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STOCHASTIC MOTION OF NONLINEAR SYSTEMS OF VAN DER POL-MATHIEU TYPE

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Abstract. *In this paper, stochastic motion in nonlinear systems of Van der Pol-Mathieu type is studied for various damping parameters. At first, a sine circle map to the nonlinear system is derived. Secondly, an approximate piecewise continuous map to the sine circle map is derived. Finally, computational experiments for small damping parameters show that if the oscillation in approximate piecewise continuous map is stochastic, that is Lyapunov exponent to the map is positive, then the oscillation in original system of Van der Pol-Mathieu type exhibits stochastic motion. The results obtained in this paper are effective to design the nonlinear systems of Van der Pol-Mathieu type so as to generate chaotic motion and periodic oscillation.*

Keywords: Stochastic motion, Nonlinear systems, Van der Pol-Mathieu type

1. Introduction.

The importance of nonlinear dynamics has been considerably intensified by researchers [1-3]. Xin et al. and Liu et al. presented the chaos synchronization method to synchronize tow identical chaotic systems with different initial values [2-4]. Studies of nonlinear systems of Van der Pol-Mathieu type are important to design systems to be structurally stable [5-8]. Recently, the Van der Pol-Mathieu equation has played in the models such as micro electronic-mechanical systems [9-15]. Parlitz and Lauterborn studied the system with large damping [16]. They obtained the results concerning with period doubling cascades, the chaotic solution, and the devil's staircase in winding number and frequency of external force. Mettin et al. also obtained the bifurcation structure [17].

Recently, remarkable works [21-23] can be found in the *International Journal of Innovative Computing, Information and Control*.

In this paper, by computational experiments, various oscillation modes are shown for variation in the damping parameter. Based on Lyapunov exponent of the nonlinear system and that of approximate piecewise continuous map (PCM) to a sine circle map associated with the original nonlinear system, the results obtained in this paper enable us to design the systems so as to generate stochastic motion and periodic oscillation.

The paper is organized as follows. In Section 2, a sine circle map and an approximate PCM are derived, and in Section 3, it is shown that the rotation numbers in the maps form a part of a Farey series. In Section 4, the dependence of multiplicity on damping parameter is shown. In Section 5, based on Lyapunov exponents to approximate PCM, computational experiments are shown to generate stochastic motion and periodic oscillation in the original nonlinear systems. In Section 6, concluding remarks of this paper are presented.

2. **Sine Circle Map and Approximate PCM.** Nonlinear systems of Van der Pol-Mathieu type are represented by the following equation:

$$\ddot{x} + \left(\beta - \delta x^2\right)\dot{x} + \left(1 + \gamma x^2 + \mu\cos(\omega t)\right)x = 0, \tag{1}$$

where β and δ are damping parametors, γ is a nonlinear parameter, μ and ω are an amplitude and an angular frequency of the periodic force $\mu\cos(\omega t)$, respectively, where $\delta = 0.01$, $\gamma = 0.01$, μ is fixed to 20.0. The phase plane portrait of the system is shown in Figure 1(a). Figures 1(b) and 1(c) show the time evolution of x and $\dot{x}$, respectively.

