise, of the form

$$x_{k+1} = A_k\, x_k + B_k\, u_k + E_k\, d_k + \zeta_k, \tag{1}$$

$$y_k = C_k\, x_k + D_k\, \beta_k, \tag{2}$$

where $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, $u_k \in \mathbb{R}^m$ and $d_k \in \mathbb{R}^\ell$ denote the state vector of the system, the observation vector, the known input vector and the unknown input vector, respectively; $A_k \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m}$, $E_k \in \mathbb{R}^{n\times \ell}$, $C_k \in \mathbb{R}^{p\times n}$, $D_k \in \mathbb{R}^{p\times q}$ are real-valued matrices; $\zeta_k \in \mathbb{R}^n$ is the independent zero mean white Gussian noise sequence having covariance matrix Q_k (i.e., $\mathcal{E}\{\zeta_k\, \zeta_j^T\} = Q_k\, \delta_{kj}$, where $\mathcal{E}\{\,\cdot\,\}$ denotes the mathematical expectation operator and δ_{kj} denotes the Kronecker delta). Moreover, $\beta_k \in \mathbb{R}^q$ represents the colored observation noise sequence, which is generated by the following linear system:

$$\beta_{k+1} = S_k\, \beta_k + w_{k+1}. \tag{3}$$

Here $S_k \in \mathbb{R}^{q\times q}$ is a known coefficient matrix, and $w_k \in \mathbb{R}^q$ denotes the independent zero mean white Gaussian noise sequence with covariance matrix R_k; explicitly, $\mathcal{E}\{w_k\, w_j^T\} = R_k\, \delta_{kj}$; and $\beta_0 = \beta_{-1} = 0$.

Here, we assume that: (i) $pair(A_k,\, C_k)$ is observable, (ii) E_k is a full column rank matrix (i.e., $\operatorname{rank} E_k = \ell$), (iii) D_k is a full raw rank matrix (i.e., $\operatorname{rank} D_k = p$), (iv) $\operatorname{rank}(C_{k+1}\, E_k) = \operatorname{rank} E_k$, and (v) $\operatorname{Ker}(D_k) \subset \operatorname{Ker}(D_{k+1}\, S_k)$.

To apply the approach proposed by Ohsumi and Sawada ([10, 11]) to the system given by (1)-(3), we introduce a new observation sequence:

$$\bar{y}_{k+1} = y_{k+1} - D_{k+1}\, S_k\, D_k^+\, y_k, \tag{4}$$

where superscript + represents the pseudo-inverse matrix defined by $M^+ := M^T(MM^T)^{-1}$. Substituting (2) and (3) into (4), the observation can be rewritten as an observation system having a white Gaussian noise sequence:

$$\bar{y}_{k+1} = C_{k+1}x_{k+1} - D_{k+1}S_kD_k^+C_kx_k + D_{k+1}\, w_{k+1}. \tag{5}$$

Let us now define a new state vector by $v_{k+1} := [x_k^T, x_{k+1}^T]^T$. With this definition, an augmented system that is subject to the white Gaussian noise sequences is obtained:

$$v_{k+1} = \bar{A}_k\, v_k + \bar{B}_k\, u_k + \bar{E}_k\, d_k + \bar{G}_k\, \zeta_k, \tag{6}$$

$$\bar{y}_k = \bar{C}_k\, v_k + D_k\, w_k, \tag{7}$$

where

$$\bar{A}_k = \begin{bmatrix} 0 & I \\ 0 & A_k \end{bmatrix}, \quad \bar{B}_k = \begin{bmatrix} 0 \\ B_k \end{bmatrix},$$
$$\bar{E}_k = \begin{bmatrix} 0 \\ E_k \end{bmatrix}, \quad \bar{G}_k = \begin{bmatrix} 0 \\ I \end{bmatrix},$$

and

$$\bar{C}_{k+1} = [-D_{k+1}\, S_k\, D_k^+\, C_k \quad C_{k+1}].$$

3. **Derivation of Robust Optimal Filters.** In this paper, we apply the improved version of the ODDO algorithm presented by Tanikawa and Sawada [16] to the augmented system given by (6) and (7). This augmented system is directly related to the discrete-time system having colored observation noise described by (1)-(3). Our optimal filter algorithm is given as follows.