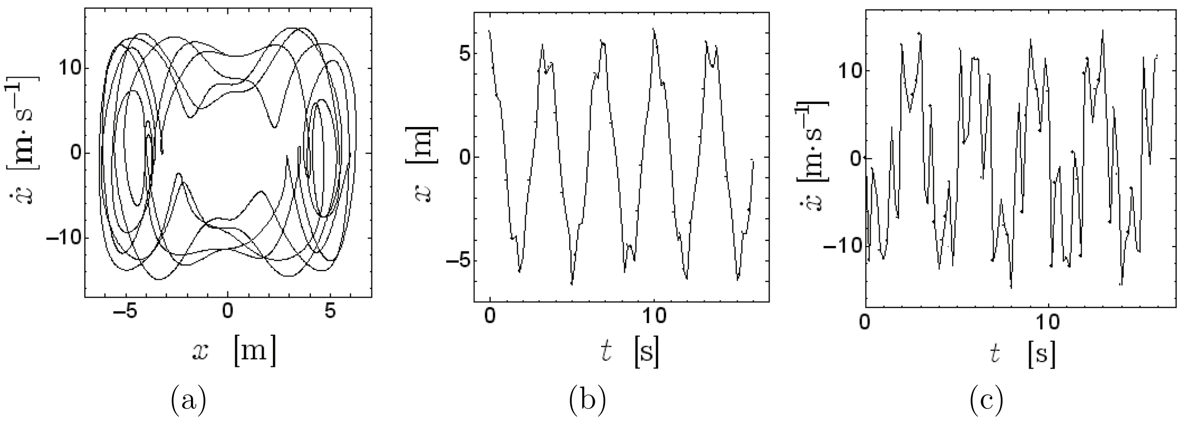


Figure 1. Oscillation and phase plane portrait of nonlinear systems of Van der Pol-Mathieu type: (a) phase plane portrait, (b) time evolution of x, (c) time evolution of $\dot{x}$, $\beta = 0.07875$ [s^{-1}], $\delta = 0.01$ [m$^{-2} \cdot s^{-1}$], $\gamma = 0.01$ [m$^{-2} \cdot$ s^{-2}], $\omega = 10.0$ [rad $\cdot$ s^{-1}], $\mu = 20.0$ [s^{-2}]

3. **Rotation Number and Farey Series in Sine Circle Map.** In order to obtain the picture of the dynamics, a sine circle map of the system is introduced. The angle variable of the system is defined as

$$\theta_n = \tan^{-1}\left(\frac{x(nT)}{\dot{x}(nT)}\right), \qquad n = 0, 1, \dots \tag{2a}$$

where

$$T = \frac{2\pi}{\omega}. \tag{2b}$$

The sine circle map F can be obtained by

$$\theta_{n+1} = F(\theta_n), \qquad n = 0, 1, \dots. \tag{3}$$

Figure 2 shows an example of evolution of an angle variable. The sine circle map is obtained by the angle variables. The oscillation mode $\pi_{r,s}(m, m+1)$ consisting of r times of the m peak oscillation (m-PO) and s times of $(m+1)$-PO of sine circle map is obtained for varying damping parameter β.

A rotation number W is defined by

$$W = \frac{P}{M}, \tag{4a}$$

where P and M denote the mixing degree and the multiplicity of the oscillation, respectively, P and M satisfy the relation:

$$P = r + s, \tag{4b}$$

$$M = mr + (m+1)s = mP + s. \tag{4c}$$

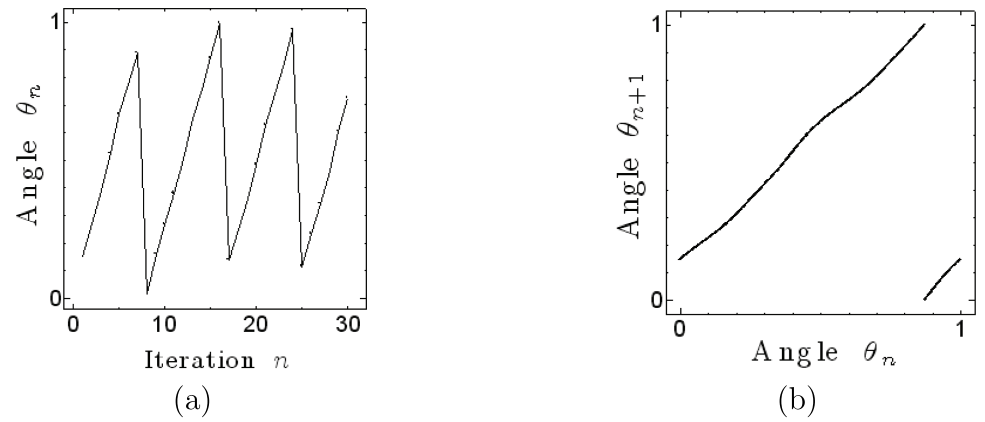


FIGURE 2. Evolution of angle: (a) mapping pattern, (b) sine circle map

The rotation number of the oscillation mode $\pi_{r,s}(m, m+1)$ consisting of r times of the m peak oscillation (m-PO) and s times (($m+1$)-PO) is obtained for varying damping parameter β.

$$W = \frac{P}{M} = \frac{r+s}{mr + (m+1)s}. \tag{5}$$

The mixing degree and the multiplicity P and M are studied for varying values of the damping parameter β. For the damping parameter β, the series of the rotation number W is as follows:
for β from 0.005 to 0.033,

$$F_{36} := \left\{ \frac{7}{36}, \frac{6}{31}, \frac{5}{26}, \frac{4}{21}, \frac{3}{16}, \frac{2}{11}, \frac{4}{23}, \frac{5}{29}, \frac{6}{35}, \frac{7}{41}, \frac{8}{47}, \frac{1}{6} \right\}, \tag{6a}$$

for β from 0.033 to 0.0575,

$$F_{104} := \left\{ \frac{1}{6}, \frac{4}{29}, \frac{3}{19}, \frac{2}{13}, \frac{3}{20}, \frac{4}{27}, \frac{5}{34}, \frac{6}{41}, \frac{7}{48}, \frac{8}{55}, \frac{9}{62}, \frac{10}{69}, \frac{11}{76}, \frac{12}{83}, \frac{13}{90}, \frac{14}{97}, \frac{15}{104}, \frac{1}{7} \right\}, \tag{6b}$$

for β from 0.0575 to 0.0576,

$$F_{99} := \left\{ \frac{1}{7}, \frac{14}{99}, \frac{23}{92}, \frac{12}{85}, \frac{11}{78}, \frac{10}{71}, \frac{9}{64}, \frac{8}{57}, \frac{7}{50}, \frac{6}{43}, \frac{5}{36}, \frac{4}{29}, \frac{3}{22}, \frac{2}{15}, \frac{3}{23}, \frac{4}{31}, \frac{5}{39}, \frac{6}{47}, \frac{7}{55}, \frac{8}{63}, \frac{1}{8} \right\}, \tag{6c}$$

for β from 0.0576 to 0.0906,

$$F_{81} := \left\{ \frac{1}{8}, \frac{14}{81}, \frac{9}{73}, \frac{8}{65}, \frac{7}{57}, \frac{6}{49}, \frac{5}{41}, \frac{4}{31}, \frac{3}{25}, \frac{2}{17}, \frac{3}{26}, \frac{4}{35}, \frac{5}{44}, \frac{6}{53}, \frac{7}{62}, \frac{8}{71}, \frac{1}{9} \right\}, \tag{6d}$$

for β from 0.0906 to 0.10250,

$$F_{29} := \left\{ \frac{1}{1}, \frac{1}{9}, \frac{3}{28}, \frac{2}{19}, \frac{3}{29}, \frac{1}{10} \right\}. \tag{6e}$$

In order to see the order of appearance of the oscillation modes of the system, a Farey series is introduced (see [18-20]). The Farey series F_M of order M is the monotonically increasing sequence of all irreducible rationales between 0 and 1. The Farey series can be represented as

$$F_M := \left\{ \frac{P_1}{M_1}, \frac{P_2}{M_2}, \frac{P_3}{M_3}, \frac{P_4}{M_4}, \ldots, \frac{P_i}{M_i}, \ldots, \frac{P_I}{M_I} \right\} \tag{7}$$

where P_i, M_i are positive integers ($M_i \leq M_I$). If P_i/M_i and P_{i+1}/M_{i+1} are consecutive terms of F_M, then the following relations hold;

$$P_i M_{i+1} - P_{i+1} M_i = 1. \quad (i = 1, 2, \ldots, I-1) \tag{8}$$

For example, the sets F_M for $M = 5$ and 10 are

$$\tilde{F}_5 := \left\{ \frac{1}{1}, \frac{4}{5}, \frac{3}{4}, \frac{2}{3}, \frac{3}{5}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \right\}, \tag{9}$$

$$\begin{aligned} \tilde{F}_{10} := \Big\{ & \frac{1}{1}, \frac{9}{10}, \frac{8}{9}, \frac{7}{8}, \frac{3}{5}, \frac{6}{7}, \frac{5}{6}, \frac{4}{5}, \frac{7}{9}, \frac{3}{4}, \frac{5}{7}, \frac{7}{10}, \frac{2}{3}, \frac{5}{8}, \frac{3}{5}, \frac{4}{7}, \frac{5}{9}, \\ & \frac{1}{2}, \frac{4}{9}, \frac{3}{7}, \frac{2}{5}, \frac{3}{8}, \frac{1}{3}, \frac{3}{10}, \frac{2}{7}, \frac{1}{4}, \frac{2}{9}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10} \Big\} \end{aligned} \tag{10}$$

and it follows that

$$\tilde{F}_5 \subset \tilde{F}_{10}. \tag{11}$$

The structure of Farey series $\tilde{F}_5$ is shown in Figure 3.