The form of the optimal observer can be described by

$$\bar{z}_{k+1} = \bar{F}_{k+1}\, \bar{z}_k + \bar{T}_{k+1}\, \bar{B}_k\, u_k + \bar{K}_{k+1}\, \bar{y}_k, \tag{8}$$
$$\hat{v}_{k+1} = \bar{z}_{k+1} + \bar{H}_{k+1}\, \bar{y}_{k+1}, \tag{9}$$

where $\hat{v}_k \in \mathbb{R}^{2n}$ is the state estimate of v_k; and coefficient matrices $\bar{F}_{k+1} \in \mathbb{R}^{2n\times 2n}$, $\bar{T}_{k+1} \in \mathbb{R}^{2n\times 2n}$, and $\bar{K}_{k+1} \in \mathbb{R}^{2n\times p}$, $\bar{H}_{k+1} \in \mathbb{R}^{2n\times p}$ are to be determined such that the estimation error variance is minimized and that the effect of the unknown input on the estimation error is decoupled. For this purpose, we explicitly consider the initial condition of H_k as $H_0 = 0$. Then, the gain matrix $\bar{K}_{k+1}$ can be defined as $\bar{K}_{k+1} := \bar{K}^1_{k+1} + \bar{K}^2_{k+1}$, for $\bar{K}^1_{k+1}$, $\bar{K}^2_{k+1} \in \mathbb{R}^{2n\times p}$. To construct the optimal filter, the coefficient matrices $\bar{F}_{k+1}$, $\bar{T}_{k+1}$, and $\bar{K}^2_{k+1}$ must satisfy the following relations:

$$\bar{F}_{k+1} = \bar{A}_k - \bar{H}_{k+1}\, \bar{C}_{k+1}\, \bar{A}_k - \bar{K}^1_{k+1}\, \bar{C}_k, \tag{10}$$
$$\bar{T}_{k+1} = I - \bar{H}_{k+1}\, \bar{C}_{k+1}, \tag{11}$$
$$\bar{K}^2_{k+1} = \bar{F}_{k+1}\, \bar{H}_k. \tag{12}$$

The decoupling of unknown input, d_k, can be achieved by satisfying the relation

$$\bar{E}_k = \bar{H}_{k+1}\, \bar{C}_{k+1}\, \bar{E}_k. \tag{13}$$

Here, the necessary and sufficient condition for this relation to hold is rank $(\bar{C}_{k+1}\, \bar{E}_k) =$ rank $(\bar{E}_k)$, which is equivalent to the condition given in assumption (iv). Thus, if assumption (iv) is valid, $\bar{H}_{k+1}$ can be given by

$$\bar{H}_{k+1} = \bar{E}_k(\bar{C}_{k+1}\, \bar{E}_k)^+, \tag{14}$$

where the superscript + again denotes the pseudo-inverse matrix. The gain matrix, $\bar{K}^1_{k+1}$, can be determined from calculation of the weighted error covariance matrix given by $\tilde{P}_k = \mathcal{E}\left\{\Lambda e_k e_k^T \Lambda\right\}$, for the state estimation error, $e_k = v_k - \hat{v}_k$; and the weight matrix, Λ, defined by a diagonal matrix. The gain matrix, $\bar{K}^1_{k+1}$, required to minimize the estimation error variance, var$\{e_{k+1}\}$, and the revised recursive algorithm, $\tilde{P}_k$, are given by