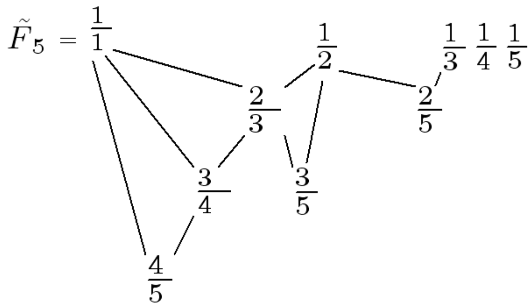


FIGURE 3. Farey series

Figure 3 shows that the rotation numbers W form a part of the Farey series.

4. **Dependence of Multiplicity on Damping Parameter.** The points in the diagram:

$$\left(\frac{M}{r+s}, \beta \right) \tag{12}$$

fall on a curve for a given value of P for $\pi_{r,s}(m, m+1)$ as

$P = 1$, on curve A: $\pi_{1,0}(m, m+1) \equiv \pi(m)$,
$P = 2$, on curve B: $\pi_{1,1}(m, m+1)$,
$P = 3$, on curve C: $\pi_{2,1}(m, m+1)$,
$P = 4$, on curve D: $\pi_{3,1}(m, m+1)$,

for the damping parameter β.

Figure 4 shows that as β approaches a certain value, say β_0, the value of M tends to ∞. For the curves A, B, C and D we suppose that the relation between M and β is

$$\left(\frac{M}{r+s} \right)^p (\beta_0 - \beta) = C_0, \tag{13}$$

for β, where p, β_0, C_0 are constants, respectively, and can be determined from the curves A, the curve B, the curve C and the curve D, respectively. Assuming the value of β_0 as the

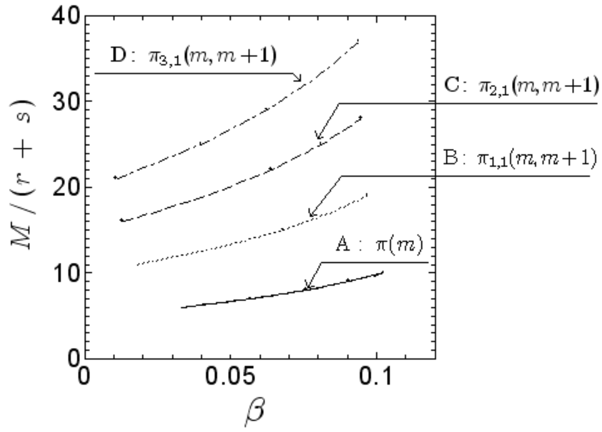


FIGURE 4. Multiplicity of oscillation mode $\pi_{r,s}(m, m+1)$ (for β from 0.009 to 0.080 m = 5, 6, 7, 8)

critical point where $M/(r+s)$ tends to ∞, we take $\beta_0 = 0.15$ for the curve A, $\beta_0 = 0.17$ for the curve B, $\beta_0 = 0.19$ for the curve C and $\beta_0 = 0.21$ for the curve D, respectively.

Equation (13) can change into the equation as follows:

$$\log(\beta_0 - \beta) = \log C_0 - p \log\left(\frac{M}{r+s}\right). \tag{14}$$

For the curve A, we put $r + s = 1$, for the curve B, $r + s = 2$, for the curve C, $r + s = 3$ and the curve D, $r + s = 4$, respectively. Then from Equation (14), we obtain

$$|\log(\beta_0 - \beta)| = p \log\left(\frac{M}{r+s}\right) - \log C_0. \tag{15}$$

Here, in Equation (15), we write as

$$\log\left(\frac{M}{r+s}\right) = X, \tag{16}$$

and

$$|\log(\beta_0 - \beta)| = Y. \tag{17}$$

Thus, $\log(M/(r+s))$ vs $|\log(\beta_0 - \beta)|$ is written as

$$Y = pX + b, \tag{18}$$

where $b = -\log C_0$. The relation of $|\log(\beta_0 - \beta)|$ to $\log(M/(r+s))$ is plotted by open squares in Figure 5. A line drawn through the squares points with least mean squares method.

Figure 5 shows that p and b values for the curve A as $p = 1.8486$, $b = 0.5526$, for the curve B as $p = 1.3402$, $b = 0.1783$, for the curve C as $p = 1.1084$, $b = 0.0567$, for the curve D as $p = 0.9448$, $b = 0.019$, respectively. Thus, the relation between the multiplicity M and the damping parameter β for the oscillation modes, can be written as follows:

$$\left(\frac{M}{r+s}\right)^p (\beta_0 - \beta) = C_0. \tag{19}$$

5. **Approximate PCM and Lyapunov Exponent.** By using least mean square method, PCM to the sine circle map is obtained as follows:

$$\theta_{n+1} := f(\theta_n) = A(\theta_n + C), \quad (0 \le \theta_n < \tilde{C}) \tag{20a}$$

$$\theta_{n+1} := g(\theta_n) = B(\theta_n - \tilde{C}), \quad (\tilde{C} < \theta_n < 1) \tag{20b}$$

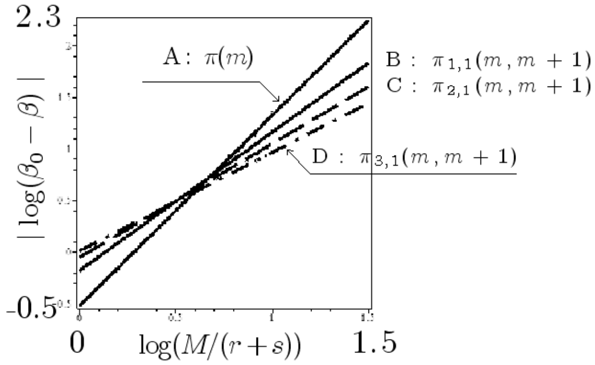


FIGURE 5. $\log(M/(r+s))$ vs $|\log(\beta_0 - \beta)|$ for β from 0.013 to 0.095 ($m = 5, 6, 7, 8$)

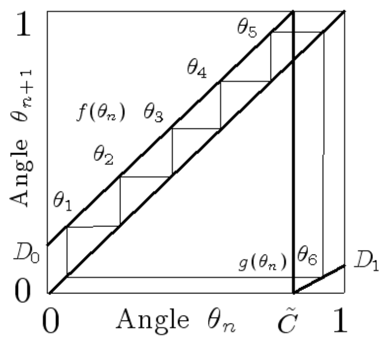


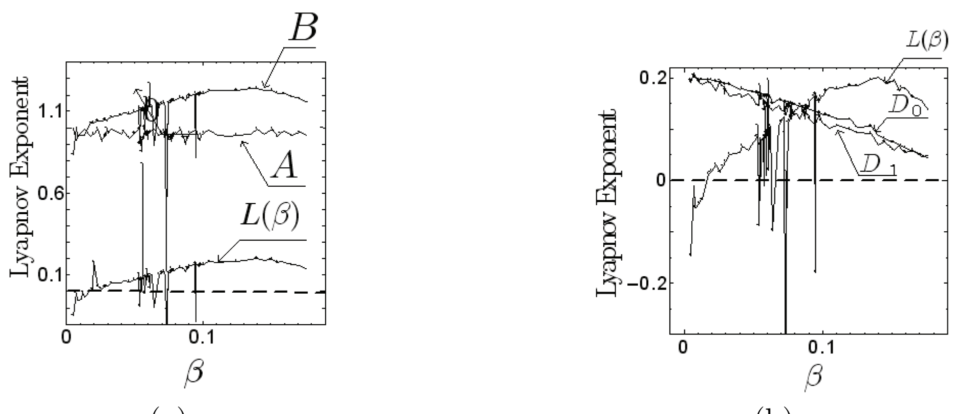
FIGURE 6. Approximate PCM

where $\tilde{C} = 1 - C$.