$$\bar{K}^1_{k+1} = \bar{A}^1_{k+1}(\Lambda^{-1}\tilde{P}_k\Lambda^{-1}\, \bar{C}_k^T - \bar{H}_k D_k R_k D_k^T) \times (\bar{C}_k\, \Lambda^{-1}\tilde{P}_k\Lambda^{-1}\, \bar{C}_k^T + D_k R_k D_k^T)^{-1}, \tag{15}$$

$$\begin{aligned}\tilde{P}_{k+1} = \Lambda \Big[\bar{A}^1_{k+1}\Big\{&\Lambda^{-1}\tilde{P}_k\Lambda^{-1} - (\Lambda^{-1}\tilde{P}_k\Lambda^{-1}C_k^T \\ &- \bar{H}_k D_k R_k D_k^T)(C_k\Lambda^{-1}\tilde{P}_k\Lambda^{-1}C_k^T + D_k R_k D_k^T)^{-1} \\ &\times(C_k\Lambda^{-1}\tilde{P}_k\Lambda^{-1} - D_k R_k D_k^T \bar{H}_k^T)\Big\}\, \bar{A}^{1\,T}_{k+1} \\ &+ \bar{T}_{k+1}\bar{G}_k Q_k \bar{G}_k^T \bar{T}_{k+1}^T + \bar{H}_{k+1} D_{k+1} R_{k+1} D_{k+1}^T \bar{H}_{k+1}^T\Big]\, \Lambda,\end{aligned} \tag{16}$$

where $\bar{A}^1_{k+1} = \bar{A}_k - \bar{H}_{k+1}\,\bar{C}_{k+1}\,\bar{A}_k$.

4. **Fault Detection.** Assuming that actuator and sensor faults can occur in the system given by (1)-(3), the system can be rewritten as

$$x_{k+1} = A_k\,x_k + B_k\,u_k + E_k\,d_k + \zeta_k + B_k\,f^a_k, \tag{17}$$
$$y_k = C_k\,x_k + D_k\,\beta_k + f^s_k, \tag{18}$$
$$\beta_{k+1} = S_k\,\beta_k + w_{k+1}, \tag{19}$$

where $f^a_k \in \mathbb{R}^m$ and $f^s_k \in \mathbb{R}^r$ are the actuator and sensor fault vectors, respectively. In this section, we consider the construction of an algorithm to detect actuator and sensor faults by using the proposed optimal filter.

Firstly, the sequence indicating the fault (called the residual), $\bar{r}_k$, is

$$\bar{r}_k = \bar{y}_k - \bar{C}_k\,\hat{v}_k, \tag{20}$$

and the observation system consisting of the augmented vector, v_k, with sensor faults is given by

$$\bar{y}_k = \bar{C}_k\,v_k + D_k\,w_k + f^s_k. \tag{21}$$

Substituting (21) into (20), the residual can thus be expressed as

$$\bar{r}_k = \bar{C}_k\,\bar{e}_k + D_k\,w_k + f^s_k, \tag{22}$$

and (22) is a robust residual signal for the stochastic system having colored observation noise, since the estimation error is unaffected by the unknown input $E_k\,d_k$.

When an actuator or a sensor fault occurs in the system, the statistics of r_k changes sharply. Hence, determination of the existence of faults can be achieved by hypothesis testing of the residual. The two hypotheses for the residual are: i) H_0: the system has no fault (normal mode); ii) H_1: actuator or sensor faults occur in the system (fault mode). When the system is under the normal (no fault) condition (i.e., H_0), the residual statistics are

$$H_0 \begin{cases} \mathcal{E}\{\bar{r}_k\} = 0 \\ \mathrm{cov}\{\bar{r}_k\} = \bar{C}_k\,\Lambda^{-1}\,\tilde{P}_k\,\Lambda^{-1}\,\bar{C}^T_k + D_k\,R_k\,D^T_k (=: \bar{W}_k), \end{cases} \tag{23}$$

and if the condition of system swaps to H_1, the statistics of the residual change from (23).

Since the noise sequences ζ_k and w_k are, from Section 2, assumed to be white Gaussian noise, a chi-square test is applied to discriminate between hypotheses H_0 and H_1. The test for the occurrence of faults consists of comparing the scalar test statistic

$$\lambda_k := \bar{r}^T_k\,\bar{W}^{-1}_k\,\bar{r}_k \tag{24}$$

to a constant threshold, where λ_k is chi-square distributed with p degrees of freedom, and

$$\begin{cases} \lambda_k \geq T_D & \text{fault} \\ \lambda_k < T_D & \text{no fault,} \end{cases} \tag{25}$$

for threshold, T_D, determined from the chi-square distribution table with $\mathrm{Prob}\{\lambda_k \geq T_D | H_0\} = P_f$ so that the probability of a false alarm, P_f, is maintained.

5. **Numerical Simulations.** To demonstrate the above filter algorithm, we consider as an example the following discrete-time time-invariant linear system with actuator and sensor faults:

$$A_k = \begin{bmatrix} 0.9944 & -0.1203 & -0.4302 \\ 0.0017 & 0.9902 & -0.0747 \\ 0 & 0.8187 & 0 \end{bmatrix},$$

$B_k = [0.4252, -0.0082, 0.1813]^T$, $C_k = I_{3\times 3}$, $S_k = \text{diag}\{-0.1, -0.2, -0.8\}$, $D_k = \text{diag}\{0.5,\ 0.5, 0.5\}$, $x_k = [x_k^{(1)},\ x_k^{(2)}, x_k^{(3)}]^T \in \mathbb{R}^3$, and $y_k \in \mathbb{R}^3$. The covariance matrices of the white Gaussian noise sequences, ζ_k and w_k, are set as $Q_k \equiv \text{diag}\{5 \times 10^{-3}, 5 \times 10^{-3}, 5 \times 10^{-5}\}$ and $R_k \equiv 0.01\, I_{3\times 3}$, respectively. Due to the model error [2], the unknown input term, $E_k\, d_k$, is assumed to be of the form:

$$\begin{aligned} E_k\, d_k &= \Delta A_k\, x_k + \Delta B_k\, u_k \\ &= E\left\{ \begin{bmatrix} -0.8950 & 0.1083 & 0.3872 \\ -0.0015 & -0.8912 & 0.0672 \end{bmatrix} x_k + \begin{bmatrix} 0.0085 \\ -0.0002 \end{bmatrix} u_k \right\}, \end{aligned} \tag{26}$$

and $E = [I_{2\times 2}\ 0]^T$.

For the simulation, the actuator fault case is performed, and the actuator fault actually arises at $k = 35$ such that $f_k^a \equiv 0$:

$$f_k^a = \begin{cases} 0 & \text{for } 0 \le k < 35 \\ 2 & \text{for } 35 \le k. \end{cases}$$

To perform the simulation with an actuator fault, the known input is set as $u_k \equiv 10$, and the initial values are $x_0 = 0$ and $\bar{P}_0 = 0.01\, I_{6\times 6}$. Furthermore, the weight matrix with respect to $\bar{P}_k$ is given by $\Lambda = \text{diag}\{1, 1, 1, 10, 10, 10\}$.

In obtaining the numerical result, the performance of the proposed filter algorithm is similar to our previous approach [13]. However, quantitative analysis to substantiate this will be required in the future. The simulation results are shown in Figures 1-6, where the superscript i ($i = 1,\ 2,\ 3$) denotes each element of a variable. In Figure 1, the behavior of state, x_k, is depicted. At $k = 35$, it can be seen that the values of state variables $x_k^{(1)}$ and $x_k^{(2)}$ sharply increase due to the actuator fault. Figure 2 shows sample runs of the colored observation noise sequence, β_k, generated by the subsystem given in (3), and in Figure 3 we see that the observation data, y_k, is disturbed by this colored noise. $y_k^{(1)}$ and $y_k^{(2)}$ also observed an sharp change of state at $k = 35$ due to the effects of the actuator fault. The output of the proposed optimal filter is shown in Figure 4, where $\hat{x}_k$, denotes the state estimate with respect to x_k, and the norm of the estimation error, e_k, is depicted in Figure 5. From Figures 4 and 5, we can establish that estimation errors of the proposed optimal filter are sufficiently small before the actuator fault occurs. However, after the fault arises, the e_k values increase substantially. In contrast, the unknown input $E_k\, d_k$ does not affect the state estimate.