Figures 2(b) and 6 show that sine circle map in the system can be well approximated by the PCM. The Lyapunov exponent $L(\beta)$ to the PCM is introduced as follows:

$$L(\beta) = \frac{P}{P+M} \log\left(A^P B^M\right), \tag{21}$$

where P denotes the number of iteration for the mapping f in Equation (20a), which is degree of mixing, and M denotes the number of iteration for the mapping g in Equation (20b), which is degree of multiplicity. If $L > 0$, then the motion of the mapped points of sine circle map is stochastic, and that if $L < 0$, then the motion of the mapped points of sine circle map is not stochastic. Figure 7(a) shows that the gradients A, B, Lyapunov exponent vs the damping parameter β. Figure 7(b) shows that parameter D_0, D_1, Lyapunov exponent vs the damping parameter β. Figure 8 shows that the maximum Lyapunov exponent of nonlinear systems of Van der Pol-Mathieu type, Lyapunov exponent of approximate PCM for the sine circle map vs the damping parameter β. It is known that the Lyapunov exponent is positive to the dynamical systems, then the oscillation exhibits stochastic or chaotic (see [18]). The bifurcation diagram of the original nonlinear systems is shown in Figure 9. From Figures 7, 8 and 9, it follows that if oscillation in approximate PCM is stochastic, then the motion of original nonlinear systems of Van der Pol-Mathieu type is stochastic.



(a) (b)

FIGURE 7. Lyapunov exponent: (a) A, B: gradients of approximate PCM, (b) D_0, D_1: cut off parameters of approximate PCM

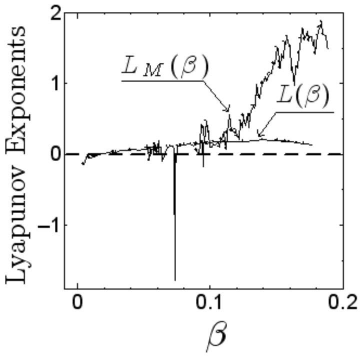


FIGURE 8. Lyapunov exponents ($L(\beta)$ Lyapunov exponent to approximate PCM $L_M(\beta)$: maximum Lyapunov exponent to original nonlinear system)

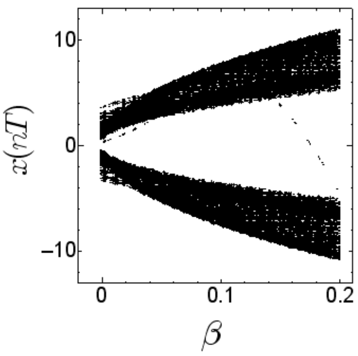


FIGURE 9. Bifurcation diagram

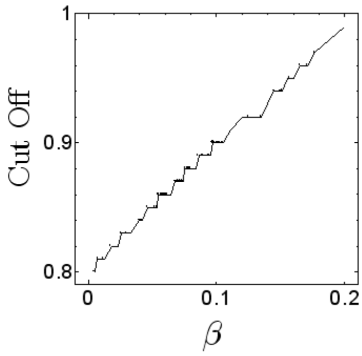


FIGURE 10. Damping parameter β vs cut off parameter $\tilde{C}$

Figure 10 shows that the cut off parameter $\tilde{C}$ vs the damping parameter β. From Figure 10, it follows that the dependence of the cut off parameter in the approximate PCM to the damping parameter is of the form of devil's stair.

6. **Concluding Remarks.** Concluding remarks of this paper are summarized as follows:

(i) The sine circle map and the approximate PCM of nonlinear systems of Van der Pol-Mathieu type are obtained.

(ii) It is shown that the series of the rotation number of oscillation in sine circle maps form a subset of Farey series in the small region of the damping parameter β.

(iii) The dependence of multiplicity M on the damping parameter β is obtained. For the small value of the damping parameter β, the value of the multiplicity M increases hyperbolically for increasing the damping parameter β. The relation between the multiplicity M and the damping parameter β in the oscillation is represented by Equation (19).

(iv) It is shown that if the Lyapunov exponent $L > 0$ in the approximate PCM, then the motion of the original nonlinear systems of Van der Pol-Mathieu type is stochastic.

(v) The results obtained in this paper are effective to design the nonlinear systems of Van der Pol-Mathieu type so as to generate stochastic motion and periodic oscillation.

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