Finally, Figure 6 shows the behavior of the value of the test statistic function with respect to the faults, r_k. At the start of the simulation, r_k has a large value, which is caused by the immediate change of the state from the known input, u_k. To use this function for detecting faults, this large initial value must be negated. However, when the actuator fault occurs at $k = 35$, r_k attains a value larger than that for $2 < k < 35$ (no fault conditions).

6. **Conclusions.** In this paper, a new optimal filter for discrete-time linear stochastic systems having unknown inputs and colored observation noise has been proposed. We applied our modified version [16] of Chen and Patton's algorithm [2] to the optimal filter

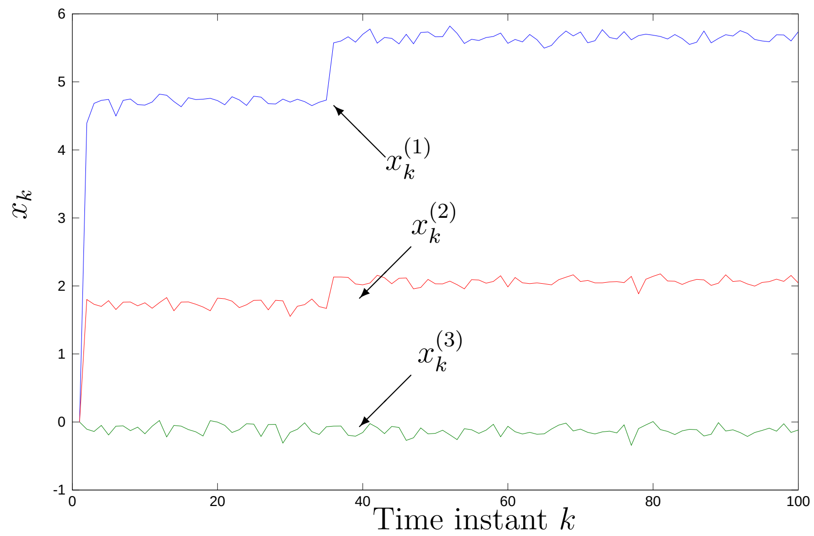

FIGURE 1. Sample runs of state vector, x_k, when an actuator fault occurs at $k = 35$

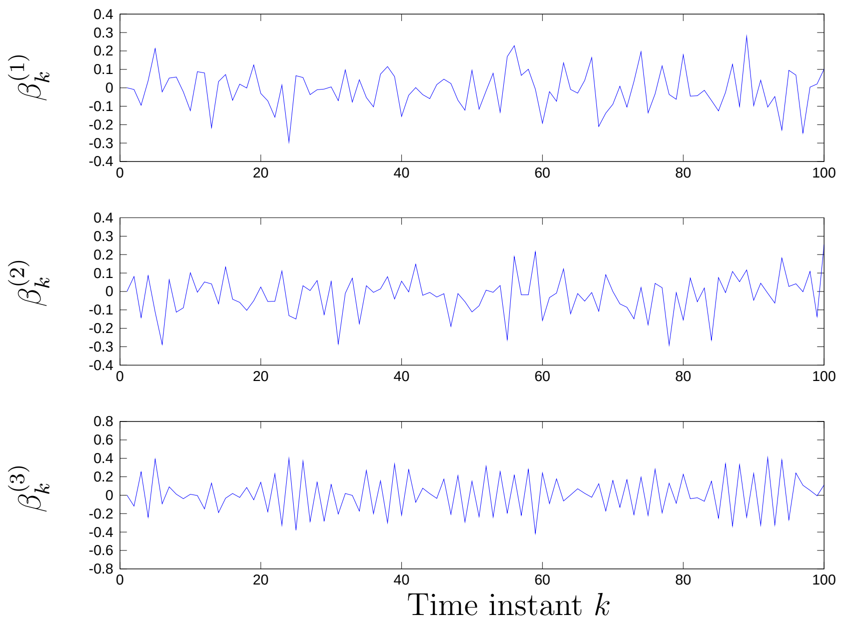

FIGURE 2. Colored observation noise sequence, β_k, generated from (3)

for systems with colored noise formulated previously in [13]. Here, the error covariance matrix, $\tilde{P}_k$, of the obtained optimal filter has additional terms, and the gain matrix, $\bar{K}^1_{k+1}$, is different from that employed before in [13].

Acknowledgment. This research was supported in part by the Japan Society for Promotion of Science (JSPS) under a Grant-in-Aid for Scientific Research (C)-22540158.

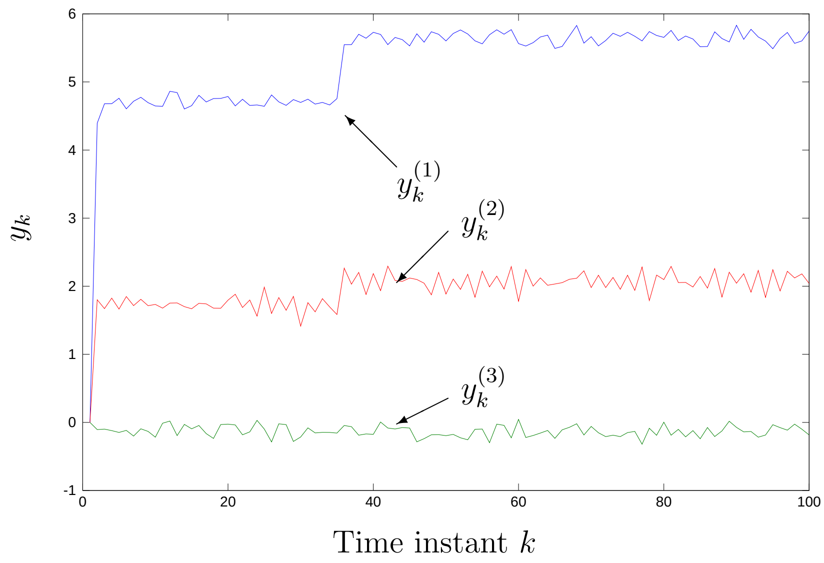

FIGURE 3. Observation vector, y_k, when an actuator fault occurs at $k = 35$

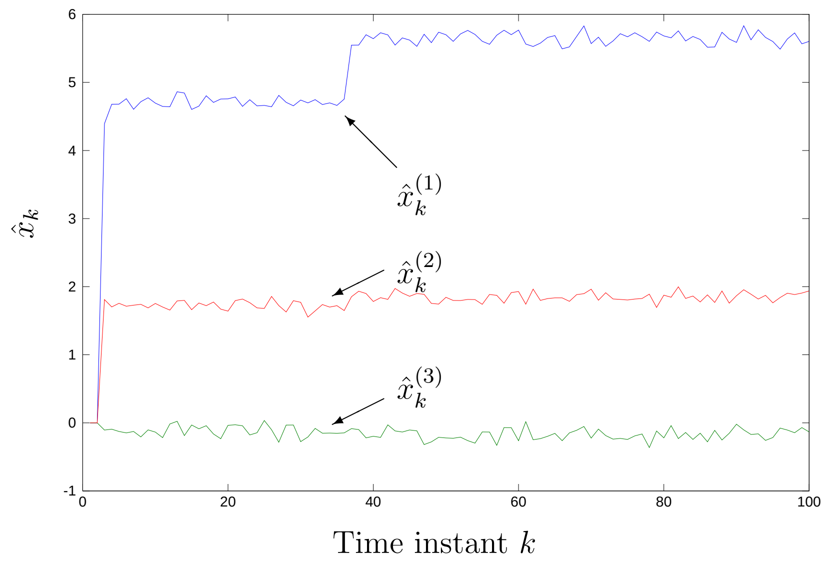

FIGURE 4. State estimate, $\hat{x}_k$, when an actuator fault occurs at $k = 35$

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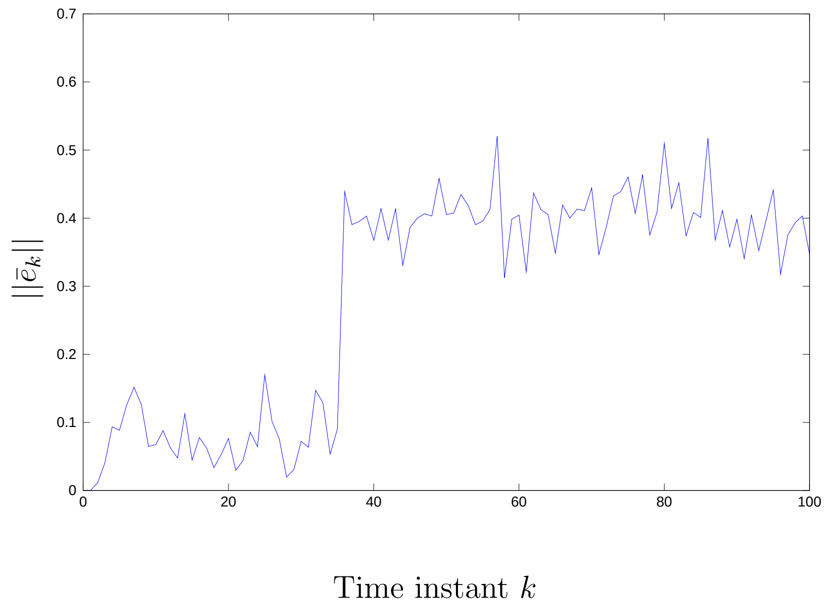

FIGURE 5. Norm of the estimation error, $||\bar{e}_k||$

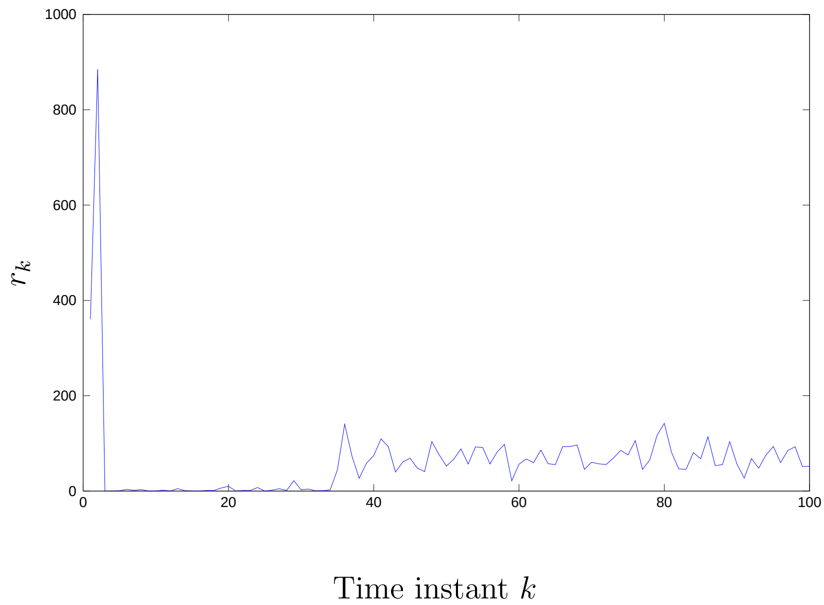

FIGURE 6. Value of the test statistics function, r_k

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